Rational Points of Bounded Height on Compactifications of Anisotropic Tori

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Introduction

In this paper, we prove new results about the distribution of $K$-rational points of bounded height on algebraic varieties $X$ defined over a number field $K$ [3], [17].
Let \( \mathcal{L} = (L, \| \cdot \|_v) \) be an ample metrized invertible sheaf on \( X \) with a family \( \{ \| \cdot \|_v \} \) of \( v \)-adic metrics (\( v \) runs over the set \( \text{Val}(K) \) of all valuation of \( K \)), and \( H_{\mathcal{L}} : X(K) \to \mathbb{R}_{>0} \) the height function associated with \( (L, \| \cdot \|_v) \). We study the height zeta-function defined by the series

\[
Z_{\mathcal{L}}(s) = \sum_{x \in X(K)} H_{\mathcal{L}}(x)^{-s}.
\]

The investigation of the analytical properties of \( Z_{\mathcal{L}}(s) \) was initiated by Arakelov and Faltings [1, 16], who considered the case when \( X = S^d(C) \) is the \( d \)-th symmetric power of an algebraic curve \( C \) of genus \( g < d - 1 \). It was shown in [17] that height zeta-functions of generalized flag varieties are Langlands-Eisenstein series.

In this paper we are interested in analytical properties of height zeta-functions in the case when \( X \) is an equivariant compactification of an anisotropic torus \( T \). It is easy to show (see 1.3.7) that all \( K \)-rational points of \( X \) are contained in the torus \( T \) itself. Therefore,

\[
Z_{\mathcal{L}}(s) = \sum_{x \in T(K)} H_{\mathcal{L}}(x)^{-s}.
\]

Now the main idea for the computation of the height zeta-function \( Z_{\mathcal{L}}(s) \) is to use the group structure on \( T(K) \) and to apply the Poisson formula in the following form:

Let \( \mathcal{G} \) be a locally compact topological abelian group with a Haar measure \( dx \), \( \mathcal{H} \subset \mathcal{G} \) a discrete subgroup such that \( \mathcal{G}/\mathcal{H} \) is compact, and \( F : \mathcal{H} \to \mathbb{R} \) a function on \( \mathcal{H} \) which can be extended to an \( L^1 \)-function on \( \mathcal{G} \). Let \( \text{vol}(\mathcal{G}/\mathcal{H}) \) be the \( dx \)-volume of the fundamental domain of \( \mathcal{H} \) in \( \mathcal{G} \). Denote by \( \hat{F} \) the Fourier transform of \( F \) with respect to the Haar measure \( dx \). Suppose that \( \hat{F} \in L^1(\mathcal{H}^\perp) \). Then

\[
\sum_{x \in \mathcal{H}} F(x) = \frac{1}{\text{vol}(\mathcal{G}/\mathcal{H})} \sum_{x \in \mathcal{G}^\perp} \hat{F}(x).
\]

where \( \mathcal{G}^\perp \) is the group of characters of \( \mathcal{G} \) which are trivial on \( \mathcal{H} \).

We will apply this formula in the case when \( \mathcal{G} = T(A_K) \) is the adele group of an anisotropic algebraic torus \( T \), \( \mathcal{H} \) is the subgroup \( T(K) \subset T(A_K) \) of all \( K \)-rational points of \( T \). The adelic method turns out to be very convenient, because the height function \( H_{\mathcal{L}} \) splits into the product of local Weil functions \( H_{\mathcal{L},v} : T(K_v) \to \mathbb{R}_{>0} \). This allows us to extend \( H_{\mathcal{L}}^{-s} \) to a function on the whole adele group \( \mathcal{G} \). Moreover, the local Weil functions \( H_{\mathcal{L},v} \) can be chosen to be \( T(O_v) \)-invariant. Therefore, the Fourier transform

\[
\int_{T(A_K)} H_{\mathcal{L}}(x)^{-s} \chi(x) d\mu = \prod_{v \in \text{Val}(K)} \int_{T(K_v)} H_{\mathcal{L},v}(x)^{-s} \chi_v(x) d\mu_v,
\]

equals 0 unless the restriction of \( \chi_v \) on \( T(O_v) \) is trivial for all \( v \in \text{Val}(K) \). The Fourier transform of \( H_{\mathcal{L}}(x)^{-s} \) can be calculated separately for each local factor \( H_{\mathcal{L},v}(x)^{-s} \). The
analytic properties of the Fourier transform of $H_L(x)^{-s}$ can be investigated by the method of Draxl [15]. This method was first applied to arithmetical problems by Moroz [25].

In Section 1, we recall basic facts from theories of algebraic tori and toric varieties $P_{\Sigma}$ associated with fans $\Sigma$ over arbitrary fields.

In Section 2, following ideas in [17], we define canonical families of $v$-adic metrics on all $T$-linearized invertible sheaves $L$ on $P_{\Sigma}$ simultaneously. This allows us to construct the complex height $H_L(x, \varphi)$ and the associated zeta-function $Z_L(\varphi)$ as a function of $\varphi$, where $\varphi$ represents an element of the complexified Picard group $\text{Pic}(P_{\Sigma})_\mathbb{C} := \text{Pic}(P_{\Sigma}) \otimes \mathbb{C}$. One obtains the 1-parameter zeta-function $Z_L(s)$ via the restriction of $Z_L(\varphi)$ to the complex line $s[l], s \in \mathbb{C}$.

In Section 3, we investigate the analytic properties of zeta-functions in order to obtain an asymptotic formula for the number of rational points of bounded height. As one of our main results, we prove the following refinement of the conjecture of Manin:

Let $P_\Sigma$ be a smooth projective compactification of an anisotropic torus $T$. Let $r$ be the rank of $\text{Pic}(P_{\Sigma},K)$. Then there exist only a finite number $N(P_\Sigma, K^{-1}, B)$ of $K$-rational points $x \in T(K)$ having the anticanonical height $H_{K^{-1}}(x) \leq B$. Moreover,

$$N(P_\Sigma, K^{-1}, B) = \Theta(\Sigma, K) \cdot B(\log B)^{r-1}(1 + o(1)), \quad B \to \infty,$$

where the constant $\Theta(\Sigma, K)$ depends on:

1. the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma) \subset \text{Pic}(P_{\Sigma})_\mathbb{R}$;
2. the Brauer group of $P_{\Sigma}$;
3. the Tamagawa number $\tau_K(P_{\Sigma})$ associated with the metrized canonical sheaf on $P_{\Sigma}$ (as it was defined by E. Peyre in [29]).

We also prove the Batyrev-Manin conjecture [3] describing the asymptotic for the number of $K$-rational points $x \in T(K)$ such that $H_{\Sigma}(x) \leq B$ in terms of $[L]$ and the geometry of the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma)$. Since, for compactifications of anisotropic tori, the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma) \subset \text{Pic}(P_{\Sigma})_\mathbb{R}$ is simplicial (see 1.3.11), our situation is very close to the case of generalized flag varieties considered in [3], [17].

In [29], Peyre refined the conjectures from [3] by proposing an adelic interpretation of the asymptotic constant for the metrized anticanonical line bundle for Fano varieties satisfying weak approximation. He checked the compatibility of his conjecture with results obtained by the circle method and in the case of split generalized flag varieties. He also calculated the constant for some split and non-split toric blowing ups of projective spaces. In our paper, we verify the Tamagawa number conjecture of Peyre for the constant in the asymptotical formula in the case of smooth compactifications of anisotropic tori.

We observe that our results provide the first examples of asymptotics for the
number of rational points of bounded height on unirational varieties of small dimension which are not rational (nonrational anisotropic tori in dimension 3 were constructed in [20]) and on varieties where weak approximation does not hold in general. We also do not assume that the considered toric varieties $\mathbf{P}_\Sigma$ are Fano. Another new phenomenon is the appearance of the nontrivial part of the Brauer group $\text{Br}(\mathbf{P}_\Sigma) / \text{Br}(K)$ in the asymptotic formulas.

1 Toric varieties over arbitrary fields

1.1 Algebraic tori

Let $K$ be an arbitrary field, $\overline{K}$ the algebraic closure of $K$, and $G_m(\overline{K}) = \overline{K}^*$ the multiplicative group of $\overline{K}$.

Let $X$ be a variety over $K$ and $E$ an extension of $K$. We shall then denote by $X(E)$ the set of $E$-points of $X$ and by $X_E$ the $E$-variety obtained by base extension.

Definition 1.1.1. A linear algebraic group $T$ over $K$ is called a $d$-dimensional algebraic torus if its base extension $T_E$ is isomorphic to $G_m^d$ over $E$.

We notice that this isomorphism is always defined over a finite Galois extension $E$ of $K$.

Definition 1.1.2. Let $T$ be an algebraic torus over $K$. A finite Galois extension $E$ of $K$ such that $T_E$ is isomorphic to $G_m^d$ over $E$ is called a splitting field of $T$.

Definition 1.1.3. We denote by $\hat{T} = \text{Hom}(T, \overline{K}^*)$ the group of regular $\overline{K}$-rational characters of $T$. For any subfield $E \subset \overline{K}$ containing $K$, we denote by $\hat{T}_E$ the group of characters of $T$ defined over $E$.

Let us formulate a well-known correspondence between Galois representations by integral matrices and algebraic tori [4], [14], [27], [39].

Theorem 1.1.4. There is a contravariant equivalence (duality) between the category of algebraic tori defined over a number field $K$ and the category of discrete continuous torsion-free $\text{Gal}(\overline{K}/K)$-modules of finite rank over $\mathbb{Z}$. The functors are given by

$$M \rightarrow T = \text{Spec } K[M]; \quad T \rightarrow \hat{T}.$$

The above contravariant equivalence is functorial under field extensions of $K$. This functoriality induces a contravariant equivalence between the categories of $E$-split algebraic tori over $K$ and $\text{Gal}(E/K)$-modules.
If \( G = \text{Gal}(E/K) \) is the Galois group of the splitting field \( E \) of a \( d \)-dimensional torus \( T \), then the structure of a \( G \)-module on the free abelian group \( \hat{T} \) of rank \( d \) is defined by the natural integral representation

\[
\rho : G \to \text{Aut}(\hat{T}) \cong GL(d, \mathbb{Z}).
\]

Remark 1.1.5. The group \( \hat{T}_K \) is a sublattice in \( \hat{T} \cong \mathbb{Z}^d \) consisting of all \( G \)-invariant elements.

Definition 1.1.6. An algebraic torus \( T \) over \( K \) is called anisotropic if \( \hat{T}_K \) has rank zero.

Example 1.1.7. Let \( K' \) be a finite separable extension of \( K \), and \( E \) the minimal Galois extension of \( K' \) containing \( K' \). Then the multiplicative group \((K')^*\) of \( K' \) is a \([K' : K]\)-dimensional algebraic torus over \( K \) which is usually denoted by \( R_{K'/K}(\mathbb{G}_m) \). The torus \( R_{K'/K}(\mathbb{G}_m) \) splits over \( E \). Let \( G = \text{Gal}(E/K) \) and \( G' = \text{Gal}(E/K') \subset G \). Then the \( G \)-module of characters of \( R_{K'/K}(\mathbb{G}_m) \) is isomorphic to the permutational \( G \)-module \( M = \mathbb{Z}[G/G'] \).

Let \( \{x_1, \ldots, x_n\} = G/G' \). Then one has the natural \( G \)-epimorphism \( \varepsilon : M \to \mathbb{Z} \), which is defined by

\[
\varepsilon \left( \sum_{i=1}^{n} a_i x_i \right) = \sum_{i=1}^{n} a_i
\]

where \( \mathbb{Z} \) is considered as the trivial \( G \)-module. Let \( I \) be the kernel of \( \varepsilon \). Then \( I = \hat{T} \), where \( T = R_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m \).

The subtorus of \( R_{K'/K}(\mathbb{G}_m) \) consisting of elements with norm 1 is usually denoted by \( R_{K'/K}^1(\mathbb{G}_m) \). It corresponds to the dual \( G \)-module \( I^* \), which is obtained from the dual short exact sequence

\[
0 \to \mathbb{Z} \to M^* \to I^* \to 0.
\]

Both tori \( R_{K'/K}^1(\mathbb{G}_m) \) and \( R_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m \) are anisotropic.

1.2 Compactifications of split tori

We recall standard facts about toric varieties over algebraically closed fields [10], [13], [18], [26]. Let \( M \) be a free abelian group of rank \( d \) and \( N = \text{Hom}(M, \mathbb{Z}) \) the dual abelian group.

Definition 1.2.1. A finite set \( \Sigma \) consisting of convex rational polyhedral cones in \( N_\mathbb{R} = N \otimes \mathbb{R} \) is called a \( d \)-dimensional fan if the following conditions are satisfied:

(i) Every cone \( \sigma \in \Sigma \) contains 0 \( \in N_\mathbb{R} \).
(ii) Every face $\sigma'$ of a cone $\sigma \in \Sigma$ belongs to $\Sigma$.

(iii) The intersection of any two cones in $\Sigma$ is a face of both cones.

Definition 1.2.2. A $d$-dimensional fan $\Sigma$ is called complete and regular if the following additional conditions are satisfied:

(i) $N_{\mathbb{R}}$ is the union of cones from $\Sigma$.

(ii) Every cone $\sigma \in \Sigma$ is generated by a part of a $\mathbb{Z}$-basis of $\mathbb{N}$.

We denote by $\Sigma(i)$ the set of all $i$-dimensional cones in $\Sigma$. For each cone $\sigma \in \Sigma$, we denote by $N_{\sigma,\mathbb{R}}$ the minimal linear subspace containing $\sigma$.

Every complete regular $d$-dimensional fan defines a smooth equivariant compactification $P_\Sigma$ of the split $d$-dimensional algebraic torus $T$. The variety $P_\Sigma$ has the following two geometric properties.

**Proposition 1.2.3.** The toric variety $P_\Sigma$ is the union of split algebraic tori $T_\sigma$ (dim $T_\sigma = d - \dim \sigma$):

$$P_\Sigma = \bigcup_{\sigma \in \Sigma} T_\sigma.$$ 

For each $k$-dimensional cone $\sigma \in \Sigma(k)$, $T_\sigma$ is the kernel of a homomorphism $T \to G_k^1$ defined by a $\mathbb{Z}$-basis of the sublattice $N \cap N_{\sigma,\mathbb{R}} \subset N$. 

Let $\check{\sigma}$ denote the cone in $M_{\mathbb{R}}$ which is dual to $\sigma$.

**Proposition 1.2.4.** The toric variety $P_\Sigma$ has a $T$-invariant open covering by affine subsets $U_\sigma$:

$$P_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$$ 

where $U_\sigma = \text{Spec} \mathcal{K}[M \cap \check{\sigma}]$. 

**Definition 1.2.5.** A continuous function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ is called $\Sigma$-piecewise linear if the restriction $\varphi_\sigma$ of $\varphi$ to every cone $\sigma \in \Sigma$ is a linear function. It is called integral if $\varphi(N) \subset \mathbb{Z}$.

**Definition 1.2.6.** For any integral $\Sigma$-piecewise linear function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$ and any cone $\sigma \in \Sigma(d)$, we denote by $m_{\sigma,\varphi}$ the restriction of $\varphi$ to $\sigma$ considered as an element in $M$. We put $m_{\sigma',\varphi} = m_{\sigma,\varphi}$ if $\sigma'$ is a face of a $d$-dimensional cone $\sigma \in \Sigma$.

**Definition 1.2.7.** For any integral $\Sigma$-piecewise linear function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$, we define the invertible sheaf $L(\varphi)$ as the subsheaf of the constant sheaf of rational functions on $P_\Sigma$ generated over $U_\sigma$ by the element $-m_{\sigma,\varphi}$ considered as a character of $T \subset P_\Sigma$. 
Remark 1.2.8. The \( T \)-action on the sheaf of rational functions restricts to the subsheaf \( L(\varphi) \), so that we can consider \( L(\varphi) \) as a \( T \)-linearized line bundle over \( P_\Sigma \).

Denote by \( e_1, \ldots, e_n \) the primitive integral generators of all 1-dimensional cones in \( \Sigma \). Let \( T_i \) (\( i = 1, \ldots, n \)) be the \((d - 1)\)-dimensional torus orbit corresponding to the cone \( \mathbb{R}_{\geq 0} e_i \in \Sigma \), and \( D_i \) the Zariski closure of \( T_i \) in \( P_\Sigma \). Define \( D(\Sigma) \cong \mathbb{Z}^n \) as the free abelian group of \( T \)-invariant Weil divisors on \( P_\Sigma \) with the basis \( D_1, \ldots, D_n \).

**Proposition 1.2.9.** The correspondence \( \varphi \to L(\varphi) \) gives rise to an isomorphism between the group of \( T \)-linearized line bundles on \( P_\Sigma \) and the group \( PL(\Sigma) \) of all \( \Sigma \)-piecewise linear integral functions on \( N_\mathbb{R} \). There is the canonical isomorphism

\[
PL(\Sigma) \cong D(\Sigma), \quad \varphi \mapsto (\varphi(e_1), \ldots, \varphi(e_n)).
\]

The Picard group \( \text{Pic}(P_\Sigma) \) is isomorphic to \( PL(\Sigma)/M \) where elements of \( M \) are considered as globally linear integral functions on \( N_\mathbb{R} \), so that we have the exact sequence

\[
0 \to M \to D(\Sigma) \to \text{Pic}(P_\Sigma) \to 0. \tag{1}
\]

**Definition 1.2.10.** Let \( \Lambda_{\text{eff}}(\Sigma) \) be the cone in \( \text{Pic}(P_\Sigma) \) generated by classes of effective divisors on \( P_\Sigma \). Denote by \( \Lambda^*_{\text{eff}}(\Sigma) \) the dual to \( \Lambda_{\text{eff}}(\Sigma) \) cone.

**Proposition 1.2.11.** \( \Lambda_{\text{eff}}(\Sigma) \) is generated by the classes \([D_1], \ldots, [D_n] \).

**Proof.** Since \( \text{Pic}(T) = \text{Pic}(P_\Sigma \setminus \bigcup_{i=1}^n D_i) = 0 \), any divisor \( D \) on \( P_\Sigma \) is linearly equivalent to an integral linear combination of \( D_1, \ldots, D_n \). Assume that \( D = a_1 D_1 + \cdots + a_n D_n \) is effective. Then there exists a rational function \( f \) on \( P_\Sigma \) having no poles and zeros on \( T \) such that

\[
(f) + D \geq 0. \tag{2}
\]

Hence we can assume that \( f \) is a character of \( T \) defined by an element \( m_i \in M \). Since \( (f) = \sum_{i=1}^n (m_i, e_i) D_i \), the condition (2) is equivalent to

\[
b_i = (m_i, e_i) + a_i \geq 0, \quad i = 1, \ldots, n.
\]

Then \( D' = b_1 D_1 + \cdots + b_n D_n \) is linearly equivalent to \( D \). So every effective class \([D]\) contains a nonnegative integral linear combination of \([D_1], \ldots, [D_n] \). \( \Box \)

**Proposition 1.2.12.** Let \( \varphi_\Sigma \) be the \( \Sigma \)-piecewise linear integral function such that \( \varphi(e_1) = \cdots = \varphi(e_n) = 1 \). Then \( L(\varphi_\Sigma) \) is isomorphic to the \( T \)-linearized anticanonical line bundle on the toric variety \( P_\Sigma \). \( \Box \)
Example 1.2.13. Projective spaces. Consider a $d$-dimensional fan $\Sigma$ whose 1-dimensional cones are generated by $d+1$ elements $e_1, \ldots, e_d, e_{d+1} = -(e_1 + \cdots + e_d)$, where $\{e_1, \ldots, e_d\}$ is a $\mathbb{Z}$-basis of $d$-dimensional lattice $N$, and $k$-dimensional cones in $\Sigma$ are generated by all possible $k$-element subsets in $\{e_1, \ldots, e_{d+1}\}$ ($0 \leq k \leq d$). Then the corresponding compactification $P_\Sigma$ of the $d$-dimensional split torus is $P^d$.

Remark 1.2.14. Let $z_1, \ldots, z_n$ be affine coordinates on $A^n$, and $V(\Sigma) \subset A^n$ the closed subset defined by the equations

$$\prod_{e_i \not\in \sigma} z_i = 0$$

where $\sigma$ runs over all cones in $\Sigma$. There exists an alternative description of the complete algebraic variety $P_\Sigma$ as a quotient of the open subset $U(\Sigma) = A^n \setminus V(\Sigma)$ in $A^n$ by the action of the $(n-d)$-dimensional torus $T_{NS}$ having $\text{Pic}(P_\Sigma)$ as its group of characters [2], [9], [12]. In Example 1.2.13, $U(\Sigma) = A^{d+1} \setminus 0$ and $T_{NS} \cong G_m$. The algebraic torus $T_{NS}$ is usually called the Neron-Severi torus (see, e.g., [21]). The $d$-dimensional torus $T \subset P_\Sigma$ is the quotient of $G_m^n \subset U(\Sigma)$ by the Neron-Severi torus $T_{NS}$.

Remark 1.2.15. It is easy to see that the combinatorial construction of toric varieties $P_\Sigma$ immediately extends to arbitrary fields $K$; i.e., using a rational complete polyhedral fan $\Sigma$, one can define the toric variety $P_{\Sigma,K}$ as the equivariant compactification of the split torus $G_m^d$.

1.3 Compactifications of nonsplit tori

Let $T$ be a $d$-dimensional algebraic torus over $K$ with a splitting field $E$ and $G = \text{Gal}(E/K)$. Denote by $M$ the lattice $\hat{T}$ and put $N = \text{Hom}(M, \mathbb{Z})$. Let $\rho^*$ be the integral representation of $G$ in $\text{Aut}(N) \cong \text{GL}(d, \mathbb{Z})$, which is dual to $\rho$. In order to construct a compactification of $T$ over $K$, we need a complete fan $\Sigma$ of cones having an additional combinatorial structure: an action of the Galois group $G$ [40].

Definition 1.3.1. A complete fan $\Sigma \subset N_R$ is called $G$-invariant if, for any $g \in G$ and for any $\sigma \in \Sigma$, one has $\rho^*(g)(\sigma) \in \Sigma$. Let $N^G$ (resp., $M^G$, $N^G_R$, $M^G_R$, and $\Sigma^G$) be the subset of $G$-invariant elements in $N$ (resp., in $M$, $N_R$, $M_R$, and $\Sigma$). Denote by $\Sigma^G \subset N^G_R$ the fan consisting of all possible intersections $\sigma \cap N^G_R$ where $\sigma$ runs over all cones in $\Sigma$.

Proposition 1.3.2. Let $G$ be an arbitrary finite subgroup of $\text{Aut}(N)$, and $\Sigma \subset N_R$ a complete regular $G$-invariant fan. Then $\Sigma^G$ is again a complete regular fan in $N^G_R$ with respect to the lattice $N^G_R$. \qed
Proof. Since \( N_R = \bigcup_{\sigma \in \Sigma} \sigma \), the intersections \( \sigma \cap N_R \) cover the whole space \( N_R \). It remains to prove that every cone in \( \Sigma_G \) is generated by a part of an integral basis of \( N^G \).

Let \( \overline{\sigma} \) be a cone of \( \Sigma_G \), and \( \sigma \) the minimal cone of \( \Sigma \) containing \( \overline{\sigma} \). Since \( G \) acts by identity on \( \overline{\sigma} \), \( \sigma \) must be \( G \)-invariant. (Otherwise, there exists a face of \( \sigma \) containing \( \overline{\sigma} \).) Therefore, \( G \) acts by permutations on the set \( A = \{ e_1, \ldots, e_n \} \) of all generators of \( \sigma \). Let \( A = A_1 \cup \cdots \cup A_l \) be the splitting of \( A \) into \( G \)-orbits. We define lattice vectors \( e_1, \ldots, e_l \in N^G \) as

\[
\overline{e}_i = \sum_{e_j \in A_i} e_j, \quad i = 1, \ldots, l.
\]

Since \( \overline{\sigma} = \sigma \cap N^G_R \), \( \overline{e}_1, \ldots, \overline{e}_l \) must be generators of \( \overline{\sigma} \). Since \( A \) is a part of an integral basis of \( N \), the system \( \{ \overline{e}_1, \ldots, \overline{e}_l \} \) is also a part of an integral basis of \( N \). Since \( N^G \) is a direct summand of \( N \), \( \{ \overline{e}_1, \ldots, \overline{e}_l \} \) is a part of an integral basis of \( N^G \).

**Corollary 1.3.3.** The set of generators of 1-dimensional cones in \( \Sigma_G \) consists of all vectors

\[
\overline{e} = e_{i_1} + \cdots + e_{i_k}
\]

such that \( \{ e_{i_1}, \ldots, e_{i_k} \} \) is a \( G \)-orbit of the \( G \)-action on \( \{ e_1, \ldots, e_n \} \), and \( e_{i_1}, \ldots, e_{i_k} \) are generators of a \( k \)-dimensional cone \( \sigma \in \Sigma_G \).

The \( G \) action on the fan \( \Sigma \) induces a \( G \)-action on \( P_{\Sigma,E} \), which permutes orbits of the split torus \( T_E \) acting on \( P_{\Sigma,E} \). In particular, this \( G \)-action sends the open subset \( T_E \) given by the 0-cone (see 1.2.3) into itself. The following theorem is due to Voskresenskii [40].

**Theorem 1.3.4.** Let \( \Sigma \) be a complete regular \( G \)-invariant fan in \( N_R \). Assume that the complete toric variety \( P_{\Sigma,K} \) defined over the splitting field \( E \) by the \( G \)-invariant fan \( \Sigma \) is projective. Then there exists a complete algebraic \( K \)-variety \( P_{\Sigma,K} \) (unique up to \( K \)-isomorphism) such that \( P_{\Sigma,K} \otimes_{\text{Spec } K} \text{ Spec } E \) and \( P_{\Sigma,E} \) are isomorphic as \( E \)-varieties with \( G \)-action. There is also a natural open immersion \( T \subset P_{\Sigma,K} \) compatible with the inclusion \( T_E \subset P_{\Sigma,E} \) and a unique \( K \)-morphism \( T \times P_{\Sigma,K} \rightarrow P_{\Sigma,K} \) extending the multiplications \( T \times T \rightarrow T \).

Proof. The \( G \)-action on \( P_{\Sigma,E} \) defines a descent datum on \( P_{\Sigma,E} \). This descent datum is *effective* if and only if the \( G \)-orbit of each point of \( P_{\Sigma,E} \) is contained in a quasi-affine subset [5, pp. 139–141], thereby giving the existence (and uniqueness) of \( P_{\Sigma,K} \) when \( P_{\Sigma,E} \) is projective. It also gives an open embedding \( T \subset P_{\Sigma,K} \), since the restriction of the \( G \)-action to \( T_E \subset P_{\Sigma,E} \) defines a descent datum on \( T_E \). Moreover, since the \( G \)-action on \( P_{\Sigma,E} \) respects
the toric $E$-morphism $T_E \times P_{\Sigma,E} \to P_{\Sigma,E}$, it is easy to see that one can descend to a toric $K$-morphism $T \times P_{\Sigma,K} \to P_{\Sigma,K}$.\footnote{We are grateful to the referee for suggesting this argument.}

Remark 1.3.5. Our definition of heights (2.1.8) and the proof of the analytic properties of height zeta-functions do not use the projectivity of the compactification $P_{\Sigma,K}$. However, we decided to omit the discussion of the technical point of existence of nonprojective compactifications of nonsplit tori in full generality.

**Proposition 1.3.6.** Let $\sigma$ be an arbitrary cone in $\Sigma^G$. Denote by $T_\sigma$ the $(d - \dim \sigma)$-dimensional algebraic torus over $K$ corresponding to the restriction of the integral $G$-representation in $GL(M)$ to the sublattice $(\hat{\sigma} \cap -\hat{\sigma}) \cap M \subset M$. Then

$$P_{\Sigma,K}(K) = \bigcup_{\sigma \in \Sigma^G} T_\sigma(K).$$

Proof. We consider $P_{\Sigma}(K)$ as the set of $G$-invariant points in $P_{\Sigma}(E)$. Since $P_{\Sigma}(E)$ is a disjoint union of $T_\sigma(E)$ and $G$ acts on the $T$-orbits $T_\sigma (\sigma \in \Sigma)$ by permutations of cones in the $G$-invariant fan $\Sigma$, the set of $G$-invariant points in $T_\sigma(E)$ is nonempty if and only if $\sigma$ is $G$-invariant.

**Corollary 1.3.7 [8].** Let $T$ be an anisotropic torus and $P_{\Sigma,K}$ a $T$-equivariant smooth $K$-compactification as described above. Then all $K$-rational points of $P_{\Sigma,K}$ are contained in $T$ itself.

Proof. By 1.3.6, it is sufficient to prove that for an anisotropic torus $T$ defined by some Galois representation of $G$ in $GL(M)$, there is no $G$-invariant cone $\sigma$ of positive dimension in $\Sigma$.

Assume that there exists a $G$-invariant $k$-dimensional cone $\sigma (k > 0)$ with the generators $\{e_1, \ldots, e_k\}$. Then $e_1 + \cdots + e_k$ is a nonzero $G$-invariant integral vector in the interior of $\sigma$. Hence the sublattice $N^G$ of $G$-invariant elements in $N$ has positive rank. Thus $M^G \cong \hat{T}_k$ also has positive rank, which gives a contradiction.\]

Taking $G$-invariant elements in the short exact sequence (1.2.9), we obtain the exact sequence

$$0 \to M^G \to D(\Sigma)^G \to \text{Pic}(P_{\Sigma,E})^G \to H^1(G,M) \to 0.$$  \hfill (3)

**Proposition 1.3.8.** The group $\text{Pic}(P_{\Sigma,E})^G$ is canonically isomorphic to the Picard group $\text{Pic}(P_{\Sigma,K})$. Moreover, $H^1(G,M)$ is a finite group which is isomorphic to the Picard group of $T$.\]

1We are grateful to the referee for suggesting this argument.
Corollary 1.3.9. The correspondence $\varphi \to L(\varphi)$ induces an isomorphism between the group of $T$-linearized invertible sheaves on $P_{\Sigma,K}$ and the group $PL(\Sigma)^G$ of all $\Sigma$-piecewise linear integral $G$-invariant functions on $N_K$. An invertible sheaf $L$ on $P_{\Sigma,K}$ admits a $T$-linearization if and only if the restriction of $L$ on $T$ is trivial. In particular, some tensor power of $L$ always admits a $T$-linearization.

Corollary 1.3.10. Let $\Lambda_{\text{eff}}(\Sigma,K)$ (resp., $\Lambda_{\text{eff}}(\Sigma,E)$) be the cone of effective divisors of $P_{\Sigma,K}$ (resp., $P_{\Sigma,E}$). Then $\Lambda_{\text{eff}}(\Sigma,K)$ consists of $G$-invariant elements in $\Lambda_{\text{eff}}(\Sigma,E)$.

Proposition 1.3.11. Let $T$ be an anisotropic torus, $P_{\Sigma,K}$ a $T$-equivariant smooth $K$-compactification as described above, and

$$\Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_r(1)$$

the decomposition of $\Sigma(1)$ into the union of $G$-orbits. Then the cone of effective divisors $\Lambda_{\text{eff}}(\Sigma,K)$ is simplicial, and it is generated by classes of $G$-invariant divisors

$$\sum_{\sigma_j \in \Sigma_i(1)} D_j, \quad i = 1, \ldots, r.$$ 

In particular, the rank of the Picard group $\text{Pic}(P_{\Sigma,K})$ equals the number of $G$-orbits in $\Sigma(1)$.

Proof. Tensoring (3) by $R$, we obtain the isomorphism

$$D(\Sigma)^G \otimes R \cong \text{Pic}(P_{\Sigma,E}) \otimes R.$$

Let $\Lambda = R^n_{\geq 0} \subset D(\Sigma) \otimes R = R^n$ be the $n$-dimensional positive octant. By 1.2.11, $\Lambda_{\text{eff}}(\Sigma,E)$ is the image of $\Lambda$ under the homomorphism in (3). By 1.3.10, $\Lambda_{\text{eff}}(\Sigma,K)$ coincides with the cone $\Lambda^G$ of $G$-invariant elements in $\Lambda$. Recall that $G$ acts by permutations on the canonical basis of $D(\Sigma)$, whose elements are in the 1-to-1 correspondence with elements of $\Sigma(1)$. Hence $\Lambda_{\text{eff}}(\Sigma,K)$ is generated by classes of $G$-invariant divisors $\sum_{\sigma_j \in \Sigma_i(1)} D_j, \quad i = 1, \ldots, r$, i.e., $\Lambda_{\text{eff}}(\Sigma,K)$ is simplicial.

1.4 Algebraic tori over local and global fields

First we fix our notation. Let $\text{Val}(K)$ be the set of all valuations of a global field $K$. For any $v \in \text{Val}(K)$, we denote by $K_v$ the completion of $K$ with respect to $v$. Let $v$ be a valuation of a number field $K$, and $E$ a finite Galois extension of $K$. Let $V$ be an extension of $v$ to $E$, and $E_V$ the completion of $E$ with respect to $V$. Then

$$\text{Gal}(E_V/K_v) \cong G_v \subset G,$$

where $G_v$ is the decomposition subgroup of $G$ and $K_v \otimes_K E \cong \prod_{V|V'} E_V$. Let $T$ be an algebraic torus over $K$ with the splitting field $E$. Denote by $T_{K_v} = T \otimes K_v$ the $v$-adic completion of $T$. 
Definition 1.4.1. We denote the group of characters \( \hat{T}_{K_v} = M^{G_v} \) by \( M_v \) and the dual group \( \text{Hom}(\hat{T}_{K_v}, \mathbb{Z}) = N^{G_v} \) by \( N_v \). We put also \( N_{R,v} = N_{R}^{G_v} \) and \( M_{R,v} = M_{R}^{G_v} \).

Let \( (K_v \otimes K E)^* \) and \( E_v^* \) be the multiplicative groups of \( K_v \otimes K E \) and \( E_v \), respectively. One has

\[
T_{K_v} = \text{Hom}_G(\hat{T}, (K_v \otimes K E)^*) = \text{Hom}_G(M, E_v^*).
\]

Denote by \( \mathcal{O}_v^* \) the maximal compact subgroup in \( E_v^* \). Let \( T(O_v) \) be the maximal compact subgroup in \( T(K_v) \).

Assume that \( v \) is a nonarchimedian valuation. Then there is a short exact sequence

\[
1 \to \mathcal{O}_v^* \to E_v^* \to \mathbb{Z} \to 1, \quad b \mapsto \text{ord} | b |_v.
\]

Applying the functor \( \text{Hom}_G(M, *) \) to the short exact sequence above, we obtain the short exact sequence

\[
1 \to N \otimes \mathcal{O}_v^* \to N \otimes E_v^* \to N \to 1,
\]

which induces an injective homomorphism

\[
\pi_v : T(K_v)/T(O_v) \to N_v = N^{G_v}.
\]

**Proposition 1.4.2 [15].** For any nonarchimedian valuation \( v \), the quotient \( T(K_v)/T(O_v) \) can be canonically identified with a sublattice of finite index in \( N_v \). Moreover, \( \pi_v(T(K_v)/T(O_v)) \) coincides with \( N_v \) if \( v \) is unramified in \( E/K \). \( \square \)

Definition 1.4.3. Let \( S \) be a finite subset of \( \text{Val}(K) \) containing all archimedean and nonarchimedean valuations of \( K \) which ramify in \( E \). We denote by \( S_{\infty}(K) \) the set of all archimedean valuations of \( K \) and put \( S_0 = S \setminus S_{\infty}(K) \).

Now we assume that \( v \) is an archimedean valuation, i.e., \( K_v \) is \( \mathbb{R} \) or \( \mathbb{C} \). It is known that any torus over \( \mathbb{R} \) is isomorphic to the product of some copies of \( \mathbb{C}^* \), \( \mathbb{R}^* \), or \( S^1 = \{ z \in \mathbb{C} | z \bar{z} = 1 \} \). The quotient \( T(K_v)/T(O_v) \) is isomorphic to the \( \mathbb{R} \)-linear space \( N_v \otimes \mathbb{R} \). The homomorphism \( T(K_v) \to T(K_v)/T(O_v) \) is simply the logarithmic mapping onto the Lie algebra of \( T(K_v) \). Hence, we obtain the following.

**Proposition 1.4.4.** For any archimedean valuation \( v \), \( T(K_v)/T(O_v) \) can be canonically identified with the real Lie algebra \( N_{R,v} \) of \( T(K_v) \). \( \square \)

Definition 1.4.5. Denote by \( T(A_K) \) the adele group of \( T \), i.e., the restricted topological product

\[
\prod_{v \in \text{Val}(K)} T(K_v)
\]
consisting of all elements \( t = \{ t_v \} \in \prod_{v \in \text{Val}(K)} T(K_v) \) such that \( t_v \in T(\mathcal{O}_v) \) for almost all \( v \in \text{Val}(K) \). Let

\[
T^1(A_K) = \left\{ t \in T(A_K) : \prod_{v \in \text{Val}(K)} |m(t_v)|_v = 1, \text{ for all } m \in \hat{T}_K \subset M \right\}.
\]

We put also

\[
K_T = \prod_{v \in \text{Val}(K)} T(\mathcal{O}_v).
\]

**Proposition 1.4.6 [27].** The groups \( T(A_K), T^1(A_K), T(K), K_T \) have the following properties, which are generalizations of the corresponding properties of the adelization of \( G_m(K) \):

(i) \( T(A_K)/T^1(A_K) \cong \mathbb{R}^k \), where \( k \) is the rank of \( \hat{T}_K \).

(ii) \( T^1(A_K)/T(K) \) is compact.

(iii) \( T^1(A_K)/K_T \cdot T(K) \) is isomorphic to the direct product of a finite group \( \text{cl}(\hat{T}_K) \) (this is an analog of the idele-classes group \( \text{Cl}(K) \)) and a connected compact abelian topological group whose dimension equals the rank \( r' \) of the group of \( \mathcal{O}_K \)-units in \( T(K) \) (this rank equals \( r_1 + r_2 - 1 \) for \( G_m \)).

(iv) \( W(T) = K_T \cap T(K) \) is a finite group of all torsion elements in \( T(K) \) (this is the analog of the group of roots of unity in \( G_m(K) \)).

The following theorem of Weil plays a fundamental role in the definition of adelic measures on algebraic varieties.

**Theorem 1.4.7 [29], [41].** Let \( X \) be a \( d \)-dimensional smooth geometrically irreducible algebraic variety over a global field \( K \). Denote by \( \mathcal{K} \) the canonical sheaf on \( X \) with a family of local metrics \( \| \cdot \|_v \). Then these local metrics uniquely define natural \( v \)-adic measures \( \omega_{\mathcal{K}, v} \) on \( X(K_v) \). If \( X \) is a projective Fano variety, then for almost all \( v \in \text{Val}(K) \), one has

\[
\int_{X(K_v)} \omega_{\mathcal{K}, v} = \frac{\text{Card}[X(k_v)]}{q_v^d},
\]

where \( k_v \) is the residue field of \( K_v \) and \( q_v = \text{Card}[k_v] \).

**Remark 1.4.8.** We notice that the structure sheaf \( \mathcal{O}_X \) of any algebraic variety \( X \) has a natural metrization defined by the standard choice of \( v \)-adic valuations of the field \( K \). If \( X = \mathcal{G} \) is an algebraic group, then there exists a natural way to define a metrization of the canonical sheaf \( \mathcal{K} \) on \( \mathcal{G} \) by choosing a \( \mathcal{G} \)-invariant algebraic differential \( d \)-form \( \Omega \). Such a form defines an isomorphism of \( \mathcal{K} \) with the structure sheaf \( \mathcal{O}_\mathcal{G} \). We denote the corresponding local measure on \( \mathcal{G}(K_v) \) by \( \omega_{\Omega, v} \). This measure depends on the choice of the normalization of the valuation \( | \cdot |_v : K_v \to \mathbb{R} \). Moreover, if \( \mathcal{G}(\mathcal{O}_v) \) denotes the maximal compact subgroup in \( \mathcal{G}(K_v) \), then for almost all \( v \in \text{Val}(K) \), one has

\[
\int_{\mathcal{G}(\mathcal{O}_v)} \omega_{\Omega, v} = \frac{\text{Card}[\mathcal{G}(k_v)]}{q_v^d}.
\]
Let $T$ be a $d$-dimensional torus over $K$ with a splitting field $E$. Take a $T$-invariant differential $d$-form $\Omega$ on $T$ (it is unique up to a constant from $K$). According to Weil (1.4.8), we obtain a family of local measures $\omega_{\Omega,v}$ on $T(K_v)$.

Definition 1.4.9 [27]. Let

$$L_S(s, T; E/K) = \prod_{v \notin S} L_v(s, T; E/K)$$

be the Artin $L$-function corresponding to the representation

$$\rho : G = \text{Gal}(E/K) \to \text{Aut}(M)$$

and a finite set $S \subset \text{Val}(K)$ containing all archimedean valuations and all nonarchimedean valuations of $K$ which are ramified in $E$. By definition, $L_v(s, T; E/K) \equiv 1$ if $v \in S$. The numbers

$$c_v = L_v(1, T; E/K) = \frac{1}{\det(\text{Id} - q_v^{-1}\Phi_v)}, \quad v \notin S$$

are called canonical correcting factors for measures $\omega_{\Omega,v}$ ($\Phi_v$ is the $\rho$-image of a local Frobenius element in $G$).

By 1.4.8 (see also 2.2.1), one has

$$c_v^{-1} = \int_{T(K_v)} \omega_{\Omega,v} = \frac{\text{Card}[T(K_v)]}{q_v^d}, \quad v \notin S.$$

Let $d\mu_v = c_v \omega_{\Omega,v}$. We put $c_v = 1$ for $v \in S$. Since

$$\int_{T(K_v)} d\mu_v = 1$$

for $v \notin S$, the $\{c_v\}$ defines the canonical measure

$$\omega_{\Omega,S} = \prod_{v \in \text{Val}(K)} d\mu_v$$

on the adele group $T(A_K)$. By the product formula, $\omega_{\Omega,S}$ does not depend on the choice of $\Omega$.

Let $dx$ be the standard Lebesgue measure on $T(A_K)/T^1(A_K)$. There exists a unique Haar measure $\omega^1_{\Omega,S}$ on $T^1(A_K)$ such that $\omega^1_{\Omega,S} dx = \omega_{\Omega,S}$. For an anisotropic torus $T$, one has $T(A_K) = T^1(A_K)$, $\omega_{\Omega,S} = \omega^1_{\Omega,S}$.

Definition 1.4.10. Let $k$ be the dimension of the $R$-space $T(A_K)/T^1(A_K) = N^S_R$. Then the Tamagawa number of $T$ is defined as

$$\tau(T) = \frac{b_S(T)}{l_S(T)}$$
where

\[ b_S(T) = \int_{T(A_K)/T(K)} \omega^1_{1,S}, \]

\[ l_S(T) = \lim_{s \to 1} (s - 1)^k L_S(s, T; E/K). \]

**Remark 1.4.11.** Although the numbers \( b_S(T) \) and \( l_S(T) \) do depend on the choice of the finite subset \( S \subset \text{Val}(K) \), the Tamagawa number \( \tau(T) \) does not depend on \( S \).

**Theorem 1.4.12** [27], [28]. The Tamagawa number \( \tau(T) \) does not depend on the choice of a splitting field \( E \). Moreover,

\[ \tau(T) = \frac{h(T)}{i(T)}, \]

where

\[ h(T) = \text{Card}[H^1(G, M)], \]

\[ i(T) = \text{Card}[\text{III}(T)], \]

and

\[ \text{III}(T) = \text{Ker}[H^1(G, T(K)) \to \prod_{v \in \text{Val}(K)} H^1(G_v, T(K_v))]. \]

In particular, \( \tau(T) \) is always a rational number, and \( \tau(G_m(K)) = 1 \).

**Definition 1.4.13** [39]. Let \( S \) be a finite subset in \( \text{Val}(K) \). Denote by \( \overline{T(K)}_S \) the closure of the image of \( T(K) \) in the product \( \prod_{v \in S} T(K_v) \) equipped with the direct product topology. Define the *obstruction group to weak approximation with respect to \( S \)* as

\[ A(T, S) = \prod_{v \in S} T(K_v)/\overline{T(K)}_S. \]

Let \( \overline{T(K)} \) be the closure of \( T(K) \) in \( T(A_K) \) in the *direct product topology*. Define the *obstruction group to weak approximation* as

\[ A(T) = T(A_K)/\overline{T(K)}. \]

**Theorem 1.4.14** [39, pp. 150–153]. If \( S \) does not contain valuations of \( K \) which ramify in \( E \), then \( A(T, S) = 0 \). Moreover, \( A(T, S) = A(T) \) for any finite \( S \) which contains all ramified in \( E \) valuations of \( K \); i.e., for these \( S \), one has

\[ \overline{T(K)} = \overline{T(K)}_S \cdot T(A_{K,S}), \]

where

\[ T(A_{K,S}) = \prod_{v \notin S} T(K_v) \cap T(A_K). \]
Theorem 1.4.15 [39], [32]. Let $\mathbf{P}_\Sigma$ be a complete smooth toric variety over $K$. There is an exact sequence

$$0 \to A(T) \to \text{Hom}(H^1(G, \text{Pic}(\mathbf{P}_\Sigma, E)), \mathbb{Q}/\mathbb{Z}) \to III(T) \to 0.$$ 

The group $H^1(G, \text{Pic}(\mathbf{P}_\Sigma, E))$ is canonically isomorphic to $\text{Br}(\mathbf{P}_\Sigma, K)/\text{Br}(K)$, where $\text{Br}(\mathbf{P}_\Sigma, K) = H^2_{\text{et}}(\mathbf{P}_\Sigma, K, \mathbb{G}_m)$. □

Corollary 1.4.16. Denote by $\beta(\mathbf{P}_\Sigma)$ the cardinality of $H^1(G, \text{Pic}(\mathbf{P}_\Sigma, E))$. Then

$$\text{Card}[A(T)] = \frac{\beta(\mathbf{P}_\Sigma)}{i(T)}.$$ □

2 Heights and their Fourier transforms

2.1 Complexified local Weil functions and heights

Let $K$ be a number field, and $K_v$ its completion with respect to a valuation $v \in \text{Val}(K)$. Denote by $| \cdot |_v : K_v \to \mathbb{R}$ the multiplier of the additive Haar measure with respect to homothety. Then one has the product formula

$$\prod_{v \in \text{Val}(K)} |a|_v = 1$$

for all $a \in K^*$.

Definition 2.1.1. A metrized very ample line bundle $\mathcal{L} = (L, \| \cdot \|_v)$ on a smooth projective algebraic variety $X$ is given by a family of norms $\| \cdot \|_v$ on all fibers $L_x \otimes K_v$ continuous in $v$-adic topology on $X(K_v)$ such that:

(i) For all $a \in K^*$ and all local sections $f \in \Gamma(U, L)$ nonvanishing in $x \in U$, we have

$$\|af(x)\|_v = |a|_v \|f(x)\|_v.$$ 

(ii) Let $\{f_0, \ldots, f_p\}$ be a basis of the space of global sections $\Gamma(L)$. Then for all $x \in X(K)$, for any rational section $f$ of $L$ nonvanishing in $x$, and for almost all valuations $v \in \text{Val}(K)$, we have

$$\|f(x)\|_v^{-1} = \max_{j=0, \ldots, p} \left| \frac{f_j(x)}{f(x)} \right|_v.$$ 

(iii) Let $D \subset X$ be the divisor of poles and zeros of the rational section $f$ of $L$. Then

$$H_v(x, D) = \|f(x)\|_v^{-1}$$

is called the local Weil function corresponding to $D$.

Remark 2.1.2. If $E$ is a finite extension of $K$, then a metrization of a very ample line bundle $L$ on $X$ over $K$ naturally extends to a metrization of $L$ over $E$. For this extension, we can use the same basis $\{f_0, \ldots, f_p\}$ and a rational section $f$ of $L$. Assume that $\mathcal{V}$ is an
extension of a valuation \( v \in \text{Val}(K) \) to a valuation of \( E \), and \( k_V/k_v \) is the extension of the corresponding residue fields. Then for any \( K \)-rational point \( x \in X(K) \) and for almost all \( v \in \text{Val}(K) \), one has

\[
\|f(x)\|_V = \|f(x)\|_V^{k_V/k_v}, \quad \text{i.e., } H_V(x, D) = H_v(x, D)^{k_V/k_v}.
\]

Heights on a projective algebraic variety \( X \) over a number field \( K \) define a unique functorial homomorphism from \( \text{Pic}(X) \) to equivalence classes of functions \( X(K) \to \mathbb{R}_{>0} \), which on metrized very ample line bundles \( L \) is given by the formula

\[
H_L(x) = \prod_{v \in \text{Val}(K)} \|f(x)\|_V^{-1} = \prod_{v \in \text{Val}(K)} H_v(x, D)
\]

where \( f \) is a rational global section of \( L \) not vanishing in \( x \in X(K) \) and \( D \) is its divisor. Two functions are equivalent if they differ by a function that is bounded on \( X(K) \). For our purposes, it will be convenient to extend these notions to the complexified Picard group \( \text{Pic}(X) \otimes \mathbb{C} \).

Let \( P_\Sigma \) be a complete toric variety over a global field \( K \). In this section we define canonical simultaneous metrizations of \( T \)-linearized line bundles on \( P_\Sigma \).

Remark 2.1.3. By 1.3.8 and 1.3.9, it suffices to define metrizations only for \( T \)-linearized line bundles on \( P_\Sigma \).

Notice that a family of local metrics on all \( T \)-linearized line bundles on \( P_\Sigma \) corresponding to \( \Sigma \)-piecewise linear \( G \)-invariant functions \( \varphi \in \text{PL}(\Sigma \otimes \mathbb{C}) \) is uniquely determined by a family of local Weil functions \( H_{\Sigma, v}(x_v, \varphi) \) on \( P_\Sigma \) corresponding to \( T \)-invariant divisors

\[
D_\varphi = \varphi(e_1)D_1 + \cdots + \varphi(e_n)D_n.
\]

Now we define even more general local Weil functions, which correspond to \( T \)-invariant Cartier divisors with complex coefficients.

Definition 2.1.4. A \( T \)-invariant \( \mathbb{C} \)-Cartier divisor is a formal linear combination \( D_s = s_1D_1 + \cdots + s_nD_n \), with \( s = (s_1, \ldots, s_n) \in \mathbb{C}^n \) or equivalently a complex piecewise linear function \( \varphi \) in \( \text{PL}(\Sigma \otimes \mathbb{C}) \) having the property \( \varphi(e_i) = s_i \) (i = 1, \ldots, n).

Definition 2.1.5. Let \( \varphi \in \text{PL}(\Sigma \otimes \mathbb{C}) \). For any point \( x_v \in T(K_v) \subset P_\Sigma(K_v) \), denote by \( \overline{x}_v \) the image of \( x_v \) in \( N_v \) (resp., \( N_v \otimes \mathbb{R} \) for archimedian valuations), where \( N_v \) is considered as a canonical lattice in the real space \( N_v^G \). Define the complexified local Weil function \( H_{\Sigma, v}(x_v, \varphi) \) by the formula

\[
H_{\Sigma, v}(x_v, \varphi) = e^{\varphi(\overline{x}_v) \log q_v},
\]

where \( q_v \) is the cardinality of the residue field \( k_v \) of \( K_v \) if \( v \) is nonarchimedian and \( \log q_v = 1 \) if \( v \) is archimedian.
Theorem 2.1.6. The complexified local Weil function $H_{\Sigma,v}(x,v)$ satisfies the following properties:

(i) $H_{\Sigma,v}(x,v)$ is $T(O_v)$-invariant.

(ii) If $\varphi = 0$, then $H_{\Sigma,v}(x,v) = 1$ for all $x \in T(K_v)$.

(iii) $H_{\Sigma,v}(x,v, \varphi + \varphi') = H_{\Sigma,v}(x,v, \varphi)H_{\Sigma,v}(x,v, \varphi')$.

(iv) If $s_i = \varphi(e_i) \in \mathbb{Z}^n$ $(i = 1, \ldots, n)$ and $D_\varphi$ is a very ample $T$-invariant Cartier divisor, then $H_{\Sigma,v}(x,v)$ is a local Weil function corresponding to a $v$-adic metrization of $L(\varphi)$ and the $T$-invariant Cartier divisor

$$D_s = s_1D_1 + \cdots + s_nD_n$$

as in 2.1.1.

Proof. The properties (i)–(iii) follow immediately from the definition. Only (iv) needs some work.

First we show that $H_{\Sigma,v}(x,v)$ has good behavior under field extensions for almost all $v \in \text{Val}(K)$ (see Remark 2.1.2). Assume that $v$ is unramified in the extension $E$ of $K$. Let $V$ be an extension of $v$ to a valuation of $E$. Then $q_V = q_E^{[k_V:k_v]}$. Therefore,

$$H_{\Sigma,V}(x,v) = H_{\Sigma,v}(x,v)^{[k_V:k_v]}.$$ 

Hence it is sufficient to prove (iv) for the case of split tori $T$.

Second, we notice that in the case of a split torus $T$, the space of global sections $\Gamma(L(\varphi))$ of a $T$-linearized line bundle over $P_{\Sigma}$ always admits a basis $f_0, \ldots, f_p$ and a rational section $f$ of $L(\varphi)$ such that $D_\varphi$ is the divisor of $f$ and

$$m_0 = \frac{f_0}{T}, \ldots, m_p = \frac{f_p}{T}$$

are $K$-rational characters of $T$. We remark that $|m_0(x)|_v, \ldots, |m_p(x)|_v$ depend only on the image of $x$ in $T(K_v)/T(O_v)$, since $|m(x)|_v = 1$ for any character $m \in \hat{T}$ and any $x \in T(O_v)$. It remains to notice that upper convex piecewise linear function $\varphi$ is the maximum of the linear functions on $N$ represented by the elements $m_0, \ldots, m_p$; i.e.,

$$\varphi(x_v) = \max_{i=0,\ldots,p} |m_i(x_v)|_v.$$ 

Definition 2.1.7. Let $\varphi \in \text{PL}(\Sigma)^G_C$. We define the complexified height function on $P_{\Sigma,K}$ by

$$H_{\Sigma}(x, \varphi) = \prod_{v \in \text{Val}(K)} H_{\Sigma,v}(x, \varphi).$$

Remark 2.1.8. Since every divisor on a smooth projective variety is a difference of two very ample divisors, by 2.1.6, $H_{\Sigma}(x, \varphi)$ is a usual global height function on $P_{\Sigma}(K)$ associated with a metrized line bundle $L(\varphi)$, whenever $\varphi$ is an integral $\Sigma$-piecewise linear function.
on $N$. For the case of nonprojective smooth compactifications $P_\Sigma$, we can again consider $H_\Sigma(x, \varphi)$ as a global height function associated with $L(\varphi)$ (in the adelic sense, see, e.g., [23], [35]), because there always exists a $G$-invariant subdivision $\Sigma'$ of the $G$-invariant fan $\Sigma$ such that $P_{\Sigma'}$ is projective (using the method from [7]), and the definition of $H_\Sigma(x, \varphi)$ is compatible with the natural lifting of the line bundle $L(\varphi)$ on $P_\Sigma$ to a line bundle on the projective toric variety $P_{\Sigma'}$.

Although all local factors $H_{\Sigma,v}(x, 0)$ of $H_\Sigma(x, \varphi)$ are functions on $PL(\Sigma)_G$, by the product formula, the global complex height function $H_\Sigma(x, \varphi)$ depends only on the class of $\varphi \in PL(\Sigma)_G$ modulo complex global linear $G$-invariant functions on $N_C$, i.e., $H_\Sigma(x, \varphi)$ depends only on the class of $\varphi$ in $\text{Pic}(P_{\Sigma, K}) \otimes \mathbb{C}$.

**Definition 2.1.9.** We define the zeta-function of the complex height function $H_\Sigma(x, \varphi)$ as

$$Z_\Sigma(\varphi) = \sum_{x \in T(K)} H_\Sigma(x, -\varphi).$$

**Remark 2.1.10.** One can see that the series $Z_\Sigma(\varphi)$ converges absolutely and uniformly in the domain $\text{Re}(\varphi(e_j)) \gg 0$ for all $j$. Since $H_\Sigma(x, \varphi)$ is the product of the local complex Weil functions $H_{\Sigma,v}(x, \varphi)$ and $H_{\Sigma,v}(x, \varphi) = 1$ for almost all $v \in \text{Pic}(P_{\Sigma,v})$, we can immediately extend $H_\Sigma(x, \varphi)$ to a function on the adelic group $T(A_K)$.

### 2.2 Fourier transforms of nonarchimedean heights

Let $v \not\in S$ be a nonarchimedean valuation of $K$ which is unramified in $E$, let $V$ be an extension of $v$ to a valuation of $E$, let $E_V$ be the corresponding extension of the local field $K_v$, and let $k_V$ be the residue field of $E_V$ which extends the residue field $k_v$ of $K_v$. We will use the same notation $G_v$ for the Galois group of $E_V/K_v$ and for the Galois group of $k_V/k_v$.

First of all, we consider properties of algebraic tori and toric varieties over the finite residue field $k_v$ containing $q_v$ elements. For any finite extension $k_V$ of $k_v$, the Galois group $G_v = \text{Gal}(k_V/k_v)$ is generated by the Frobenius automorphism $F_v : z \rightarrow z^{q_v}$. By 1.1.4, any $d$-dimensional algebraic torus $T$ over $k_v$ splitting over $k_V$ is uniquely defined by the conjugacy class of the integral matrix $\Phi_v = \rho(F_v)$ in $\text{GL}(d, \mathbb{Z})$. The characteristic polynomial of the matrix $\Phi_v$ gives the following formula (obtained by Ono in [27]) for the number of $k_v$-rational points in $T(k_v)$.

**Theorem 2.2.1.** Let $T$ be a $d$-dimensional algebraic torus defined over a finite field $k_v$. In the above notation, one has the following formula for the number of $k_v$-rational points of $T$:

$$\text{Card}[T(k_v)] = (-1)^d \det(\Phi_v - q_v \cdot \text{Id}).$$
Definition 2.2.2. Let $\Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_l(1)$ be the decomposition of $\Sigma(1)$ into a disjoint union of $G_v$-orbits. Denote by $d_i$ the length of the $G_v$-orbit $\Sigma_i(1)$ ($d_1 + \cdots + d_l = n$). We establish a 1-to-1 correspondence $\Sigma_i(1) \leftrightarrow u_i$ between the $G_v$-orbits $\Sigma_1(1), \ldots, \Sigma_l(1)$ and independent variables $u_1, \ldots, u_l$. Let $\sigma \in \Sigma G_v$ be any $G_v$-invariant cone, and $\Sigma_{i_1}(1) \cup \cdots \cup \Sigma_{i_k}(1)$ the set of all 1-dimensional faces of $\sigma$. We define the rational function $R_\sigma(u_1, \ldots, u_l)$ corresponding to $\sigma$ as follows:

$$R_\sigma(u_1, \ldots, u_l) := \frac{u_{i_1}^{d_{i_1}} \cdots u_{i_k}^{d_{i_k}}}{(1 - u_{i_1}^{d_{i_1}}) \cdots (1 - u_{i_k}^{d_{i_k}})}.$$ 

Define the polynomial $Q_\Sigma(u_1, \ldots, u_l)$ and the rational function $R_\Sigma(u_1, \ldots, u_l)$ by the formula

$$R_\Sigma(u_1, \ldots, u_l) = \sum_{\sigma \in \Sigma G_v} R_\sigma(u_1, \ldots, u_l) = \frac{Q_\Sigma(u_1, \ldots, u_l)}{(1 - u_1^{d_1}) \cdots (1 - u_l^{d_l})}.$$ 

Proposition 2.2.3. Let $\Sigma$ be a complete regular $G_v$-invariant fan. Then the polynomial $Q_\Sigma(u_1, \ldots, u_l) - 1$ contains only monomials of degree $\geq 2$.

Proof. It follows immediately from the definition that

$$(1 - u_1^{d_1}) \cdots (1 - u_l^{d_l}) R_\sigma(u_1, \ldots, u_l)$$ 

contains only monomials of degree $\geq 2$ if $\dim \sigma \geq 2$. Let $\Sigma_{i_1}(1), \ldots, \Sigma_{i_k}(1)$ be the set of all $G_v$-invariant 1-dimensional cones; i.e., $d_i = 1$ if and only if $i \in \{i_1, \ldots, i_k\}$. We know that

$$R_{\Sigma_{i_j}}(u_1, \ldots, u_l) = \frac{u_j}{1 - u_j} \text{ for } j \in \{i_1, \ldots, i_k\};$$

i.e., the only monomials of degree $\leq 1$ in the polynomial

$$(1 - u_1^{d_1}) \cdots (1 - u_l^{d_l}) R_{\Sigma_{i_j}}(u_1, \ldots, u_l)$$ 

are $u_j$ with the coefficient +1. On the other hand, $R_\sigma(u_1, \ldots, u_l) = 1$ if $\sigma = 0$. Therefore, the only monomials of degree $\leq 1$ in the polynomial

$$(1 - u_1^{d_1}) \cdots (1 - u_l^{d_l}) R_\sigma(u_1, \ldots, u_l)$$ 

are the constant 1 and the monomials $u_j$ ($j \in \{i_1, \ldots, i_k\}$) with coefficient $-1$. This shows that the monomials of degree 1 disappear in $Q_\Sigma(u_1, \ldots, u_l)$.

Proposition 2.2.4. Let $P_\Sigma$ be a toric variety over a finite field $k_v$ defined by a $G_v$-invariant fan $\Sigma \subset \mathbb{N}_R$. Then

$$\text{Card}[P_\Sigma(k_v)] = (-1)^d \det(\Phi_v - q_v \cdot \text{Id}) R_\Sigma(q_v^{-1}, \ldots, q_v^{-1}).$$
Proof. By 1.3.6,

\[ P_\Sigma(k_v) = \bigcup_{\sigma \in \Sigma G_v} T_\sigma(k_v). \]

It remains to compute the number of \( k_v \)-points in \( T_\sigma(k_v) \) for any \( G_v \)-invariant cone \( \sigma \in \Sigma G_v \). Let \( M_\sigma \subset \tilde{\sigma} \cap (-\tilde{\sigma}) \) be the maximal linear sublattice in the dual cone \( \tilde{\sigma} \subset \mathcal{R}_v \), and \( \Phi_{\sigma,v} \) the restriction of \( \Phi_v \) to an automorphism of \( M_\sigma \). (We remark that \( \text{rk} M_\sigma = \text{codim} \sigma \).) By 2.2.1,

\[ \text{Card}[T(k_v)] = (-1)^{\text{codim} \sigma} \det(\Phi_{\sigma,v} - q_v \cdot \text{Id}). \]

On the other hand, we have the short exact sequence of \( G_v \)-modules

\[ 0 \to M_\sigma \to M \to M/M_\sigma \to 0, \]

where \( M/M_\sigma \) is dual to the permutational \( G_v \)-module \( N_\sigma \) of rank \( \text{dim} \sigma \), because \( G_v \) acts by permutations on generators of \( \sigma \). Therefore, the number of \( k_v \) points in the algebraic torus associated with the \( G_v \)-module \( M/M_\sigma \) equals

\[ \left(q_\sigma^{d_{i_1}} - 1\right) \cdots \left(q_\sigma^{d_{i_k}} - 1\right) = R_\sigma^{-1}(q_v^{-1}, \ldots, q_v^{-1}). \]

So we obtain

\[ (-1)^{\text{codim} \sigma} \det(\Phi_{\sigma,v} - q_v \cdot \text{Id}) = (-1)^d \det(\Phi_v - q_v \cdot \text{Id}) R_\sigma(q_v^{-1}, \ldots, q_v^{-1}). \]

Let \( \chi \) be a topological character of \( T(A_K) \) such that its \( v \)-component \( \chi_v : T(K_v) \to S^1 \subset \mathbb{C}^\ast \) is trivial on \( T(O_v) \). For each \( j \in \{1, \ldots, l\} \), we denote by \( n_j \) one of the \( d_j \) generators of all 1-dimensional cones of the \( G_v \)-orbit \( \Sigma_j(1) \); i.e., \( G_v n_j \) is the set of generators of 1-dimensional cones in \( \Sigma_j(1) \). By 1.4.2, \( n_j \) represents an element of \( T(K_v) \) modulo \( T(O_v) \). Therefore, \( \chi_v(n_j) \) is well defined.

**Definition 2.2.5.** Denote by \( \hat{H}_{\Sigma,v}(\chi_v, -\varphi) \) the value at \( \chi_v \) of the Fourier transform of the local Weil function \( H_{\Sigma,v}(x_v, -\varphi) \) with respect to the \( v \)-adic Haar measure \( d\mu_v \) normalized by the condition \( \int_{T(O_v)} d\mu_v = 1 \).

**Theorem 2.2.6.** For any topological character \( \chi_v \) of \( T(K_v) \), one has

\[ \hat{H}_{\Sigma,v}(\chi_v, -\varphi) = \int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v \]

\[ = R_{\Sigma} \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right) = R_{\Sigma} \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right) \]

\[ = Q_{\Sigma} \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right) = \left( 1 - \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}} \right) \cdots \left( 1 - \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right) \]
if \( \chi_v \) is trivial on \( T(\mathcal{O}_v) \), and

\[
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v = 0
\]

otherwise.

Proof. Since the Haar measure \( \mu_v \) is \( T(\mathcal{O}_v) \)-invariant, one has

\[
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v = \sum_{x_v \in T(K_v)/T(\mathcal{O}_v)} H_{\Sigma,v}(\bar{x}_v, -\varphi) \chi_v(\bar{x}_v) \int_{T(\mathcal{O}_v)} \chi_v d\mu_v
\]

where \( \bar{x}_v \) denotes the image of \( x_v \) in \( T(K_v)/T(\mathcal{O}_v) = N_v \). Notice that \( \int_{T(\mathcal{O}_v)} \chi_v d\mu_v = 0 \) if \( \chi_v \) has nontrivial restriction on \( T(\mathcal{O}_v) \).

Assume now that \( \chi_v \) is trivial on \( T(\mathcal{O}_v) \), i.e., \( \chi_v(x_v) \) depends only on the image \( x_v \) of \( x_v \) in \( T(K_v)/T(\mathcal{O}_v) = N_v \). By the normalization of \( d\mu_v \), we have

\[
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v = \sum_{x_v \in N_v} H_{\Sigma,v}(\bar{x}_v, -\varphi) \chi_v(\bar{x}_v).
\]

For any \( \sigma \in \Sigma \), we denote by \( \sigma^\circ \) the relative interior of the cone \( \sigma \). (We put \( \sigma^\circ = 0 \) if \( \sigma = 0 \).) Notice that for any \( G_v \)-invariant point \( \bar{x}_v \in N_v \), there exists the unique \( G_v \)-invariant cone \( \sigma \in \Sigma_{G_v} \) such that \( \bar{x}_v \in \sigma^\circ \). Therefore, we can decompose \( N_v \) into the disjoint union

\[
N_v = \bigcup_{\sigma \in \Sigma_{G_v}} N_v \cap \sigma^\circ.
\]

Hence

\[
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v) d\mu_v = \sum_{\sigma \in \Sigma_{G_v}} \left( \sum_{x_v \in N_v \cap \sigma^\circ} H_{\Sigma,v}(\bar{x}_v, -\varphi) \chi_v(\bar{x}_v) \right).
\]

It remains to compute

\[
\sum_{\bar{x}_v \in N_v \cap \sigma^\circ} H_{\Sigma,v}(\bar{x}_v, -\varphi) \chi_v(\bar{x}_v)
\]

for every \( G_v \)-invariant cone \( \sigma \).

As in the proof of 1.3.2, we see that \( G_v \) acts by permutations on the set \( A \subset \{ e_1, \ldots, e_n \} \) of all generators of \( \sigma \). Let \( A = A_{i_1} \cup \cdots \cup A_{i_k} \) be the splitting of \( A \) into \( G_v \)-orbits (\( A_{i_p} \) is the set of primitive generators of the 1-dimensional cones in the \( G_v \)-orbit \( \Sigma_{i_p}(1) \), \( 1 \leq p \leq k \)). We define lattice vectors \( \bar{e}_1, \ldots, \bar{e}_k \in N_v \) as

\[
\bar{e}_p = \sum_{e_j \in A_{i_p}} e_j, \quad p = 1, \ldots, k.
\]
Then $\sigma^\circ \cap N_v$ is the set of the interior points in the $k$-dimensional cone $\sigma$ which is generated by a part $\xi_1, \ldots, \xi_k$ of an integral basis of $N_v$. Since $\varphi(\xi_p) = d_v \varphi(n_i_p)$ and $\chi_v(\xi_p) = \varphi(n_i_p)$, by summing the multidimensional geometric series

$$
\sum_{x_v \in N_v \cap \sigma^\circ} H_{\Sigma,v}(x_v, -\varphi) \chi_v(x_v)
$$

(we also use the fact that the restriction of $\varphi$ to $\sigma$ is a linear function), we obtain

$$
R_\sigma \left( \frac{\chi_v(n_1)}{q_v^{\varphi(n_1)}}, \ldots, \frac{\chi_v(n_l)}{q_v^{\varphi(n_l)}} \right).
$$

**Corollary 2.2.7.** The restriction of

$$
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) d\mu_v
$$

to the line $s_1 = \cdots = s_r = s$ is equal to

$$
L_v(s, T; E/K) \cdot L_v(s, T_{NS}, K_v/K) \cdot Q_\Sigma(q_v^{-s}, \ldots, q_v^{-s}).
$$

In particular, for $s = 1$, one has

$$
\int_{T(K_v)} H_{\Sigma,v}(x_v, -\varphi) d\mu_v = \frac{L_v(1, T; E/K)}{q_v^d} \text{Card}(\mathcal{P}_{\Sigma}(k_v)).
$$

**Proof.** We obtain the first statement using the equality

$$
L_v(s, T; E/K) \cdot L_v(s, T_{NS}, K_v/K) = \prod_{j=1}^t \left( 1 - \frac{1}{q_v} \right)^{-1},
$$

which follows from the short exact sequence of $G_v$-modules (1.2.9). For $s = 1$, we use 2.2.4 and the equality

$$
(-1)^d \det(\Phi_v - q_v \cdot \text{Id}) = q_v^d L_v(1, T; E/K)^{-1}.
$$

**Remark 2.2.8.** It is not easy to calculate explicitly the Fourier transforms of local heights for finitely many “bad” non-archimedian valuations $v \in S$, because we have only an embedding of finite index

$$
T(K_v)/T(\mathcal{O}_v) \hookrightarrow N_v.
$$

However, for our purposes, it will be sufficient to use upper estimates for these local Fourier transforms. In particular, one immediately sees that for all nonarchimedian valuations $v$, the local Fourier transforms of $H_{\Sigma,v}(x_v, -\varphi)$ can be bounded absolutely and uniformly in all characters by a finite combination of multidimensional geometric series in $q_v^{-1/2}$ in the domain $\text{Re}(\varphi(\xi_i)) > 1/2$. 
2.3 Fourier transforms of archimedian heights

Now we assume that \( v \) is an archimedian valuation. Then \( T(K_v)/T(0_v) = N_R^G \subset N_R \) where \( G \) is the trivial group for the case \( K_v = C \), and \( G = \text{Gal}(C/R) \cong \mathbb{Z}/2\mathbb{Z} \) for the case \( K_v = R \). Let \( \langle \cdot, \cdot \rangle \) be the pairing between \( N_R \) and \( M_R \) induced from the duality between \( N \) and \( M \). Let \( y \) be an arbitrary element of the dual \( R \)-space \( M_R^G = \text{Hom}(T(K_v)/T(0_v), R) \). Then \( \chi_y(x_v) = e^{-2\pi i \langle T(x_v), y \rangle} \) is a topological character of \( T(K_v) \), which is trivial on \( T(0_v) \). We choose the Haar measure \( d\mu_v \) on \( T(K_v) \) as the product of the Haar measure \( d\mu_0 \) on \( T(0_v) \) and the Haar measure \( d\bar{x}_v \) on \( T(K_v)/T(0_v) \), where the \( d\mu^G_0 \)-volume of \( T(0_v) \) equals 1 and \( d\bar{x}_v \) is the standard Lebesgue measure on \( N_R^G \) normalized by the full sublattice \( N_R^G \).

**Proposition 2.3.1.** The Fourier transform \( \hat{H}_{\Sigma,v}(x_v, -\varphi) \) of a local archimedian Weil function \( H_{\Sigma,v}(x_v, -\varphi) = e^{-\varphi(x_v)} \) is a rational function in variables \( s_j = \varphi(e_j) \) for \( \text{Re}(s_j) > 0 \).

**Proof.** One has

\[
\hat{H}_{\Sigma,v}(x_v, -\varphi) = \int_{N_R^G} e^{-\varphi(x_v)-2\pi i \langle x_v, y \rangle} d\bar{x}_v.
\]

First we consider the case \( K_v = C \). Since \( N_R = \bigcup_{\sigma \in \Sigma(d)} \sigma \), we obtain

\[
\int_{N_R} e^{-\varphi(x_v)-2\pi i \langle x_v, y \rangle} d\bar{x}_v = \sum_{\sigma \in \Sigma(d)} \int_{\sigma} e^{-\varphi(x_v)-2\pi i \langle x_v, y \rangle} d\bar{x}_v.
\]

On the other hand, for any \( d \)-dimensional cone \( \sigma \), we have

\[
\int_{\sigma} e^{-\varphi(x_v)-2\pi i \langle x_v, y \rangle} d\bar{x}_v = \frac{1}{\prod_{e_j \in \sigma} (s_j + 2\pi i \langle e_j, y \rangle)}.
\]

The case \( K_v = R \) can be reduced to the above situation using 1.3.2. \( \blacksquare \)

The proof of the following proposition was suggested to us by W. Hoffmann.

**Proposition 2.3.2.** Choose two arbitrary positive real numbers \( \delta_1, \delta_2 (\delta_2 > \delta_1 > 0) \). Then there exists a constant \( c(\delta_1, \delta_2, \Sigma) \) such that

\[
| \hat{H}_{\Sigma,v}(x_v, -\varphi) | \leq c(\delta_1, \delta_2, \Sigma) \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_k \in \sigma} (1 + | \langle e_k, y \rangle |)^{1+1/d}}
\]

if \( \delta_2 > \text{Re}(s_j) > \delta_1 \) for all \( j = 1, \ldots, n \).

**Proof.** It is sufficient to consider the case \( K_v = C \). (For the case \( K_v = R \), we can apply 1.3.2 and reduce the situation to the above case.)

Let \( f_1, \ldots, f_d \) be a basis of \( M \). Put \( x_i = (x_v, f_i) \). We denote by \( y_1, \ldots, y_d \) the coordinates of \( y \) in the basis \( f_1, \ldots, f_d \). Let \( \varphi_j(x_v) = \frac{\partial}{\partial x_j} \varphi(x_v) \). Then \( \varphi_j(x_v) \) is a locally constant
function having the value $\varphi_{j,\sigma}$ in the interior of a cone $\sigma \in \Sigma(d)$. We have

$$
\hat{H}_{\Sigma,v}(\chi_y, -\varphi) = \int_{\mathbb{R}^n} e^{-\varphi(v)} e^{-2\pi i (\overline{v}, y)} \, dv
= \frac{1}{2\pi i y_j} \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} e^{-\varphi(v)} e^{-2\pi i (\overline{v}, y)} \, dv
= \frac{i}{2\pi y_j} \sum_{\sigma \in \Sigma(d)} \varphi_{j,\sigma} \prod_{e_k \in \sigma} (s_k + 2\pi i \langle e_k, y \rangle).
$$

Notice that $M_R$ is covered by $d$ domains:

$$
V_j = \{ y = \sum_i y_i f_i \in M_R : \text{ } |y_j| = \max_i |y_i| \}.
$$

Therefore, it suffices to prove the statement for each $V_j$ ($j = 1, \ldots, d$). Let $\|y\|^2 = \sum_i y_i^2$. Then $\|y\| \leq \sqrt{d} |y_j|$ for $y \in V_j$. Then

$$
|\hat{H}_{\Sigma,v}(\chi_y, -\varphi)| \leq \sqrt{d} \frac{1}{\|y\|} \sum_{\sigma \in \Sigma(d)} \prod_{e_k \in \sigma} (s_k + 2\pi i \langle e_k, y \rangle)
$$

for $y \in V_j$. We use the obvious statement: Assume that $\delta_2 > \text{Re}(s) > \delta_1 > 0$. Then there exists a positive constant $C(\delta_1, \delta_2)$ such that for all $t$ one has $C(\delta_1, \delta_2)(|s + 2\pi i t|) \geq 1 + |t|$. We obtain

$$
|\hat{H}_{\Sigma,v}(\chi_y, -\varphi)| \leq \frac{C(\delta_1, \delta_2)}{\|y\|} \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_k \in \sigma} (1 + |\langle e_k, y \rangle|)}.
$$

Since $\hat{H}_{\Sigma,v}(\chi_y, -\varphi)$ is bounded at $y = 0$, there exists a constant $C'(\delta_1, \delta_2)$ such that

$$
|\hat{H}_{\Sigma,v}(\chi_y, -\varphi)| \leq \frac{C'(\delta_1, \delta_2)}{1 + \|y\|} \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_k \in \sigma} (1 + |\langle e_k, y \rangle|)}.
$$

Since $|\langle e_k, y \rangle| \leq \|y\| \|e_k\|$, it follows that there exist positive constants $c_\sigma$ ($\sigma \in \Sigma(d)$) such that

$$
c_\sigma (1 + \|y\|)^d \geq \prod_{e_k \in \sigma} (1 + |\langle e_k, y \rangle|).
$$

Finally, we obtain

$$
|\hat{H}_{\Sigma,v}(y, -\varphi)| \leq c(\delta_1, \delta_2, \Sigma) \cdot \sum_{\sigma \in \Sigma(d)} \frac{1}{\prod_{e_k \in \sigma} (1 + |\langle e_k, y \rangle|)^{1+1/d}}
$$

where

$$
c(\delta_1, \delta_2, \Sigma) = C'(\delta_1, \delta_2) \left( \sum_{\sigma \in \Sigma(d)} 1/c_\sigma^{1/d} \right).
$$
Corollary 2.3.3. Let \( g(y, -\varphi) \) be an arbitrary complex function satisfying the inequality
\[
|g(y, -\varphi)| \leq c(1 + \|y\|)^{\delta_0}
\]
for all \( y \in M_{R,v} \), all \( s_j \) in the domain \( \delta_2 > s_1 > \delta_1 > 0 \), and some constants \( c > 0 \), \( 1/d > \delta_0 > 0 \). Then, for any \( v \in S_\infty(K) \), the sum
\[
\sum_{y \in L} g(y, -\varphi) \hat{H}_{\Sigma,v}(y, -\varphi)
\]
is absolutely and uniformly convergent in this domain for any lattice \( L \subset M_{R,v} \).

Proof. The statement follows from the standard fact that the sum
\[
\sum_{y \in L} \frac{(1 + \|y\|)^{\delta_0}}{\prod_{e_k \in \sigma} (1 + |\langle e_k, y \rangle|)^{1+1/d}}
\]
is absolutely convergent for any lattice \( L \subset M_{R,v} \), any \( 1/d > \delta_0 > 0 \), and any \( \sigma \in \Sigma(d) \).

Corollary 2.3.4. Consider
\[
\hat{H}_{\Sigma,\infty}(y, -\varphi) = \prod_{v \in S_\infty(K)} \hat{H}_{\Sigma,v}(y, -\varphi)
\]
as a function on
\[
M_{R,\infty} = \bigoplus_{v \in S_\infty(K)} M_{R,v}.
\]
Then, for any \( \delta_2 > \delta_1 > 0 \) and an arbitrary complex function \( g(y, -\varphi) \) satisfying the inequality
\[
|g(y, -\varphi)| \leq c(1 + \|y\|)^{\delta_0}
\]
for all \( y \in M_{R,\infty} \), all \( s_j \) in the domain \( \delta_2 > s_1 > \delta_1 > 0 \), and some constants \( c > 0 \) and \( 1/d > \delta_0 > 0 \), we have that
\[
\sum_{y \in L} g(y, -\varphi) \hat{H}_{\Sigma,\infty}(y, -\varphi)
\]
is absolutely and uniformly convergent in this domain for any lattice \( L \subset M_{R,\infty} \).

Proof. Let \( \Sigma_\infty = \prod_{v \in S_\infty(K)} \Sigma_{\mathcal{C}_v} \subset N_{R,\infty} \) be the finite product of regular complete fans \( \Sigma_{\mathcal{C}_v} \subset N_{R,v} \). (Every cone in \( \Sigma_\infty \) is a direct product of cones in \( \Sigma_{\mathcal{C}_v} \).) This reduces the situation for \( \Sigma_\infty \) to the one considered in 2.3.3.

2.4 Characteristic functions of convex cones

Let \( V \) be an \( r \)-dimensional real vector space, \( V_\mathbb{C} \) its complex scalar extension, \( \Lambda \subset V \) a convex \( r \)-dimensional cone such that \( \Lambda \cap -\Lambda = 0 \in V \). Denote by \( \Lambda^\circ \) the interior of \( \Lambda \), by \( \Lambda^\circ_\mathbb{C} = \Lambda^\circ + iV \) the complex tube domain over \( \Lambda^\circ \), by \( V^* \) the dual space, by \( \Lambda^* \subset V^* \) the dual to the \( \Lambda \) cone, and by \( dy \) a Haar measure on \( V^* \).
Definition 2.4.1. The characteristic function of $\Lambda$ is defined as the integral

$$X_{\Lambda}(dy, u) = \int_{\Lambda^*} e^{-\langle u, y \rangle} dy,$$

where $u \in \Lambda_C^\circ$.

Remark 2.4.2. Characteristic functions of convex cones have been investigated in the theory of homogeneous cones by M. Köcher, O. S. Rothaus, and E. B. Vinberg [19], [38], [31].

Remark 2.4.3. We will be interested in characteristic functions of convex cones $\Lambda$ in real spaces $V$ which have natural lattices $L \subset V$ of maximal rank $r$. Let $L^*$ be the dual lattice in $V^*$; then we can normalize the Haar measure $dy$ on $V^*$ so that the volume of the fundamental domain $V^*/L^*$ equals 1. In this case, the corresponding characteristic function will be denoted simply by $X_{\Lambda}(u)$.

Proposition 2.4.4 [38]. Let $u \in \Lambda^* \subset V$ be an interior point of $\Lambda$, and

$$\Lambda^*_u(t) := \{y \in \Lambda^* \mid \langle u, y \rangle = t\}$$

the $(r - 1)$-dimensional convex compact.

We define the $(r - 1)$-dimensional measure $dy'_t$ on the affine hyperplane $\{\langle u, y \rangle = t\}$ in such a way that, for any function $f : V \to \mathbb{R}$ with compact support, one has

$$\int_{V^*} f(y) dy = \int_{-\infty}^{+\infty} dt \left( \int_{\langle u, y \rangle = t} f(y) dy'_t \right).$$

Then

$$X_{\Lambda}(u) = (r - 1)! \int_{\Lambda^*_u(1)} dy'_1.$$

The characteristic function $X_{\Lambda}(u)$ has the following properties [31], [38].

Proposition 2.4.5. (i) If $A$ is any invertible linear operator on $V$, then

$$X_{\Lambda}(Au) = \frac{X_{\Lambda}(u)}{\det A}.$$

(ii) If $\Lambda = \mathbb{R}_+^r$, $L = \mathbb{Z}^r \subset \mathbb{R}^r$, then

$$X_{\Lambda}(u) = (u_1 \cdots u_r)^{-1}, \quad \text{for } \Re(u_i) > 0.$$

(iii) If $z \in \Lambda^\circ$, then

$$\lim_{z \to \partial \Lambda} X_{\Lambda}(z) = \infty.$$

(iv) $X_{\Lambda}(u) \neq 0$ for all $u \in \Lambda_C^\circ$. 

Definition 2.4.6. Let $X$ be a smooth proper algebraic variety. Denote by $\Lambda_{\text{eff}} \subset \text{Pic}(X)_R$ the cone generated by classes of effective divisors on $X$. Assume that the anticanonical class $[X^{-1}] \in \text{Pic}(X)_R$ is contained in the interior of $\Lambda_{\text{eff}}$. We define the constant $\alpha(X)$ by

$$\alpha(X) = \chi_{\Lambda_{\text{eff}}}([X^{-1}]).$$

Corollary 2.4.7. If $\Lambda_{\text{eff}}$ is a finitely generated polyhedral cone, then $\alpha(X)$ is a rational number.

Remark 2.4.8. Peyre [29] introduces a similar constant $\alpha_c(X)$ related to the cone of effective divisors. By 2.4.4, $\alpha_c(X) = \alpha(X) / (r - 1)!$ (where $r = \text{rk Pic}(X)$).

Example 2.4.9. Let $P_{\Sigma,K}$ be a smooth compactification of an anisotropic torus $T_K$. By 1.3.11, $\Lambda_{\text{eff}} \subset \text{Pic}(P_{\Sigma,K}) \otimes R$ is a simplicial cone. Using 2.4.5 and the exact sequence

$$0 \to PL(\Sigma)^G \to \text{Pic}(P_{\Sigma,K}) \to H^1(G, M) \to 0,$$

we obtain

$$\chi_{\Lambda_{\text{eff}}}(u) = \frac{1}{h(T)u_1 \cdots u_r},$$

where $u = \varphi$, $\varphi(c_j) = u_j$ ($j = 1, \ldots, l$). In particular,

$$\alpha(P_{\Sigma}) = \frac{1}{h(T)}.$$

3 Distribution of rational points

3.1 The method of Draxl

Let $\Sigma$ be a $G$-invariant regular fan, and $\Sigma(1) = \Sigma_1(1) \cup \cdots \cup \Sigma_r(1)$ the decomposition of $\Sigma(1)$ into $G$-orbits. We choose a representative $c_j$ in each $\Sigma_j(1)$ ($j = 1, \ldots, r$). Let $e_j$ be the primitive integral generator of $c_j$, and $G_j \subset G$ the stabilizer of $e_j$. Denote by $k_j$ the length of the $G$-orbit of $e_j$, and by $K_j \subset E$ the subfield of $G_j$-fixed elements. One has $k_j = [K_j : K]$, $j = 1, \ldots, r$.

Consider the $n$-dimensional torus

$$T' := \prod_{j=1}^r R_{K_j/K}(G_m).$$

Notice that the group $D(\Sigma)$ can be identified with the $G$-module $\hat{T'}$. The homomorphism of $G$-modules $M \to D(\Sigma)$ induces the homomorphisms $T' \to T$ and

$$\gamma : T'(A_k) / T'(K) = \prod_{j=1}^r G_m(A_{K_j}) / G_m(K_j) \to T(A_k) / T(K).$$
So we have the dual homomorphism for characters
\[ \gamma^*: (T(A_K)/T(K))^* \to \prod_{j=1}^{r}(G_m(A_{K_j})/G_m(K_j))^*. \]

**Theorem 3.1.1 [15].** The kernel of \( \gamma^* \) is dual to the obstruction group to weak approximation \( \Lambda(T) \) defined in 1.4.13.

Let
\[ \chi: T(A_k) \to \mathbb{S} \subset \mathbb{C}^* \]
be a topological character which is trivial on \( T(K) \). Then \( \chi \circ \gamma \) defines \( r \) Hecke characters of the idele groups
\[ \chi_j: G_m(A_{K_j}) \to \mathbb{S} \subset \mathbb{C}^*, \quad j = 1, \ldots, r. \]
If \( \chi \) is trivial on \( K_T \), then all characters \( \chi_j \) \((j = 1, \ldots, r)\) are trivial on the maximal compact subgroups in \( G_m(A_{K_j}) \); i.e., \( \chi_j \) is an idele class character. We denote by \( L_{K_j}(s, \chi_j) \) the Hecke \( L \)-function corresponding to the character \( \chi_j \). The following statement is well known.

**Theorem 3.1.2.** The function \( L_{K_j}(s, \chi_j) \) is holomorphic in the whole plane unless \( \chi_j \) is trivial. In the latter case, \( L_{K_j}(s, \chi_j) \) is holomorphic for \( \text{Re}(s) > 1 \) and has a meromorphic extension to the complex plane with a pole of order 1 at \( s = 1 \).

The following statement describes the analytical properties of Fourier transforms of height functions.

**Theorem 3.1.3.** Define affine complex coordinates \( \{s_1, \ldots, s_r\} \) on the vector space \( \text{PL}(\Sigma)_C^G \) by \( s_j = \varphi(e_j) \) \((j = 1, \ldots, r)\). Then the Fourier transform \( \hat{H}_{L}(\chi, -\varphi) \) of the complex height function \( H_{L}(x, -\varphi) \) is always an analytic function for \( \text{Re}(s_j) > 1 \) \((1 \leq j \leq l)\), and
\[ \hat{H}_{L}(\chi, -\varphi) \prod_{i=1}^{r} L_{K_j}^{-1}(s_j, \chi_j) \]
has an analytic extension to the domain \( \text{Re}(s_j) > 1/2 \) \((1 \leq j \leq r)\).

**Proof.** The idea of the proof is essentially due to Draxl [15]. We have the Euler product
\[ \hat{H}_{L}(\chi, -\varphi) = \prod_{v \in \text{Val}(K)} \hat{H}_{L,v}(\chi_v, -\varphi). \]
Since, for all valuations \( v \in \text{Val}(K) \), the Fourier transform \( \hat{H}_{L,v}(\chi_v, -\varphi) \) is bounded by a constant in the domain \( \text{Re}(s) > 1/2 \) (see 2.2.8), it is sufficient to investigate the analytical properties of the product
\[ \hat{H}_{L,S}(\chi, -\varphi) = \prod_{v \notin S} \hat{H}_{L,v}(\chi_v, -\varphi). \]
Choose a valuation \( v \not\in S \). Then we obtain a cyclic subgroup \( G_v = \langle \Phi_v \rangle \subset G \) generated by a lattice automorphism \( \Phi_v : N \to N \) representing the local Frobenius element.

For every \( j \in \{1, \ldots, r\} \), the \( G \)-orbit \( \Sigma_j(1) \) decomposes into a disjoint union of \( G_v \)-orbits

\[ \Sigma_j(1) = \Sigma_{j1}(1) \cup \cdots \cup \Sigma_{jl_j}(1). \]

Let \( d_{ji} \) be the length of the \( G_v \)-orbit \( \Sigma_{ji}(1) \).

We know that \( l_j \) is the number of different valuations \( V_{j1}, \ldots, V_{jl_j} \in \text{Val}(K_j) \) over \( v \in \text{Val}(K) \). Let \( k_v \) be the residue field of \( v \in \text{Val}(K) \), and \( k_{V_{ji}} \) the residue field of \( V_{ji} \in \text{Val}(K) \).

Then

\[ d_{ji} = [k_{V_{ji}} : k_v]. \]

We denote by \( n_{ji} \) one of \( d_{ji} \) generators of all 1-dimensional cones of the \( G_v \)-orbit \( \Sigma_{ji}(1) \); i.e., \( G_v n_{ji} \) is the set of generators of 1-dimensional cones in \( \Sigma_{ji}(1) \). Then \( \chi_{j}(n_{ji}) \) is the \( V_{ji} \)-adic component of the Hecke character \( \chi_{j} \). Since \( \phi(n_{ji}) = s_j \) (\( j = 1, \ldots, l_j \)), we obtain that

\[ \prod_{i=1}^{l_j} \left( 1 - \frac{\chi_{j}(d_{ji}n_{ji})}{q_{V_{ji}}^{d_{ji}n_{ji}}} \right)^{-1} \]

equals the product of the local factors

\[ \prod_{V_{ji}} \left( 1 - \frac{\chi_{j}(n_{ji})}{q_{V_{ji}}} \right)^{-1} \]

of the Hecke \( L \)-function \( L_{K_j}(s_j, \chi_j) \).

By 2.2.6 and 2.2.3,

\[ \hat{H}_{\Sigma, S}(\chi, -\phi) \prod_{j=1}^{r} L_{K_j, S}^{-1}(s_j, \chi_j) = \prod_{v \not\in S} Q_{\Sigma} \left( \frac{\chi_{j}(n_{11})}{q_{V_{j1}}^{n_{11}}}, \ldots, \frac{\chi_{j}(n_{rl_r})}{q_{V_{jrl_r}}^{n_{rl_r}}} \right) \]

is an absolutely convergent Euler product for \( \text{Re}(s_j) > 1/2 \) (\( j = 1, \ldots, l \)) and therefore is bounded by some constant in the domain \( \text{Re}(s) > 1/2 + \delta \) (\( \delta > 0 \)) uniformly in \( \chi \). □

### 3.2 The meromorphic extension of \( Z_{\Sigma}(\phi) \)

**Definition 3.2.1** [36]. A Hecke character

\[ \chi : G_m(A_k)/G_m(K) \to S^1 \]

is called **unramified at** \( v \in \text{Val}(K) \) if the local \( v \)-component \( \chi_v \) of \( \chi \) is trivial on \( G_m(O_v) \); i.e., \( \chi_v \) is given by

\[ \chi_v(x) = |x|_v^{1/s_v}. \]
for some real number \( t_v \). A Hecke character \( \chi \) is called unramified if it is unramified at all \( v \in \text{Val}(K) \).

**Definition 3.2.2.** Let \( \chi \) be an unramified Hecke character. We set

\[
y(\chi) := \{ t_v \}_{v \in S_\infty(K)} \in \mathbb{R}^{r_1 + r_2},
\]

where \( r_1 \) (resp., \( r_2 \)) is the number of real (resp., pairs of complex) valuations of \( K \). We also set

\[
\|y(\chi)\| := \max_{v \in S_\infty(K)} |t_v|.
\]

We will use the following standard statement based on the Phragmén-Lindelöf principle and on the functional equation for Hecke L-functions (see [30, Theorem 5]).

**Theorem 3.2.3.** For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) and a constant \( c(\varepsilon) > 0 \) such that the inequality

\[
|L_K(s, \chi)| \leq c(\varepsilon)(1 + |\text{Im}(s)| + \|y(\chi)\|)^\varepsilon
\]

holds for all \( s \) with \( \text{Re}(s) > 1 - \delta \) and every Hecke L-function \( L_K(s, \chi) \) corresponding to a nontrivial unramified Hecke character \( \chi \).

**Corollary 3.2.4.** For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for any compact \( K \) in the halfplane \( \text{Re}(s) > 1 - \delta \), there exists a constant \( C(K, \varepsilon) \) depending only on \( K \) and \( \varepsilon \) such that

\[
|L_K(s, \chi)| \leq C(K, \varepsilon)(1 + \|y(\chi)\|)^\varepsilon \quad \text{for } s \in K \text{ and every } \chi \neq 1.
\]

**Theorem 3.2.5.** Let \( s_j = \varphi(e_j) \) (\( j = 1, \ldots, r \)). Then the height zeta-function \( Z_{E}(\varphi) \) is holomorphic for \( \text{Re}(s_j) > 1 \). There exists a meromorphic continuation of \( Z_{E}(\varphi) \) to the domain \( \text{Re}(s_j) > 1 - \delta \) such that the only singularities of \( Z_{E}(\varphi) \) in this domain are poles of order \( \leq 1 \) along the hyperplanes \( s_j = 1 \) (\( j = 1, \ldots, r \)).

**Proof.** Let

\[
\hat{H}_{E,s}(\chi, -\varphi) = \int_{T(K)} H_{E}(x, -\varphi)\chi(x)\omega_{\Omega,S}
\]

be the Fourier transform of the height function \( H_{E}(x, -\varphi) \) with respect to the adelic measure \( \omega_{\Omega,S} \). Recall that for anisotropic tori, \( b_S(T) \) is the volume of \( T(A_K)/T(K) \) with respect to \( \omega_{\Omega,S} \).

---

\(^2\)We are grateful to B. Z. Moroz for pointing out this fact and the reference.
By the Poisson formula,
\[ Z_\Sigma(\varphi) = \frac{1}{b_S(T)} \sum_{\chi \in (T(A_K)/T(K))^*} \hat{H}_{\Sigma,S}(\chi, -\varphi). \]
Since \( H_{\Sigma}(\chi, -\varphi) \) is \( K_T \)-invariant, we can assume that in the above formula \( \chi \) runs over the elements of the group \( P \) consisting of characters of \( T(A_K) \) that are trivial on \( K_T \cdot T(K) \).

Let \( J \) be a subset of \( I = \{1, \ldots, r\} \). Denote by \( P_J \) the subset of \( P \) consisting of all characters \( \chi \in P \) such that the corresponding Hecke character \( \chi_j \) is trivial if and only if \( j \in J \). Then
\[ Z_\Sigma(\varphi) = \sum_{J \subset I} Z_{\Sigma,J}(\varphi) \]
where
\[ Z_{\Sigma,J}(\varphi) = \frac{1}{b_S(T)} \sum_{\chi \in P_J} \hat{H}_{\Sigma,S}(\chi, -\varphi). \]

Since in the considered domain \( \prod_{j \in J} \zeta_{K_j}(s_j) \) has only poles of order 1 along the hyperplanes \( s_j = 1 \), it is sufficient to prove the following.

**Lemma 3.2.6.** There exists a \( \delta > 0 \) such that
\[ Z_{\Sigma,J}(\varphi) \prod_{j \in J} \zeta_{K_j}(s_j)^{-1} \]
is a holomorphic function for \( \text{Re}(s_j) > 1 - \delta \) and for any \( J \subset I \). \( \square \)

For the proof of Lemma 3.2.6, we consider the logarithmic space
\[ N_{R,\infty} = \bigoplus_{v \in S_\infty(K)} T(K_v)/T(O_v) = \bigoplus_{v \in S_\infty(K)} N_{R,v} \]
containing the full sublattice \( T(O_K)/W(T) \) of \( O_K \)-integral points of \( T(K) \) modulo torsion. Then the dual logarithmic space \( M_{R,\infty} = \bigoplus_{v \in S_\infty(K)} M_{R,v} \)
contains the dual lattice \( L := (T(O_K)/W(T))^* \subset M_{R,\infty} \), which is the image of the projection of \( P \) to \( M_{R,\infty} \). By 1.4.6, we have the exact sequence
\[ 0 \to \text{cl}(T) \to P \to L \to 0, \]
where \( \text{cl}(T) \) is a finite group. We denote by \( y(\chi) \) the image of \( \chi \in P \) in \( L \). We also set
\[ \|y(\chi)\| := \max_{j \in I} \|y(\chi_j)\|, \]
where \( \|y(\chi_j)\| \) are defined in 3.2.2.
By 3.2.4, for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that, for any compact \( K \) in the domain \( \text{Re}(s_j) > 1 - \delta \) \((j = 1, \ldots, r)\), we have

\[
\prod_{j \notin J} L_{K_j}(s_j, \chi_j) \leq C(K, \varepsilon) \prod_j (1 + \|y(\chi_j)\|)^\varepsilon,
\]

where \( \chi_j \) \((j \notin J)\) are nontrivial Hecke characters. We fix these \( \varepsilon < 1/rd \) and \( \delta \). We also assume that \( \delta < 1/2 \).

Recall that the Fourier transforms of local heights at nonarchimedian places of bad reduction are bounded by a constant for \( \text{Re}(s_j) > 1/2 \). We set

\[
Q_{\Sigma}(\chi, -\varphi) := \prod_{v \in S_\infty(K)} \hat{H}_{\Sigma, v}(\chi_v, \varphi) \prod_{j=1}^r L_{K_j}^{-1}(s_j, \chi_j).
\]

By 3.1.3, \( Q_{\Sigma}(\chi, -\varphi) \) is an absolutely convergent Euler product for \( \text{Re}(s_j) > 1 - \delta > 1/2 \) and therefore is also uniformly bounded by a constant \( C_0(K) \) in this domain.

Hence, the function

\[
g(\chi, -\varphi) := Q_{\Sigma}(\chi, -\varphi) \prod_{j \notin J} L_{K_j}(s_j, \chi_j)
\]

can be estimated as

\[
|g(\chi, -\varphi)| \leq C_0(K) C(K, \varepsilon)(1 + \|y(\chi)\|)^\varepsilon.
\]

Let

\[
\hat{H}_{\Sigma, \infty}(\chi, \varphi) = \prod_{v \in S_\infty(K)} \hat{H}_{\Sigma, v}(\chi_v, \varphi).
\]

Since all characters \( \chi \in \mathcal{P}_J \) are defined up to elements of the finite group \( \text{cl}(\mathcal{T}) \) by their archimedian components \( y(\chi) \in \mathcal{L} \subset \mathcal{M}_{R, \infty} \), \( g(\chi, -\varphi) \) satisfies the conditions of 2.3.4, and we obtain that

\[
\sum_{\chi \in \mathcal{P}_J} g(\chi, -\varphi) \hat{H}_{\Sigma, \infty}(\chi, \varphi)
\]

is absolutely and uniformly convergent on any compact \( K \) in the domain \( \text{Re}(s_j) > 1 - \delta \) \((j = 1, \ldots, r)\). On the other hand,

\[
\sum_{\chi \in \mathcal{P}_J} g(\chi, -\varphi) \hat{H}_{\Sigma, \infty}(\chi, \varphi) = Z_{\Sigma, J}(\varphi) \prod_j \zeta_{K_j}(s_j)^{-1}.
\]

This proves Lemma 3.2.6 and finishes the proof of Theorem 3.2.5.

3.3 Rational points of bounded height

Recall the standard tauberian statement.
Theorem 3.3.1 [11]. Let \( \xi(t) \) be a function on \([0, \infty)\) of bounded variation in each finite \( t \)-interval. Assume that we have
\[
\int_0^\infty e^{-st}\xi(t) = \frac{1}{(s-a)^r}g(s) + h(s)
\]
for \( \Re(s) > a > 0 \), with \( g(s) \) and \( h(s) \) holomorphic functions for \( \Re(s) \geq a \), \( g(a) \neq 0 \), and \( r \neq 0, -1, -2, \ldots \). Then, as \( t \to \infty \),
\[
\xi(t) = \frac{g(a)}{\Gamma(r)}e^{at^{r-1}}(1 + o(1)).
\]
We will use it in the following form.

Theorem 3.3.2. Let \( X \) be a countable set, and \( F : X \to \mathbb{R}_{>0} \) a real-valued function. Assume that
\[
ZF(s) = \sum_{x \in X} F(x)^{-s}
\]
is absolutely convergent for \( \Re(s) > a > 0 \) and has a representation
\[
ZF(s) = (s-a)^{-r}g(s) + h(s)
\]
with \( g(s) \) and \( h(s) \) holomorphic for \( \Re(s) \geq a \), \( g(a) \neq 0 \), \( r \in \mathbb{N} \). Then for any \( B > 0 \), there exists only a finite number \( N(F, B) \) of elements \( x \in X \) such that \( F(x) \leq B \). Moreover, for \( B \to \infty \) we have
\[
N(F, B) = \frac{g(a)}{a(r-1)!}B^a(\log B)^{r-1}(1 + o(1)).
\]
Proof. For \( \Re(s) > a \), we have
\[
ZF(s) = s \int_1^\infty N(F, B)B^{-s-1}dB.
\]
We can apply the theorem above with \( t = \log B \).

Let \( \mathcal{L} = \mathcal{L}(\varphi_0) \) be a metrized invertible sheaf over a smooth compactification \( \mathbf{P}_\Sigma \) of an anisotropic torus \( T_K \) defined by a \( G \)-invariant fan \( \Sigma \). We denote by \( Z_{\Sigma, \mathcal{L}}(s) = Z_{\mathcal{L}}(s \varphi_0) \) the restriction of \( Z_{\mathcal{L}}(s) \) to the line \( s[\mathcal{L}] \subset \text{Pic}(\mathbf{P}_\Sigma)_R \). Let \( a(\mathcal{L}) \) be the abscissa of convergence of \( Z_{\Sigma, \mathcal{L}}(s) \), and \( b(\mathcal{L}) \) the order of the pole of \( Z_{\Sigma, \mathcal{L}}(s) \) at \( s = a(\mathcal{L}) \). By 3.2.5,
\[
a(\mathcal{L}) \leq \min_{j=1}^r \frac{1}{\varphi(e_j)}.
\]

Theorem 3.3.3. The number \( a(\mathcal{L}) \) equals
\[
a(\mathcal{L}) = \inf\{\lambda \mid \lambda[\mathcal{L}] + [X] \in \Lambda_{\text{eff}}(\Sigma)\};
\]
Rational Points of Bounded Height

\[ a(\mathcal{L}) = \min_{j=1}^{r} \frac{1}{\varphi(e_j)}. \]

Moreover, \( b(\mathcal{L}) \) equals the codimension of the minimal face of \( \Lambda_{\text{eff}}(\Sigma) \) containing \( a(\mathcal{L})[\mathcal{L}] + [\mathcal{X}] \). \( \square \)

**Proof.** It was proved in [27] that we can always choose the finite set \( S \) such that the natural map

\[ \pi_S : T(K) \rightarrow \bigoplus_{v \in S} T(K_v)/T(\mathcal{O}_v) = \bigoplus_{v \not\in S} \mathcal{N}_v \]

is surjective. Denote by \( T(\mathcal{O}_S) \) the kernel of \( \pi_S \) consisting of all \( S \)-units in \( T(K) \). The group \( T(\mathcal{O}_S)/W(T) \) has a natural embedding in the finite-dimensional space

\[ \mathcal{N}_{R,S} = \bigoplus_{v \in S} T(K_v)/T(\mathcal{O}_v) \otimes \mathbb{R} \]

as a full sublattice.

Let \( \Delta \) be a bounded fundamental domain of \( T(\mathcal{O}_S)/W(T) \) in \( \mathcal{N}_{R,S} \). For any \( x \in T(K) \), denote by \( \overline{x}_S \) the image of \( x \) in \( \mathcal{N}_{R,S} \). Define \( \phi_S(x) \) to be the element of \( T(\mathcal{O}_S) \) such that \( \overline{x}_S - \phi_S(x) \in \Delta \). Thus, we have obtained the mapping

\[ \phi_S : T(K) \rightarrow T(\mathcal{O}_S). \]

Define a new height function \( \tilde{H}_L(x, \varphi) \) on \( T(K) \) by

\[ \tilde{H}_L(x, \varphi) = H_L(\phi_S(x), \varphi) \prod_{v \not\in S} H_{L,v}(x_v, \varphi). \]

Since \( \overline{x}_S - \phi_S(x) \) belongs to the bounded set \( \Delta \) in \( \mathcal{N}_{R,S} \) for any \( x \in T(K) \), we obtain the following statement.

**Lemma 3.3.4.** Choose a compact subset \( K \subset \mathbb{C}^* \) such that \( \text{Re}(\varphi(e_j)) > \delta \) for \( \varphi \in K \). Then there exist positive constants \( C_1, C_2 \) such that

\[ 0 < C_1 < \frac{\tilde{H}_L(x, \varphi)}{H_L(x, \varphi)} < C_2, \quad \text{for } \varphi \in K, \ x \in T(K). \] \( \square \)

Define \( \tilde{Z}_L(\varphi) \) by

\[ \tilde{Z}_L(\varphi) = \sum_{x \in T(K)} \tilde{H}_L(x, -\varphi). \]

Then \( \tilde{Z}_L(\varphi) \) splits into the product

\[ \tilde{Z}_L(\varphi) = \prod_{v \not\in S} \left( \sum_{z \in \mathcal{N}_v} H_{L,v}(z, -\varphi) \right) \cdot \left( \sum_{u \in T(\mathcal{O}_S)} H_L(u, -\varphi) \right). \]
Using the same arguments as in the proof of Theorem 3.1.3, the Euler product
\[
\prod_{j=1}^{r} \zeta_{K_j}(s_j) \prod_{v \notin S} \left( \sum_{z \in N_v} H_{\Sigma,v}(z, -\varphi) \right)
\]
is a holomorphic function without zeros for \( \text{Re}(s_j) > 1/2 \).

On the other hand,
\[
\sum_{u \in T(O_S)} H_{\Sigma}(u, -\varphi)
\]
is an absolutely convergent series, nonvanishing for \( \text{Re}(s_j) > 0 \). Therefore, \(~\tilde{Z}_{\Sigma}(\varphi)\) has a meromorphic extension to the domain \( \text{Re}(s_j) > 1/2 \) where it has poles of order 1 along the hyperplanes \( s_j = 1 \).

By 3.3.4 and 3.3.2, \(~\tilde{Z}_{\Sigma}(\varphi)\) and \(~Z_{\Sigma}(\varphi)\) must have the same poles in the domain \( \text{Re}(s_j) > 1 - \delta \). Therefore, \(~Z_{\Sigma}(\varphi)\) has poles of order 1 along the hyperplanes \( s_j = 1 \). By taking the restriction of \(~Z_{\Sigma}(\varphi)\) to the line \( \varphi = s \varphi_0 \), we obtain the statement.

By 3.3.2, we obtain the following.

**Theorem 3.3.5.** Assume that \( \varphi(e_i) > 0 \) for all \( i = 1, \ldots, r \); i.e., the class \([L]\) is contained in the interior of the cone of effective divisors \( \Lambda_{\text{eff}}(\Sigma) \). Then there exists only finite number \( N(P_{\Sigma}, L, B) \) of \( K \)-rational points \( x \in T(K) \) having the \( L \)-height \( H_{\Sigma}(x) \leq B \). Moreover,
\[
N(P_{\Sigma}, L, B) = c(L)B^{a(L)} \cdot (\log B)^{b(L)-1} + o(1), \quad B \to \infty
\]
with some constant \( c(L) \neq 0 \).

This proves that Batyrev-Manin conjectures about the distribution of rational points of bounded \( L \)-height (see [3]) are valid for smooth compactifications of anisotropic tori.

3.4 The asymptotic constant

In [29], Peyre defined the Tamagawa number of Fano varieties. This definition immediately extends to smooth algebraic varieties \( X \) with a metrized canonical sheaf \( \mathcal{K} \) which satisfy the conditions \( h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0 \). Under these assumptions, the Neron-Severi group of \( X \) (or, equivalently, the Picard group of \( X \) modulo torsion) over \( \bar{K} \) is a discrete continuous \( \text{Gal}(\bar{K}/K) \)-module of finite rank over \( \mathbb{Z} \). Let \( T_{\text{NS}} \) be the corresponding torus under the duality from 1.1.4.

Let \( x_1, \ldots, x_d \) be local analytic coordinates on \( X \). They define a homeomorphism \( \psi : \mathcal{U} \to \mathbb{K}_v \) in \( v \)-adic topology between an open subset \( \mathcal{U} \subset X \) and \( \psi(\mathcal{U}) \subset K_v \). Let
Let $dx_1 \cdots dx_d$ be the Haar measure on $K_v^d$ normalized by the condition
\[
\int_{O_v^d} dx_1 \cdots dx_d = \frac{1}{(\sqrt{\delta_v})^d}
\]
where $\delta_v$ is the absolute different of $K_v$. Denote by $dx_1 \wedge \cdots \wedge dx_d$ the standard differential form on $K_v^d$. Then $f = \psi^*(dx_1 \wedge \cdots \wedge dx_d)$ is a local analytic section of the metrized canonical sheaf $\mathcal{X}$. We define the $v$-adic measure on $U$ by
\[
\int_W \omega_{\mathcal{X}, v} = \int_{\psi(W)} \|f(\psi^{-1}(x))\|_v dx_1 \cdots dx_d,
\]
where $W$ is arbitrary open subset in $U$. The measure $\omega_{\mathcal{X}, v}$ does not depend on the choice of local coordinates and extends to a global measure on $X(K_v)$ [29]. Let $S$ be a finite set of valuations containing the archimedean places and the places of bad reduction. The adelic Tamagawa measure $\omega_{\mathcal{X}, S}$ is defined by
\[
\omega_{\mathcal{X}, S} = \prod_{v \in \text{Val}(K)} \lambda_v^{-1} \omega_{\mathcal{X}, v}
\]
where $\lambda_v = L_v(1, T_{NS}; \overline{K}/K)$ if $v \not\in S$, $\lambda_v = 1$ if $v \in S$.

Definition 3.4.1 [29]. Let $r$ be the rank of the Picard group of $X$ over $K$, and $\overline{X(K)}$ the closure of $X(K) \subset X(A_K)$ in the direct product topology. Then the Tamagawa number of $X$ is defined by
\[
\tau_{\mathcal{X}}(X) = \frac{b_S(X)}{l_S(X)}
\]
where
\[
b_S(X) = \int_{X(K)} \omega_{\mathcal{X}, S}
\]
whenever the adelic integral converges, and
\[
l_S^{-1}(X) = \lim_{s \to 1} (s - 1)^r L_S(s, T_{NS}; \overline{K}/K).
\]

Remark 3.4.2. We notice that there is a difference in the definitions of convergence factors for the Tamagawa measure on an algebraic variety $X$ and for the Tamagawa measure on an algebraic torus $T$. In the first case, we choose $L_v^{-1}(1, T_{NS}; \overline{K}/K)$, whereas in the second case one uses $L_v(1, T; \overline{K}/K)$. This explains the difference in the definitions of $l_S(X)$ and $l_S(T)$.

Remark 3.4.3. Peyre [29] proves the existence of the Tamagawa number for Fano varieties by using the Weil conjectures. In our case we show the convergence of the adelic integral directly.
Proposition 3.4.4. Let \( \mathcal{K} = \mathcal{L}(-\varphi) \) be the metrized canonical sheaf on a toric variety \( \mathbf{P}_\Sigma \). Then the restriction of the \( \nu \)-adic measure \( \omega_{\mathcal{K},\nu} \) to \( T(K_v) \subset \mathbf{P}_\Sigma(K_v) \) coincides with the measure
\[
H_{\Sigma,\nu}(x, -\varphi) \omega_{\Omega,\nu},
\]
where \( \varphi \) (resp., \( \omega_{\Omega,\nu} \)) is defined in 1.2.12 (resp., in 1.4.8).

Proof. The \( T \)-invariant rational differential \( d \)-form \( \Omega \) is a rational section of the canonical sheaf \( \mathcal{K} \). By definition of the \( \nu \)-adic metric on \( \mathcal{L}(-\varphi) \), \( H_{\Sigma,\nu}(x, -\varphi) \) equals the norm \( \| \Omega \|_\nu \) of the \( T \)-invariant section \( \Omega \) of \( \mathcal{K} \). This implies the statement. \( \square \)

Proposition 3.4.5. One has
\[
\int_{T(K)} \omega_{\mathcal{K},S} = \int_{\mathbf{P}_\Sigma(K)} \omega_{\mathcal{K},S}.
\]

Proof. Notice that \( \mathbf{P}_\Sigma(K_v) \setminus T(K_v) \) has zero \( \nu \)-adic measure for every \( \nu \in \text{Val}(K) \), i.e.,
\[
\int_{T(K_v)} \omega_{\mathcal{K},\nu} = \int_{\mathbf{P}_\Sigma(K_v)} \omega_{\mathcal{K},\nu},
\]
because \( \mathbf{P}_\Sigma \setminus T \) is a proper Zariski closed subset in a smooth variety (see [6, pp. 36–38]). By 1.4.14,
\[
\overline{T(K)} = \overline{T(K)}_S \times T(A_{K,S}).
\]
Therefore,
\[
\overline{\mathbf{P}_\Sigma(K)} = \overline{\mathbf{P}_\Sigma(K)}_S \times \mathbf{P}_\Sigma(A_{K,S}),
\]
where \( \overline{\mathbf{P}_\Sigma(K)}_S \) stands for the closure of \( \mathbf{P}_\Sigma(K) \) in the finite direct product \( \prod_{\nu \in S} \mathbf{P}_\Sigma(K_v) \). It remains to show that
\[
\overline{\mathbf{P}_\Sigma(K)}_S \setminus \overline{T(K)}_S
\]
has zero measure.

Assume that \( x = \{x_v\}_{\nu \in S} \) is an element of \( \overline{\mathbf{P}_\Sigma(K)}_S \setminus \overline{T(K)}_S \). If \( x_v \in T(K_v) \) for all \( \nu \in S \), then there exists an open neighbourhood of \( x \) in \( \prod_{\nu \in S} T(K_v) \). Hence, almost all elements of any sequence of \( K \)-rational points in \( \mathbf{P}_\Sigma(K) \) converging to \( x \) must be in \( T(K) \), which gives us a contradiction. Therefore, there exists a \( \nu \in S \) such that \( x_{\nu} \in \mathbf{P}_\Sigma(K_{\nu}) \setminus T(K_{\nu}) \). By Fubini’s theorem, the set of all such \( x \in \prod_{\nu \in S} \mathbf{P}_\Sigma(K_{\nu}) \) has zero measure. \( \square \)

Theorem 3.4.6. There exists a \( \delta > 0 \) such that the height zeta-function \( Z_{\Sigma}(s) \) obtained by restriction of the zeta-function \( Z_{\Sigma}(\varphi) \) to the complex line \( \varphi(e_j) = s \) for all \( j = 1, \ldots, r \) has a representation of the form
\[
Z_{\Sigma}(s) = \frac{g(s)}{(s - 1)^r} + h(s)
\]
with \( g(s) \) and \( h(s) \) holomorphic functions in the domain \( \text{Re}(s) > 1 - \delta \). Moreover, we have

\[
g(1) = \Theta(\Sigma, K) = \alpha(P) \beta(P) \tau(K). \quad \square
\]

**Proof.** Using the notation from the proof of Theorem 3.2.5, we have

\[
Z_{\Sigma}(\varphi) = \sum_{J \subset I} Z_{\Sigma, J}(\varphi).
\]

By 3.2.6, the pole of the highest order \( r \) of \( Z_{\Sigma}(\varphi) \) at \( s_1 = \cdots = s_r = 1 \) appears from the term \( Z_{\Sigma, I}(\varphi) \), which contains only the Fourier transforms \( \hat{H}_{\Sigma}(\chi, -s\varphi) \) such that \( \chi_1, \ldots, \chi_r \) are trivial characters of \( G_m(A_{K_j})/G_m(K_j) \) (\( j = 1, \ldots, r \)); i.e., \( \chi \) is a character of the finite group \( A(T) \) (see 3.1.1).

We assume that \( \varphi(e_1) = \cdots = \varphi(e_r) = s \), i.e., \( \varphi = s\varphi_{\Sigma} \). For \( \text{Re}(s) > 1 \), we can apply the Poisson formula to the finite group \( A(T) = T(A_{K_i}/T(K_i))_{1.4.16} \) and rewrite the sum

\[
Z_{\Sigma, I}(s\varphi_{\Sigma}) = \frac{1}{b_S(T)} \sum_{\chi \in (A(T))^*} \hat{H}_{\Sigma,S}(\chi, -s\varphi_{\Sigma})
\]

as an integral over \( T(K) \):

\[
Z_{\Sigma, I}(s\varphi_{\Sigma}) = \frac{\beta(P)}{i(b_S(T))} \int_{T(K)} \hat{H}_{\Sigma,S}(x, -s\varphi_{\Sigma}) \omega_{\Omega,S}
\]

(see 1.4.16).

Our purpose is to compute the constant

\[
\Theta(\Sigma, K) = \lim_{s \to 1} (s - 1)^r Z_{\Sigma, I}(s\varphi_{\Sigma}).
\]

By 1.4.14,

\[
\overline{T(K)} = \overline{T(K)}_S \times T(A_{K,S}),
\]

where \( \overline{T(K)}_S \) is the image of \( \overline{T(K)} \) in \( \prod_{v \in S} T(K_v) \). It follows that

\[
\int_{\overline{T(K)}} \hat{H}_{\Sigma}(x, -s\varphi_{\Sigma}) \omega_{\Omega,S} = \prod_{v \in S} \left( \prod_{v \in S} \hat{H}_{\Sigma,v}(x_v, -s\varphi_{\Sigma}) \omega_{\Omega,v} \cdot \prod_{v \in S} \int_{T(K_v)} \hat{H}_{\Sigma,v}(x_v, -s\varphi_{\Sigma}) d\mu_v. \right)
\]

(Recall that \( d\mu_v = c_v \omega_{\Omega,v} \) and \( c_v = 1 \) for \( v \in S \).) From our calculations of the Fourier
transform of local height functions for \(v \notin S\) (2.2.7), we have

\[
\prod_{\nu \notin S} \int_{T(K_{\nu})} H_{\Sigma,\nu}(x_{\nu}, -s\varphi_{\Sigma}) \, d\mu_{\nu} \tag{5}
\]

\[
= L_S(s, T; E/K) \cdot L_S(s, T_{NS}; \overline{K}/K) \prod_{\nu \notin S} Q_{\Sigma}(q_{\nu}^{-s}, \ldots, q_{\nu}^{-s}).
\]

By 2.2.3,

\[
\prod_{\nu \notin S} Q_{\Sigma}(q_{\nu}^{-s}, \ldots, q_{\nu}^{-s})
\]

is an absolutely convergent Euler product for \(s = 1\). By 2.2.8,

\[
\int_{T(K)} \prod_{\nu \in S} H_{\Sigma,\nu}(x_{\nu}, -s\varphi_{\Sigma}) \omega_{\Omega,\nu}
\]

is also absolutely convergent for \(s = 1\). Moreover, \(L_S(s, T; E/K)\) equals the nonzero constant \(l_S(T)\) for \(s = 1\). Using (4) and (5), we obtain

\[
\lim_{s \to 1} (s - 1)^r \int_{T(K)} H_{\Sigma}(x, -s\varphi_{\Sigma}) \omega_{\Omega,S} \tag{6}
\]

\[
= \frac{l_S(T)}{l_S(P_\Sigma)} \int_{T(K)} \prod_{\nu \in S} H_{\Sigma,\nu}(x_{\nu}, -\varphi_{\Sigma}) \omega_{\Omega,\nu} \cdot \prod_{\nu \notin S} Q_{\Sigma}(q_{\nu}^{-1}, \ldots, q_{\nu}^{-1}).
\]

Now we remark that

\[
\prod_{\nu \notin S} Q_{\Sigma}(q_{\nu}^{-1}, \ldots, q_{\nu}^{-1}) = \prod_{\nu \notin S} \int_{T(K_{\nu})} \lambda_{\nu}^{-1} \omega_{\K_{\nu}},
\]

since, by 2.2.7 and 3.4.4,

\[
Q_{\Sigma}(q_{\nu}^{-1}, \ldots, q_{\nu}^{-1}) = \int_{T(K_{\nu})} \lambda_{\nu}^{-1} H_{\Sigma,\nu}(x_{\nu}, -\varphi_{\Sigma}) \omega_{\Omega,\nu}.
\]

By 3.4.4, we also have

\[
\int_{T(K)} \prod_{\nu \in S} H_{\Sigma,\nu}(x_{\nu}, -\varphi_{\Sigma}) \omega_{\Omega,\nu} = \int_{T(K)} \prod_{\nu \in S} \omega_{\K_{\nu}}.
\]

Using the splitting \(T(K) = T(K)_{E} \times T(A_{K,S})\) (1.4.14), and multiplying (7) and (8),

\[
\int_{T(K)} \omega_{\K_{S}} = \int_{T(K)} \prod_{\nu \in S} \omega_{\K_{\nu}} \cdot \prod_{\nu \notin S} \lambda_{\nu}^{-1} \omega_{\K_{\nu}}.
\]

On the other hand, by 3.4.5, we have

\[
\int_{T(K)} \omega_{\K_{S}} = \int_{P_\Sigma} \omega_{\K_{S}} = b_S(P_\Sigma).
\]
Hence,
\[ b_S(P_L) = \int_{T(K)} \prod_{v \in S} H_L(x, -\varphi_L) \omega_{L_v} \cdot \prod_{v \notin S} Q_L(q_v^{-1}, \ldots, q_v^{-1}). \]

Therefore,
\[ \Theta(\Sigma, K) = \frac{\beta(P_L)}{i(T)b_S(T)} \cdot \frac{l_S(T)}{l_S(P_L)} \cdot b_S(P_L). \]

By 1.4.10 and 1.4.12, we have the equality
\[ i(T)b_S(T) = h(T)l_S(T). \]

This implies
\[ \Theta(\Sigma, K) = \alpha(P_L)\beta(P_L)\tau_K(P_L). \]

Using the Tauberian theorem 3.3.2, we obtain the following.

**Corollary 3.4.7.** Let \( T \) be an anisotropic torus, and \( P_L \) its smooth projective compactification. (Notice that we do not need to assume that \( P_L \) is a Fano variety.) Let \( r \) be the rank of \( \text{Pic}(P_{L,K}) \). Then the number \( N(P_L, X^{-1}, B) \) of \( K \)-rational points \( x \in T(K) \) having the anticanonical height \( H_{X^{-1}}(x) \leq B \) has the asymptotic
\[ N(P_L, X^{-1}, B) = \frac{\Theta(\Sigma, K)}{(r - 1)!} \cdot B(\log B)^{r-1}(1 + o(1)), \quad B \to \infty. \]

3.5 Examples

**Example 3.5.1. Projective spaces.** Consider a \( d \)-dimensional fan \( \Sigma \) as in 1.2.13. It has a natural action of the symmetric group \( S_{d+1} \). Let \( K' \) be a finite extension of \( K \) of degree \( d + 1 \), and \( E \) the minimal normal extension of \( K \) containing \( K' \). Assume that the Galois group \( \text{Gal}(E/K) \) is a subgroup of \( S_{d+1} \). (For instance, \( K' \) is a simple algebraic extension defined by a \( K \)-irreducible polynomial \( f \) of degree \( d + 1 \).) Then the action of \( G \) on \( \Sigma \) defines a \( d \)-dimensional toric variety \( P_{L,K} \), which over \( E \) is isomorphic to \( d \)-dimensional projective space; i.e., \( P_{L,K} \) is a Severi-Brauer variety. Since \( P_{L,K} \) contains infinitely many \( K \)-rational points, \( P_{L,K} \) is in fact isomorphic to \( P^d \) over \( K \). Since for any number field \( K \) there exists an irreducible polynomial \( f \) of degree \( d + 1 \), we can always consider a \( d \)-dimensional projective space over \( K \) as an equivariant compactification of an anisotropic torus. This shows that our methods give another proof of the result of Schanuel [33]. We remark that the constant in the asymptotic formula obtained by the above method depends on \( K' \). Moreover, our choice of archimedian metrics is different from the standard one used in Schanuel’s paper.
Now we give a simple method for constructing infinitely many examples of compactifications of anisotropic tori in arbitrary fixed dimension $d$.

Example 3.5.2. A complete fan $\Sigma$ is called centrally symmetric if it is invariant under the map $-\text{Id}$ of $N_R$. Let $\Sigma$ be a centrally symmetric $d$-dimensional fan, and let $E$ be an extension of $K$ of degree 2. The $d$-dimensional torus $T$ corresponding to the integral representation of $\text{Gal}(E/K) \cong \mathbb{Z}/2\mathbb{Z}$ by $\text{Id}$ and $-\text{Id}$ is isomorphic to the anisotropic torus $(\mathbb{R}_E^1/K)^d$. The $\mathbb{Z}/2\mathbb{Z}$-invariant fan $\Sigma$ defines the compactification $\mathbb{P}_{\Sigma,K}$ of $(\mathbb{R}_E^1/K)^d$.

For special choices of $E$ and $K$, the zeta-function $Z_{\Sigma}(\varphi)$ can be computed explicitly as some Euler product.

**Proposition 3.5.3.** Let $\Sigma$ be a $d$-dimensional centrally symmetric fan as above. We set $K = \mathbb{Q}$, $E = \mathbb{Q}[\sqrt{-1}]$, $S := R_E^1/G_m$, and $T = S^d$. Then the zeta-function $Z_{\Sigma}(\varphi)$ has the form

$$Z_{\Sigma}(\varphi) = 4^d \cdot \prod_{p=4k+1} \frac{Q_{\Sigma}(1/p^{s_1}, \ldots, 1/p^{s_n})}{(1 - 1/p^{s_1}) \cdots (1 - 1/p^{s_n})}.$$ 

**Proof.** The anisotropic torus $S$ over $K$ is defined by the equation $x^2 + y^2 = 1$. Since $\mathbb{Z}[\sqrt{-1}]$ is a unique factorization domain, the natural homomorphism

$$S(K) \rightarrow \bigoplus_{v \in \text{Val}(K)} S(K_v)/S(\mathcal{O}_v)$$

is surjective, and its kernel consists of 4 elements: $\pm 1, \pm \sqrt{-1} \in \mathbb{Z}[\sqrt{-1}]$. Therefore, we obtain the natural isomorphism

$$T(K)/W(T) \cong \bigoplus_{v \in \text{Val}(K)} T(K_v)/T(\mathcal{O}_v),$$

where $W(T)$ is the group of order $4^d$ consisting of torsion elements in $T(K)$. It is obvious that $S(K_v) = S(\mathcal{O}_v)$ and $T(K_v) = T(\mathcal{O}_v)$ unless $v$ corresponds to a prime number $p$ of type $4k + 1$. It remains to apply 2.2.6.

Example 3.5.4. Let $\Sigma$ be the 2-dimensional complete regular centrally symmetric fan with 6 generators $\pm e_1, \pm e_2, \pm (e_1 + e_2)$, and $\mathbb{P}_{\Sigma}$ the corresponding compactification of the 2-dimensional anisotropic torus $S \times S$ over $K = \mathbb{Q}$. (One easily sees that $\mathbb{P}_{\Sigma}$ is isomorphic over $E = \mathbb{Q}[\sqrt{-1}]$ to the blow up of 3 points in $\mathbb{P}^2$.) Then, by 3.5.3, we have

$$Z_{\Sigma}(s\varphi_{\Sigma}) = 2^4 \cdot \prod_{p=4k+1} \frac{(1 - 9/p^{2s} + 16/p^{3s} - 9/p^{4s} + 1/p^{6s})}{(1 - 1/p^{2s})^6}$$

$$= 2^4 \cdot \prod_{p=4k+1} \frac{(1 + 4/p^s + 1/p^{2s})}{(1 - 1/p^s)^2}.$$
Let $\chi$ be the quadratic Dirichlet character with the following properties: $\chi(p) = 1$ if $p = 4k + 1$, and $\chi(p) = -1$ if $p = 4k + 3$. Define the quadratic polynomial $H_p(t)$ as

$$H_p(t) = 1 + (\chi(p) + 3)t + t^2.$$  

Then $Z(\varphi)$ can be rewritten as

$$Z(\varphi) = 2^4 \prod_{p \neq 2} H_p(p^{-1} \frac{1 - 1/p^2}{(1 - 1/p)^2}).$$

Since for $p = 4k + 1$ the absolute values of the roots of the polynomial $H_p(t)$ are not equal to 1, by a result of N. Kurokawa [22], the Euler product $Z(\varphi)$ does not have a meromorphic extension to the whole complex plane $C$. In particular, $Z(\varphi)$ does not satisfy any functional equation.

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This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.