RATIONAL POINTS ON COMPACTIFICATIONS OF SEMI-SIMPLE GROUPS

by

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ABSTRACT. — We prove Manin's conjecture concerning the distribution of rational points of bounded height, and its refinement by Peyre, for wonderful compactifications of semi-simple algebraic groups over number fields. The proof proceeds via the study of the associated height zeta function and its spectral expansion.

Contents

Introduction

Let F be a number field and G a connected linear algebraic group over F. We are interested in distribution properties of rational points on smooth projective equivariant compactifications X of G with respect to heights. More precisely, we assume that X contains G as a Zariski open subset and that the left- and right action of G on itself extends to X. Equivalently, X is an equivariant compactification of $\mathsf{G} \times \mathsf{G}/\mathsf{G}$. A split over F semi-simple group G of adjoint type has a canonical (wonderful) compactification, constructed over an algebraically closed field in [15] and over arbitrary fields (and over $\mathbb{Z}$) in [39] and [16]. In this paper we consider wonderful compactifications of forms of such groups. One of our main results is a proof of Manin's conjecture for this class of varieties:

THEOREM 1. —   *Let X be the wonderful compactification of a semi-simple group G over F of adjoint type and $\mathcal{L} = (L, \|\cdot\|_v)$ an adelically metrized line bundle such that its class $[L]$ in the Picard group $\mathrm{Pic}(X)$ of X is contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$. Then*

$$(0.1) \qquad \mathcal{Z}(\mathcal{L}, s) := \sum_{x \in \mathsf{G}(F)} H_{\mathcal{L}}(x)^{-s} = \frac{c(\mathcal{L})}{(s - a(L))^{b(L)}} + \frac{h(s)}{(s - a(L))^{b(L)-1}}$$

and, consequently,

$$\mathcal{N}(\mathcal{L}, B) := \#\{x \in \mathsf{G}(F) \,|\, H_{\mathcal{L}}(x) \leq B\} \sim \frac{c(\mathcal{L})}{a(L)(b(L)-1)!} B^{a(L)} \log(B)^{b(L)-1}$$

as $B \to \infty$. Here

- $a(L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$ *(where K_X is the canonical line bundle of X);*
- $b(L)$ *is the (maximal) codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(L)[L] + [K_X]$;*
- $c(\mathcal{L}) \in \mathbb{R}_{>0}$ *and*
- $h(s)$ *is a holomorphic function (for $\Re(s) > a(L) - \epsilon$, some $\epsilon > 0$).*

Moreover, $c(-\mathcal{K}_X)$ is the constant defined in [29].

The proof is based on the study of analytic properties of the height zeta function (0.1). The zeta function is first realized as a special value of an automorphic form on the group $\mathsf{G}(\mathbb{A})$. Then we write down the automorphic Fourier expansion in terms of a basis of automorphic forms. In general, we also have a contribution from the continuous spectrum. The Fourier expansion gives an identity of continuous functions which we use to prove a meromorphic extension of the height zeta function.

This paper is part of a program, initiated in [18] and developed in [4], [29], [7], to relate asymptotics of points of bounded height to geometric and arithmetic invariants of X. For further results and motivating examples we refer to the book [30] and the

papers [6], [37], [14] and [31]. The recent preprint [20] gives an independent proof of Theorem 1, based on adelic mixing.

The paper is organized as follows. In Part One we collect various results on algebraic groups and automorphic forms. Section 2 contains a theorem about transfer of one-dimensional automorphic representations among inner forms of a group. Section 3.1 is concerned with Eisenstein series and automorphic Fourier expansions. The key results are Lemma 3.1 and Proposition 3.5. In Section 4.1 we recall a theorem of Oh (Theorem 4.2) which we use for estimates in the spectral expansion. In Part Two we apply this theory to Manin's conjecture. In Section 5 we review the construction of wonderful compactifications and their geometric properties. In Section 6 we study height functions and their integrals in the local situation. The problem of regularizing global height integrals is considered in Section 7. In Section 8 we establish analytic properties of height zeta functions in several complex variables. Tauberian theorems imply asymptotics for the number of rational points of bounded height.

Part I. ALGEBRAIC GROUPS AND AUTOMORPHIC FORMS

1. Algebraic groups

1.1. Basic notation. — For a number field F, let $\mathrm{Val}(F)$ be the set of all places and $S_\infty = S_\infty(F)$ the set of archimedean places. For $v \in \mathrm{Val}(F)$, let F_v be the completion of F with respect to v, $\mathcal{O}_F$ (resp. $\mathcal{O}_v$) the ring of integers in F (resp. F_v) and $q = q_v$ the order of the residue field $\mathbf{k} = \mathbf{k}_v$ of $\mathcal{O}_v$. For any finite set of places S (containing S_∞) we denote by $\mathcal{O}_S$ the ring of S-integers of F. We denote by $\mathbb{A}$ the ring of adeles, by $\mathbb{A}_S = \prod'_{v \notin S} F_v$ and by $\mathbb{A}_f = \mathbb{A}_{S_\infty}$. Let Γ_F and Γ_v the Galois groups of $\bar{F}/F$ and $\bar{F}_v/F_v$, respectively. For L/F a Galois extension, with $w \in \mathrm{Val}(L)$ and $v \in \mathrm{Val}(F)$ satisfying $w|v$, set $\Gamma_{w/v} = \mathrm{Gal}(L_w/F_v)$.

For $X = X_F$ an algebraic variety over F and L/F an extension write $X_L = X \times_F L$ for the base-change to L and $X(F)$ for the set of F-rational points of X. For $v \in \mathrm{Val}(F)$, we abbreviate $X_v = X_{F_v}$. We typically identify $X_v(F_v)$ and $X(F_v)$.

We denote by $\mathrm{Pic}(X)$ the Picard group of X and by $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_\mathbb{R}$ the (closed) cone of effective divisors on X. We often identify line bundles, divisors and their classes in $\mathrm{Pic}(X)$. If X has an action by an algebraic group H we write $\mathrm{Pic}^{\mathsf{H}}(X)$ for the group of isomorphism classes of H-linearized line bundles on X.

1.2. Setup. — Throughout this paper, G is a fixed connected semi-simple group of adjoint type defined over a number field F and G' a quasi-split group over F of

which G is an inner form. The groups G and G' have a unique split F-form G^{sp}. We may assume that G^{sp} is obtained by base-change from a split group, again denoted G^{sp}, over $\mathbb{Q}$. We fix a Galois extension E/F such that $\mathsf{G}(E) = \mathsf{G}'(E) = \mathsf{G}^{sp}(E)$. Its Galois group is denoted by $\Gamma = \mathrm{Gal}(E/F)$. We denote the simply-connected cover of G by G^{sc}. Let T be a maximal torus of G viewed as an algebraic group over F. For each $v \in \mathrm{Val}(F)$, we denote by S_v the maximal F_v-split subtorus of T considered as an algebraic group over F_v. The corresponding objects for G' and G^{sp} are written T', S'_v, and T^{sp}, respectively; for G^{sp}, we make the choice in such a way that T^{sp} is split. We choose maximal tori $\mathsf{T}, \mathsf{T}', \mathsf{T}^{sp}$ in such a way that they have the same set of rational points over E. In particular, T' is split over E. We let S' be the maximal split torus in G' contained in T'. We choose a finite set $S_E \subset \mathrm{Val}(E)$ containing all the archimedean places so that if $w \notin S_E$ is non-archimedean then

- the extension E_w/F_v is unramified for all $v \in \mathrm{Val}(F)$ with $w|v$;
- if $v \in \mathrm{Val}(F)$ is not divisible by any $w \in S_E$, then G is quasi-split over F_v and $\mathsf{G}(F_v) = \mathsf{G}'(F_v)$.

We will denote by S_F the collection of all places of F which are divisible by some $w \in S_E$. Later, we will need to enlarge S_E, and S_F, to satisfy extra assumptions; c.f. 1.4. We have a *standard* and a *twisted* action of $\Gamma = \mathrm{Gal}(E/F)$ on $\mathsf{G}'(E)$, denoted by

$$g \mapsto \sigma(g), \quad \text{resp.} \quad g \mapsto \tilde{\sigma}(g).$$

Here

$$\tilde{\sigma} = c(\sigma) \cdot \sigma,$$

where $c \in Z^1(\Gamma, \mathrm{Inn}(\mathsf{G}')(E))$ is a cocycle with values in the group of inner automorphisms of G'.

Pick a Borel subgroup B^{sp} in G^{sp} which contains T^{sp}. As G' is quasi-split it contains a Borel subgroup B', containing T', which is defined over F. If $v \notin S_F$, then by transfer of structure we obtain a Borel subgroup $\mathsf{B}(F_v)$ in $\mathsf{G}(F_v)$. If $v \in S_F$, we pick a minimal parabolic F_v-subgroup $\mathsf{B}(F_v)$ of $\mathsf{G}(F_v)$ containing $\mathsf{S}(F_v)$. For each place v, denote by $\tilde{\Phi}(\mathsf{G}(F_v), \mathsf{S}(F_v))$ the set of roots of $\mathsf{S}(F_v)$ in $\mathsf{G}(F_v)$ and by $\Phi(\mathsf{G}(F_v), \mathsf{S}(F_v))$ the set of non-multipliable roots in $\tilde{\Phi}$ with the ordering given by $\mathsf{B}(F_v)$. Let $\mathfrak{X}^*(\mathsf{S}(F_v))$ denote the set of characters of $\mathsf{S}(F_v)$ defined over F_v. Denote by $\mathfrak{X}^+$ (resp. Φ^+) the set of positive characters (resp. roots) in $\mathfrak{X}^*(\mathsf{S}(F_v))$ (resp. $\Phi(\mathsf{G}(F_v), \mathsf{S}(F_v))$) with respect to that ordering. We also let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots of for the pair $(\mathsf{G}^{sp}, \mathsf{T}^{sp})$ with respect to the ordering introduced by B^{sp}, and let ρ be half the sum of the positive roots for the same pair. For any subset $I \subset [1, \ldots, r]$ we denote by P^{sp}_I the corresponding parabolic subgroup of G^{sp} and by L^{sp}_I its Levi subgroup. Let W be the Weyl group of the pair $(\mathsf{G}^{sp}, \mathsf{T}^{sp})$. Denote by $\{\mathsf{s}_1, ..., \mathsf{s}_r\}$ the set of simple reflections, by ℓ the length function on W and by $\mathsf{w}_0 \in \mathsf{W}$ the longest element with respect to ℓ. We let $\{\omega_1, \ldots, \omega_r\}$ be the collection of the fundamental weights of the simply-connected cover of G^{sp}. We will also occasionally speak of the root system of the pair $(\mathsf{G}', \mathsf{T}')$, the ordering given by the choice of B', and the collection of simple roots $\Delta(\mathsf{G}', \mathsf{T}')$; the collection of Γ_F-orbits in $\Delta(\mathsf{G}', \mathsf{T}')$ is the same as $\Delta(\mathsf{G}', \mathsf{S}')$.

If v is archimedean, we set

$$(1.1) \qquad F_v^0 = \{x \in \mathbb{R} \mid x \geq 0\}$$

and

$$(1.2) \qquad \hat{F}_v = \{x \in \mathbb{R} \mid x \geq 1\}.$$

If v is non-archimedean, we fix a uniformizer ϖ of F_v, and set

$$(1.3) \qquad F_v^0 = \{\varpi^n \mid n \in \mathbb{Z}\}$$

and

$$(1.4) \qquad \hat{F}_v = \{\varpi^{-n} \mid n \in \mathbb{N}\}.$$

We set

$$(1.5) \qquad \mathsf{S}_v(F_v)^0 = \{a \in \mathsf{S}_v(F_v) \mid \alpha(a) \in F_v^0 \text{ for each } \alpha \in \mathfrak{X}^*(\mathsf{S}(F_v))\}$$

and

$$(1.6) \qquad \mathsf{S}_v(F_v)^+ = \{a \in \mathsf{S}(F_v) \mid \alpha(a) \in \hat{F}_v \text{ for each } \alpha \in \Phi^+\}.$$

Similarly we may define $\mathsf{S}'_v(F_v)^+$. We define constants κ_α for $\alpha \in \Delta(\mathsf{G}^{sp}, \mathsf{T}^{sp})$ by:

$$(1.7) \qquad \sum_{\alpha > 0, \alpha \in \Phi(\mathsf{G},\mathsf{T})} \alpha = \sum_{\alpha \in \Delta(\mathsf{G},\mathsf{T})} \kappa_\alpha \alpha.$$

1.3. Integrality. — By assumption, G and G' are of adjoint type and we can identify $\mathrm{Inn}(\mathsf{G}') = \mathsf{G}'$. Then, for $g \in \mathsf{G}'(E)$, we have

$$c(\sigma) \cdot g = a_\sigma g a_\sigma^{-1}$$

for a uniquely defined element $a_\sigma \in \mathsf{G}'(E)$. For $w \in \mathrm{Val}(E)$ and $v \in \mathrm{Val}(F)$ with $w|v$, we have a restriction homomorphism

$$\iota_w : \quad Z^1(\Gamma, \mathsf{G}'(E)) \quad \rightarrow \quad Z^1(\Gamma_{w/v}, \mathsf{G}'(E_w)),$$
$$c \qquad \mapsto \qquad c_w.$$

The group $\mathsf{G}_v = \mathsf{G}_{F_v}$ is obtained from the corresponding quasi-split group G'_v by twisting with c_w. With our choice of the set S_E from 1.1, the local cocycle c_w splits for $w \notin S_E$. Thus for $w \notin S_E$, $\sigma \in \Gamma_w$, we have

$$c_w(\sigma) = a_w \cdot \sigma(a_w^{-1}) \,,$$

for some element $a_w \in \mathsf{G}'(E_w)$. Identifying $\mathsf{G}'(E) = \mathsf{G}(E)$ we may regard

- $\mathsf{G}'(F)$ as the Γ-fixed points of $\mathsf{G}'(E)$ for the standard action and
- $\mathsf{G}(F)$ as the Γ-fixed points of $\mathsf{G}'(E)$ for the twisted action.

Locally, we have a similar situation:

$$
\begin{array}{ccc}
\mathsf{G}(E_w) & =\!=\!=\!= & \mathsf{G}'(E_w) \\
\uparrow & & \uparrow \\
\mathsf{G}(F_v) & & \mathsf{G}'(F_v)
\end{array}
$$

and further

$$a_w^{-1}\mathsf{G}(F_v)a_w = \mathsf{G}'(F_v) \ .$$

Recall that the global group G' splits over E and put

$$\mathfrak{g}' = \mathrm{Lie}(\mathsf{G}') \quad \text{and} \quad \mathfrak{g}'_E = \mathfrak{g}'(E) \ .$$

Let $\mathsf{T}' \subset \mathsf{G}'$ be a maximal split torus defined over E. Let

$$\mathfrak{g}^{sp} = \mathrm{Lie}(\mathsf{G}^{sp}), \quad \mathfrak{g}^{sp}_{\mathbb{Q}} = \mathfrak{g}^{sp}(\mathbb{Q}) \ .$$

Let $\Lambda_{\mathbb{Q}}$ be a Chevalley lattice in $\mathfrak{g}^{sp}_{\mathbb{Q}}$ adapted to T^{sp}. Set

$$\Lambda'_E := \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathcal{O}_E$$

and also set

$$\Lambda'_w := \Lambda'_E \otimes_{\mathfrak{o}_E} \mathcal{O}_w.$$

Then Λ'_w is a Chevalley lattice in $\mathfrak{g}'_w = \mathfrak{g}'(E_w)$. Moreover, Λ'_w is adapted to the torus T'_w obtained from T' by base extension to E_w. Let K_w be the stabilizer of Λ_w in $\mathsf{G}(E_w) = \mathsf{G}'(E_w)$. We have (see [**22**], Proposition 2.33):

$$\mathsf{G}_w = \mathsf{K}_w \cdot \mathsf{B}(E_w) = \mathsf{B}(E_w) \cdot \mathsf{K}_w \ .$$

PROPOSITION 1.1. — *Let v be a finite place of F and E_w an unramified extension of F_v. An element $a_w \in \mathsf{G}(E_w) = \mathsf{G}'(E_w)$ satisfies*

$$a_w \sigma(a_w)^{-1} \in \mathsf{K}_w$$

if and only if it is of the form

$$a_w = k_w \cdot \gamma_w,$$

with $k_w \in \mathsf{K}_w$ and $\gamma_w \in \mathsf{G}'(F_v)$.

We first recall the following

THEOREM 1.2 (see [**32**], p. 292). — *Let G_v be a connected semi-simple group over F_v. Suppose that G_v is defined over $\mathfrak{o}_v$ and that as a group scheme over $\mathfrak{o}_v$, it has a connected smooth reduction modulo $\mathfrak{p}_v$. Let E_w be a finite unramified extension of F_v. Then*

$$H^1(\Gamma_w, \mathsf{G}(\mathcal{O}_w)) = 1.$$

Proof of Proposition 1.1. — Consider the exact sequence of (pointed) sets

$$1 \to \mathsf{G}'_v(\mathcal{O}_w) \to \mathsf{G}'_v(E_w) \to \mathsf{G}'_v(\mathcal{O}_w)\backslash\mathsf{G}'_v(E_w) \to 1$$

and the corresponding exact sequence in cohomology

$$1 \to \mathsf{G}'_v(\mathcal{O}_v) \to \mathsf{G}'_v(F_v) \to (\mathsf{G}'_v(\mathcal{O}_w)\backslash\mathsf{G}'_v(E_w))^{\Gamma_w} \to H^1(\Gamma, \mathsf{G}'_v(\mathcal{O}_w)) = 1,$$

by Theorem 1.2. Consequently,

$$\mathsf{G}'_v(\mathcal{O}_v)\backslash\mathsf{G}'_v(F_v) = (\mathsf{G}'_v(\mathcal{O}_w)\backslash\mathsf{G}'_v(E_w))^{\Gamma_w} \ \ ,$$

precisely the statement of Proposition 1.1.   □

1.4. Good primes. — To apply Proposition 1.1 we will enlarge S_E, and S_F, so that for all $v \notin S_F$ one has:

- G' has a group-scheme structure via the Chevalley lattice Λ_F;
- G' has a connected smooth reduction modulo $\mathfrak{p}_v$;
- for all $\sigma \in \Gamma$
$$c(\sigma) \in \mathsf{G}'(\mathcal{O}_w) = \mathsf{K}'_w \ ;$$
- the local cocycle
$$c_w \in Z^1(\Gamma_w, \mathsf{G}'(E_w))$$
splits.

Under these assumptions, for $\sigma \in \Gamma_w$ we may write

$$c_w(\sigma) = a_w \cdot \sigma(a_w^{-1})$$

for some $a_w \in \mathsf{G}'(E_w)$. Then, for $w \notin S$,

$$a_w \cdot \sigma(a_w^{-1}) \in \mathsf{K}'_w \ .$$

By Proposition 1.1, we have $a_w = k_w \cdot \gamma_w$, for some $k_w \in \mathsf{K}'_w, \gamma_w \in \mathsf{G}'(F_v)$. Since $\sigma(\gamma_w) = \gamma_w$ for $\sigma \in \Gamma_w$, we may assume that

$$c_w(\sigma) = a_w \cdot \sigma(a_w^{-1})$$

with $a_w \in \mathsf{K}'_w$. Our main conclusion is

COROLLARY 1.3. — *There is a unique element $a_w \in \mathsf{K}'_w$ such that*

$$a_w^{-1}\mathsf{G}(F_v)a_w = \mathsf{G}'(F_v).$$

Proof. — We only need to prove uniqueness. Suppose that we have another element $b \in \mathsf{G}(E_w)$ such that

$$b^{-1}\mathsf{G}(F_v)b = \mathsf{G}'(F_v).$$

Let $z = ba^{-1}$. Then

(1.8) $$z\mathsf{G}(F_v)z^{-1} = \mathsf{G}(F_v).$$

Next, let $\sigma \in \mathrm{Gal}(E_w/F_v)$. Then from (1.8) we have

$$\tilde{\sigma}(zgz^{-1}) = zgz^{-1}$$

for all $g \in \mathsf{G}(F_v)$. Hence, $z^{-1}\sigma(z)$ belongs to the centralizer of $\mathsf{G}(F_v)$ in $\mathsf{G}(\bar{F}_v)$. But then by a theorem of Rosenlicht, $\mathsf{G}(F_v)$ is Zariski dense in $\mathsf{G}(\bar{F}_v)$ (see [8], p. 11). Thus $z \in \mathsf{Z}(\bar{F}_v)$ (the center of $\mathsf{G}(\bar{F}_v)$). Since G is an adjoint group, we have then $z = 1$ and $a = b$. $\square$

We set $\mathsf{K}'_v = \mathsf{K}'_w \cap \mathsf{G}'(F_v)$. For $v \notin S_F$, this is a maximal compact subgroup in $\mathsf{G}'(F_v)$. Put $\mathsf{K}_v = a_w \mathsf{K}'_v a_w^{-1}$. Then

COROLLARY 1.4 (Cartan Decomposition). — *For $v \notin S_F$, we have*

$$(1.9) \qquad\qquad \mathsf{G}(F_v) = \mathsf{K}_v \mathsf{S}(F_v)^+ \mathsf{K}_v.$$

REMARK 1.5. — If $v \in S_F$, then there is a maximal compact subgroup K_v of $\mathsf{G}(F_v)$ and a finite set $\Omega_v \subset \mathsf{G}(F_v)$ such that

$$(1.10) \qquad\qquad \mathsf{G}(F_v) = \mathsf{K}_v \mathsf{S}(F_v)^+ \Omega_v \mathsf{K}_v.$$

The set Ω_v is trivial if v is archimedean, or $\mathsf{G}(F_v)$ is split, or $\mathsf{G}(F_v)$ quasi-split and split over an unramified extension.

1.5. Galois cohomology of simply-connected groups. — Here we recall basics facts about simply-connected semi-simple groups over the adeles.

THEOREM 1.6 (Kneser). — *Let H be a simply connected semi-simple group over F. Then, for $v \nmid \infty$,*

$$H^1(\Gamma_v, \mathsf{H}(\bar{F}_v)) = 1$$

and the natural map

$$H^1(\Gamma, \mathsf{H}^{sc}(F)) \to \prod_{v|\infty} H^1(\Gamma_v, H(\bar{F}_v))$$

is a bijection.

It is useful to keep in mind the following:

LEMMA 1.7 (Theorem 6.17 of [32]). — *Let U be a connected compact algebraic group over $\mathbb{R}$. Then*

$$H^1(\Gamma_\mathbb{R}, \mathsf{U}(\mathbb{C})) = \mathsf{A}'/\mathsf{W}$$

where

$$\mathsf{A}' := \{t \in \mathsf{A} \,|\, t^2 = 1\},$$

$\mathsf{A} \subset \mathsf{U}$ *is the maximal $\mathbb{R}$-torus and* $\mathsf{W} = N_\mathsf{U}(\mathsf{A})/\mathsf{A}$ *is the corresponding Weyl group. Here, $N_\mathsf{U}(\mathsf{A})$ is the normalizer of A in U.*

1.6. Norm maps. — Let T be an algebraic torus over F, splitting over a normal extension E/F with Galois group Γ, v a place of F, $w \in \mathrm{Val}(E)$ a place over v, and $\Gamma_{w/v}$ the corresponding local Galois group. We have the following natural *norm* homomorphisms:

$$
\begin{array}{ccccccc}
\mathsf{T}(E_w) & \xrightarrow{\mathrm{N}_{w/v}} & \mathsf{T}(F_v), & & \mathsf{T}(E_w) & \xrightarrow{\mathrm{N}_v} & \mathsf{T}(F_v) \\
t_w & \mapsto & \prod_{\sigma \in \Gamma_{w/v}} \sigma(t_w), & & t_w & \mapsto & \prod_{w|v} \mathrm{N}_{w/v}(t_w), \\
\mathsf{T}(E) & \xrightarrow{\mathrm{N}_{E/F}} & \mathsf{T}(F), & & \mathsf{T}(\mathbb{A}_E) & \xrightarrow{\mathrm{N}_{\mathbb{A}_E/\mathbb{A}_F}} & \mathsf{T}(\mathbb{A}_F) \\
t & \mapsto & \prod_{\sigma \in \Gamma} \sigma(t), & & (t_w)_w & \mapsto & (\mathrm{N}_v(t_w))_v.
\end{array}
$$

Lemma 1.8. — *For $t \in \mathsf{T}(E)$ one has $\mathrm{N}_{E/F}(t) = \mathrm{N}_{\mathbb{A}_E/\mathbb{A}_F}(t)$.*

Proof. — Let w be a place of E over v and $\Gamma_w \subset \Gamma$ the fixer of w. For any $\sigma \in \Gamma$ we have $\sigma \Gamma_w \sigma^{-1} = \Gamma_{\sigma(w)}$ and $|x|_w = |\sigma(x)|_{\sigma(w)}$. For a fixed w write

$$
\Gamma = \cup_{j=1}^g \sigma_j \Gamma_{w_1} = \cup_{j=1}^g \Gamma_{w_1} \sigma_j^{-1}.
$$

We have, for $t \in \mathsf{T}(E)$,

$$
\mathrm{N}_{E/F}(t) = \prod_{j=1}^g \prod_{\rho \in \Gamma_{w_1}} \rho \sigma_j^{-1}(t)
$$

$$
= \prod_{j=1}^g \sigma_j^{-1} \prod_{\rho \in \Gamma_{w_1}} \sigma_j \rho \sigma_j^{-1}(t)
$$

$$
= \prod_{j=1}^g \sigma_j^{-1} \prod_{\rho \in \Gamma_{w_j}} \rho(t).
$$

We have the commutative diagram

$$
\begin{array}{ccccccc}
E & \xrightarrow{\iota_w} & E_w & \longleftarrow & F_v & \xleftarrow{\iota_v} & F \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle id} & & \\
E & \xrightarrow{\iota_{\sigma(w)}} & E_{\sigma(w)} & \longleftarrow & F_v & &
\end{array}
$$

where ι_v, ι_w are the canonical maps and σ^* is the continuous extension of σ.

For $t_w := \prod_{\rho \in \Gamma_w} \rho(t)$ we have

$$
\iota_v(\mathrm{N}_{E/F}(t)) = \iota_{w_1}(\mathrm{N}_{E/F}(t))
$$

$$
= \prod_{j=1}^g \iota_{w_1} \sigma_j^{-1}(a_j)
$$

$$
= \prod_{j=1}^g (\sigma_j^*)^{-1} \cdot \iota_{w_j}(a_j).
$$

Next, let for each w,
$$\Gamma_w^* := \{\rho^* \mid \rho \in \Gamma_w\}.$$
If $\rho \in \Gamma_{w_j}$ then $\rho(a_j) = a_j$. Thus
$$\iota_{w_j}(a_j) = \iota_{w_j}\rho(a_j) = \rho^*(\iota_{w_j}(a_j)).$$
Thus $\iota_{w_j}(a)$ is fixed by Γ_w^* and therefore belongs to F_v. But then
$$(\sigma_j^*)^{-1}(\iota_{w_j}(a_j)) = \iota_{w_j}(a_j).$$
Thus
$$\iota_v(\mathrm{N}_{E/F}(t)) = \prod_{j=1}^g \iota_{w_j}(a_j).$$
On the other hand,
$$\mathrm{N}_{\mathbb{A}_E/\mathbb{A}_F}(t) = \prod_{j=1}^g \mathrm{N}_{E_{w_j}/F_v}(\iota_{w_j}(t))$$
$$= \prod_{j=1}^g \prod_{\rho^* \in \Gamma_{w_j}^*} \rho^* \iota_{w_j}(t)$$
$$= \prod_{j=1}^g \prod_{\rho \in \Gamma_{w_j}} \iota_{w_j}(\rho(t))$$
$$= \prod_{j=1}^g \iota_{w_j}(a_j).$$
This completes the proof. $\square$

2. One-dimensional automorphic representations

In this section we define a transfer associating to each one-dimensional automorphic representation χ of $\mathsf{G}(\mathbb{A})$ a one-dimensional automorphic representation χ' of $\mathsf{G}'(\mathbb{A})$ in such a way that it respects local isomorphisms. The basic idea is to define a local transfer, and then use invariance under $\mathsf{G}(F)$ and weak approximation to extend the definition to the entire $\mathsf{G}(\mathbb{A})$. We will also define certain Euler products that will later be used to regularize height integrals.

2.1. Automorphic characters for simply-connected groups. —

PROPOSITION 2.1. — *Let F be a number field and H a connected simply-connected semi-simple group over F. Let χ be a one-dimensional automorphic representation of $\mathsf{H}(\mathbb{A})$. Then $\chi = 1$.*

Proof. — Let E/F be a finite Galois extension splitting H. There are infinitely many $v \in \mathrm{Val}(F)$ which split completely in E. For such v the local group $\mathsf{H}(F_v)$ is split. Since H is simply-connected, $\mathsf{H}(F_v)$, as an abstract group, is its own derived group (see [**9**], esp. 3.3.5, p. A-16). Thus $\chi|_{\mathsf{H}(F_v)} = 1$. Also, $\chi|_{\mathsf{H}(F)} = 1$. By strong approximation (see Theorem 7.12, p. 427 of [**32**]), $\mathsf{H}(F_v)\mathsf{H}(F)$ is dense in $\mathsf{H}(\mathbb{A})$. It follows that $\chi = 1$. $\qquad\square$

2.2. The local transfer. — Let G be as in 1.2 and

$$j_v \;:\; \mathsf{G}^{sc}(F_v) \to \mathsf{G}(F_v)$$

be the canonical homomorphism from its simply-connected covering. Let v be a finite place of F and $\hat{\mathsf{G}}_v$ the group of characters χ_v of $\mathsf{G}(F_v)$ which are trivial on $j_v(\mathsf{G}^{sc}(F_v))$.

By Kneser's theorem 1.6, $H^1(F_v, \mathsf{G}^{sc}(\bar{F}_v)) = 1$. We have an exact sequence

$$1 \to \mathsf{Z}(F_v) \to \mathsf{G}^{sc}(F_v) \xrightarrow{j_v} \mathsf{G}(F_v) \xrightarrow{\delta_v} H^1(F_v, \mathsf{Z}(\bar{F}_v)) \to 1$$

and a similar sequence for G'. Here Z is the center of G^{sc}. We may identify

$$(2.1) \qquad\qquad \hat{\mathsf{G}}_v = H^1(F_v, \mathsf{Z}(\bar{F}_v))^\vee,$$

(the character group of $H^1(F_v, \mathsf{Z}(\bar{F}_v))$. Recall that G is obtained from G' by replacing the standard Galois action by a twisted action, via some representative of $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Inn}(\mathsf{G}'(\bar{F})))$. Since inner automorphisms of $\mathsf{G}'(\bar{F})$ fix $\mathsf{Z}(\bar{F})$, these two actions coincide on $\mathsf{Z}(F)$, and similarly for $\mathsf{Z}(\bar{F}_v)$. By (2.1) we get then, for non-archimedean v, a natural isomorphism

$$(2.2) \qquad\qquad \mathrm{tr}_v \;:\; \hat{\mathsf{G}}_v \to \hat{\mathsf{G}}'_v.$$

REMARK 2.2. — We observe that equation (2.1) implies that there is a number n, independent of v, such that every occurring character of $\mathsf{G}(F_v)$ satisfies $\chi^n = 1$. This follows from a case-by-case analysis of $H^1(F_v, Z(\bar{F}_v))$ as carried out in [**32**], Section 6.5. We will come back to this point in 8.8.

2.3. Weak approximation. — We now apply weak approximation (Theorem 7.7, p. 415 of [**32**]):

THEOREM 2.3. — *The group* $\mathsf{G}(F)$ *is dense in* $\mathsf{G}_\infty := \prod_{v|\infty} \mathsf{G}(F_v)$.

For $v|\infty$, the map $j_v \;:\; \mathsf{G}^{sc}(F_v) \to \mathsf{G}(F_v)$ is submersive. Write $\mathsf{G}^*_\infty \subset \mathsf{G}_\infty$ for the image of $\prod_{v|\infty} \mathsf{G}^{sc}(F_v)$ under $\prod_{v|\infty} j_v$, it is an open subgroup. Thus $pr_\infty(\mathsf{G}(F)) \cdot \mathsf{G}^*_\infty = \mathsf{G}_\infty$ and

$$\mathsf{G}(\mathbb{A}) = \mathsf{G}(F) \cdot \mathsf{G}_0 \cdot \mathsf{G}^*_\infty.$$

Here pr_∞ is the product of the projection maps $pr_v : \mathsf{G}(F) \to \mathsf{G}(F_v)$ for $v|\infty$.

2.4. Global transfer. — Let $\chi = \prod_v \chi_v$ be a one-dimensional automorphic representation of $\mathsf{G}(\mathbb{A})$ such that χ_v is trivial on $j_v(\mathsf{G}^{sc}(F_v))$, for all $v \in \mathrm{Val}(F)$. In particular, $\chi_\infty := \prod_{v|\infty} \chi_v$ is trivial on G^*_∞. Using (2.2) we define

$$\chi'_0 = \prod_{v \nmid \infty} \mathrm{tr}_v(\chi_v).$$

We also have

$$\mathsf{G}'(\mathbb{A}) = \mathsf{G}'(F) \cdot \mathsf{G}'_0 \cdot (\mathsf{G}'_\infty)^*.$$

We extend χ'_0 to a character χ' of $\mathsf{G}'(\mathbb{A})$ by setting $\chi' = 1$ on $\mathsf{G}'(F) \cdot (\mathsf{G}'_\infty)^*$. Then χ' is well-defined since it is trivial on the intersection

$$\mathsf{G}'_0 \cap \mathsf{G}'(F) \cdot (\mathsf{G}'_\infty)^*.$$

Indeed, write $\mathsf{G}(F)^* := \{\gamma \in \mathsf{G}(F) \mid pr_\infty(\gamma) \in \mathsf{G}^*_\infty\}$, and similarly for G'. Note that

$$pr_0(\mathsf{G}(F)^*) = \mathsf{G}(\mathbb{A}_f) \cap \mathsf{G}(F) \cdot \mathsf{G}^*_\infty,$$

and similarly for G'. It suffices to check that

$$(2.3) \qquad\qquad \delta_0(pr_0(\mathsf{G}(F)^*) = \delta'_0(pr_0(\mathsf{G}'(F)^*)),$$

where $\delta_0 := \prod_{v \nmid \infty} \delta_v$ and similarly, $\delta'_0 = \prod_{v \nmid \infty} \delta'_v$. In fact, the character χ is trivial on $\mathsf{G}(F) \cdot \mathsf{G}^*_\infty$.

Next, the sequence

$$(2.4) \qquad\qquad \mathsf{G}^{sc}(F_v) \xrightarrow{j_v} \mathsf{G}(F_v) \xrightarrow{\delta_v} H^1(F_v, \mathsf{Z}(\bar{F}_v))$$

is exact for all v. Set $\delta_\infty = \prod_{v|\infty} \delta_v$. Then in particular, $\delta_\infty(\mathsf{G}^*_\infty) = \{e\}$. We also set $\delta_\mathbb{A} = \delta_0 \times \delta_\infty = \prod_v \delta_v$. Then (2.3) is equivalent to

$$(2.5) \qquad\qquad \delta_\mathbb{A}(\mathsf{G}(F)^*) = \delta'_\mathbb{A}(\mathsf{G}'(F)^*).$$

Let p_v be the canonical map

$$p_v \,:\, H^1(F, \mathsf{Z}(\bar{F})) \to H^1(F_v, \mathsf{Z}(\bar{F}_v))$$

and

$$p \,:\, H^1(F, \mathsf{Z}(\bar{F})) \to \prod_v H^1(F_v, \mathsf{Z}(\bar{F}_v))$$

the corresponding global map. We have $p \circ \delta_F = \delta_\mathbb{A}$, where δ_F is the coboundary map $\mathsf{G}(F) \to H^1(F, \mathsf{Z}(\bar{F}))$.

PROPOSITION 2.4. — *Let $c \in H^1(F, \mathsf{Z}(\bar{F}))$. Then $c \in \delta_F(\mathsf{G}(F)^*)$ if and only if*

$$p_v(c) = c_v = 0 \quad \text{for all} \quad v \mid \infty.$$

Proof. — We have a diagram

$$\begin{array}{ccccc}
\mathsf{G}(F) & \xrightarrow{\;\delta_F\;} & H^1(F, \mathsf{Z}(\bar{F})) & \xrightarrow{\;k_F\;} & H^1(F, \mathsf{G}^{sc}(\bar{F})) \\
\downarrow & & z \downarrow & & \downarrow g \\
\mathsf{G}(\mathbb{A}) & \xrightarrow{\;\delta_{\mathbb{A}}\;} & \prod_v H^1(F_v, \mathsf{Z}(\bar{F}_v)) & \xrightarrow{\;k_{\mathbb{A}}\;} & \prod_{v \nmid \infty} H^1(F_v, \mathsf{G}^{sc}(\bar{F}_v)).
\end{array}$$

where g is bijective (see Th. 6.6, p. 286 of [32]), both squares are commutative and the top row is exact. Recall also that $H^1(F_v, \mathsf{G}^{sc}(\bar{F}_v)) = \{e\}$, if v is finite.

Suppose that $c_v = z_v(c) = 0$ for v infinite, $c \in H^1(F, \mathsf{Z}(\bar{F}))$. Then $k_{\mathbb{A}}((c_v)_v) = e$. Consequently, $k_{\mathbb{A}} \cdot z(c) = g \cdot k_F(c) = e$. Since g is bijective, we have $k_F(c) = 0$ and consequently

$$c = \delta_F(\gamma)$$

for some $\gamma \in \mathsf{G}(F)$. We prove that $\gamma \in \mathsf{G}(F)^*$. In fact, we have

$$c_v = \delta_v(\gamma_v) = e, \quad \text{for} \quad v \mid \infty.$$

But then, by the exactness of the sequence (2.4), $\gamma_v \in \mathrm{Im}(j_v)$, for any infinite v. Therefore, $\gamma \in \mathsf{G}_\infty^*$. The converse assertion also follows from the exactness of (2.4). $\qquad\square$

COROLLARY 2.5. — *As before, let*

$$\delta_F \; : \; \mathsf{G}(F) \to H^1(F, \mathsf{Z}(\bar{F}))$$

and similarly δ_F' be the respective co-boundary maps. Then

$$\delta_F(\mathsf{G}(F)^*) = \delta_F'(\mathsf{G}'(F)^*).$$

Proof. — By the Proposition, each of the sets $\delta_F(\mathsf{G}(F)^*)$ and $\delta_F'(\mathsf{G}'(F)^*)$ coincides with the set of $c \in H^1(F, \mathsf{Z}(\bar{F}))$ satisfying $c_v = e$ for $v \mid \infty$. Our assertion is now an immediate consequence. $\qquad\square$

2.5. Compatibility. — The following lemma shows that the above definition of transfer is the correct one:

LEMMA 2.6. — *Let $v \notin S_F$ and $w \in \mathrm{Val}(E)$ be such that $w \mid v$. Let a_w be as in Corollary 1.3. Then for all $x \in \mathsf{G}(F_v)$ one has*

$$\chi_v'(a_w^{-1} x a_w) = \chi_v(x).$$

Proof. — In this proof, we write a for a_w. There is a character ξ of the discrete group $H^1(F_v, \mathsf{Z}(\bar{F}_v))$ such that

$$\chi(x) = \xi(\delta_v(x)), \; x \in \mathsf{G}(F_v) \quad \text{and} \quad \chi'(x) = \xi(\delta_v'(x)), \; x \in \mathsf{G}'(F_v).$$

It suffices to prove that $\delta_v'(a^{-1} x a) = \delta_v(x)$. We calculate the two co-boundary maps, δ_v and δ_v'. Choose $\tilde{x} \in \mathsf{G}^{sc}(\bar{F}_v)$ such that $j_v(\tilde{x}) = x$. Then

$$\delta_v(x) = \tilde{x}^{-1}\sigma(\tilde{x}) = \tilde{x}^{-1}c(\sigma) \circ \sigma'(\tilde{x}) = \tilde{x}^{-1}\tilde{a}\sigma'(\tilde{a})^{-1}\sigma'(\tilde{x})\sigma'(\tilde{a})\tilde{a}^{-1}.$$

Recall that, for $v \notin S_F$,

$$c_v(\sigma) = a^{-1}\sigma'(a).$$

We calculate $\delta'_v(a^{-1}xa)$, for $x \in \mathsf{G}(F_v)$. Choose $\tilde{a} \in \mathsf{G}^{sc}(\bar{F}_v)$ such that $\mathrm{Ad}(\tilde{a}) = a$. We have then

$$\delta'_v(a^{-1}xa) = (\tilde{a}^1\tilde{x}\tilde{a})^{-1}\sigma'(\tilde{a}^{-1}\tilde{x}\tilde{a}) = \tilde{a}^1\tilde{x}^{-1}\tilde{a}\sigma'(\tilde{a}^{-1})\sigma'(\tilde{x})\sigma'(\tilde{a}).$$

Since $\delta'_v(x) \in \mathsf{Z}(\bar{F}_v)$, we have

$$a\delta'_v(x) = \delta'_v(x)a,$$

or

$$\tilde{a}\delta'_v(x)\tilde{a}^{-1} = \delta'_v(x).$$

Thus

$$\delta'_v(a^{-1}xa) = \tilde{x}^{-1}\tilde{a}\sigma'(\tilde{a}^{-1})\sigma'(\tilde{x}^{-1})\sigma'(\tilde{a})\tilde{a}^{-1} = \delta_v(x),$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.6. Continuity. — Since Z is commutative, $H^1(F_v, \mathsf{Z}(\bar{F}_v))$ is a (discrete) group and the co-boundary map

$$\delta_v : \mathsf{G}(F_v) \to H^1(F_v, \mathsf{Z}(\bar{F}_v))$$

is a group homomorphism.

LEMMA 2.7. — *The map δ_v is continuous.*

Proof. — The map

$$j_v : \mathsf{G}^{sc}(F_v) \to \mathsf{G}(F_v)$$

is a submersion of v-adic Lie groups (the differential of j_v is the identity on the Lie algebra $\mathrm{Lie}(\mathsf{G}^{sc}(F_v) = \mathrm{Lie}(\mathsf{G}(F_v))$. Hence, $\mathrm{Im}(j_v) = \mathrm{Ker}(\delta_v)$ is an open subgroup of $\mathsf{G}(F_v)$ and the claim follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In any case, if $\chi_v : \mathsf{G}(F_v) \to \mathbb{C}^*$ is a homomorphism of groups which is trivial on the image of j_v, then χ_v is automatically continuous.

2.7. Automorphy. — We have associated to each one-dimensional automorphic representation χ of $\mathsf{G}(\mathbb{A})$, trivial on $j(\mathsf{G}^{sc}(\mathbb{A}))$, a "formal" automorphic character

$$\chi' : \mathsf{G}'(\mathbb{A}) \to \mathbb{C}^\times,$$

i.e., a homomorphism, trivial on $\mathsf{G}'(F)$.

THEOREM 2.8. — *The character χ' is an automorphic character.*

Proof. — We need to show that χ' is continuous. Since the map

$$\delta'_v : \mathsf{G}'(F_v) \to H^1(F_v, \mathsf{Z}(\bar{F}_v))$$

is continuous, each χ'_v is also continuous, for v finite. By construction, $\chi'_\infty = \prod_{v|\infty} \chi'_v$ is trivial on the open subgroup $(\mathsf{G}'_\infty)^* \subset \mathsf{G}'_\infty$. Thus, χ'_∞ is continuous. The character χ_v is trivial on K_v, for almost all v. We claim that χ' is trivial on K'_v for almost all v. This immediately follows from Lemma 2.6 and considerations of Section 1.4.

We have already noted that χ'_v is continuous for all v. It follows finally that χ' is continuous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.8. Hecke characters. — We now associate to χ' a function $L(s, \chi')$, a product of Hecke L-functions. This function will later be used to regularize Fourier transforms of global height functions.

Suppose first that $\mathsf{G}'_F = \mathsf{G}^{sp}$ is split. Let $\mathsf{T}^{sp} \subset \mathsf{B}^{sp} \subset \mathsf{G}^{sp}$ be a maximal split torus (over F), and a Borel subgroup containing T. Let $\Delta = \{\alpha\}$ be the associated set of simple roots and $\{\check{\alpha}\}$ the dual basis of the co-characters $\mathfrak{X}_*(\mathsf{T}^{sp})$. For $\lambda \in \mathbb{G}_m(\mathbb{A})$ and $\alpha \in \Delta$ define

$$\xi_\alpha(\lambda) = \chi'(\check{\alpha}(\lambda)).$$

This ξ_α is a character of $\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(F)$ (Hecke-character). We define

$$L(s, \chi') = \prod_{\alpha \in \Delta} L(s, \xi_\alpha).$$

In general, when G' is quasi-split, and not necessarily split over F, we have

$$\Phi(\mathsf{G}', \mathsf{T}') \subset \mathfrak{X}^*_E(\mathsf{T}') = \mathfrak{X}^*(\mathsf{T}') \ \ \text{and} \ \ \Phi(\mathsf{G}', \mathsf{S}') \subset \mathfrak{X}^*_F(\mathsf{T}').$$

Then the restriction map

(2.6) $$r \ : \ \Phi(\mathsf{G}', \mathsf{T}') \to \Phi(\mathsf{G}', \mathsf{S}')$$

is surjective, and

$$r(\Delta(\mathsf{G}', \mathsf{T}')) = \Delta(\mathsf{G}', \mathsf{S}').$$

The Galois group Γ acts transitively on the fibers of

$$r \ : \ \Delta(\mathsf{G}', \mathsf{T}') \to \Delta(\mathsf{G}', \mathsf{S}').$$

Fix a simple root α and let Γ_α be the stabilizer of α in Γ. Let E_α be the fixed field of Γ_α. Since G is adjoint, for each $\alpha \in \Delta(\mathsf{G}', \mathsf{T}')$ we have an associated co-root $\check{\alpha}$ uniquely characterized by

$$(\check{\alpha}, \beta) = \mathbf{1}_{\alpha\beta}$$

where $(\,,\,)$ is the natural Γ-equivariant pairing

$$\mathfrak{X}_*(\mathsf{T}') \times \mathfrak{X}^*(\mathsf{T}') \to \mathbb{Z}$$

(and $\mathbf{1}$ is the delta function).

Since α is defined over $K := E_\alpha$, so is $\check{\alpha}$. We have then a morphism

$$\check{\alpha} \ : \ \mathbb{G}_m \to \mathsf{T}',$$

defined over K and consequently a continuous homomorphism

$$\check{\alpha}_\mathbb{A} \ : \ \mathbb{G}_m(\mathbb{A}_K) \to \mathsf{T}'(\mathbb{A}_K).$$

Let ϕ_α be the composite

$$\phi_\alpha \ : \ \mathbb{G}_m(\mathbb{A}_K) \xrightarrow{\check{\alpha}_\mathbb{A}} \mathsf{T}'(\mathbb{A}_K) \xrightarrow{\mathrm{N}_{\mathbb{A}_K/\mathbb{A}_F}} \mathsf{T}'(\mathbb{A}_F).$$

By Section 1.6,
$$\phi_\alpha \, : \, \mathbb{G}_m(K) \to \mathsf{T}'(F).$$
Thus, if χ is a character of $\mathsf{T}'(\mathbb{A}_F)/\mathsf{T}'(F)$ then $\xi_\alpha = \chi \circ \phi_\alpha$ is an automorphic character of $\mathbb{G}_m$ over K, i.e., a Hecke character. Write $\xi_\alpha = \prod_{w \in \mathrm{Val}(K)} \xi_{\alpha,w}$.

Let v be a place of F, w a place of K lying over v and u a place of E lying over w. Let $F_v \subset K_w \subset E_u$ be the corresponding completions and $\iota_w, \iota_w, \iota_u$ the respective embeddings, with $\iota_u|_K = \iota_w$, $\iota_w|_F = \iota_v$. Write $\mathfrak{X}^*_{E_u}(\mathsf{T}'(F_v)) = \mathfrak{X}^*(\mathsf{T}'(F_v))$ and let
$$\iota^* \, : \, \mathfrak{X}^*(\mathsf{T}') \xrightarrow{\ \sim\ } \mathfrak{X}^*(\mathsf{T}'(F_v))$$
be the isomorphism induced by ι_u. As above, we have a restriction map
$$r_v \, : \, \Phi(\mathsf{G}'(F_v), \mathsf{T}'(F_v)) \to \Phi(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$$
mapping $\Delta(\mathsf{G}'(F_v), \mathsf{T}'(F_v))$ to $\Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$. The local Galois group $\Gamma_{u/v}$ acts transitively on the fibers of
$$r_v \, : \, \Delta(\mathsf{G}'(F_v), \mathsf{T}'(F_v)) \to \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v)).$$

Next suppose that K_w/F_v is unramified. Suppose also that $\xi_{\alpha,w}$ is unramified. Let
$$L_w(s, \xi_{\alpha,w}) = (1 - \xi_{\alpha,w}(\varpi_w)q_w^{-s})^{-1}$$
where ϖ_w is a prime element for K_w. Let
$$\alpha_u = \iota_u^*(\alpha), \ \ \vartheta_v = r_v(\alpha_u) \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v)).$$
Let $\check{\vartheta}_v \in \mathfrak{X}_*(\mathsf{S}'_v(F_v))$ be the associated co-character. Let
$$\ell(\vartheta_v) := \#\{\beta \in \Delta(\mathsf{G}'(F_v), \mathsf{T}'(F_v)) \ \ \text{with} \ \ r_u(\beta) = \vartheta_u\}$$
$$= \#\{\Gamma_{u/v} - \text{orbit of} \ \ \alpha_u \ \ \text{in} \ \ \Delta(\mathsf{G}'(F_v), \mathsf{T}'(F_v))\}$$

PROPOSITION 2.9. — *With the above notations, we have*
$$L_w(s, \xi_{\alpha,w}) = (1 - \chi_v(\check{\vartheta}_v(\varpi)) \cdot q_v^{-\ell(\vartheta_v)s})^{-1}.$$

Proof. — It suffices to prove
- $\ell(\vartheta_u) = [K_w : F_v]$;
- $\xi_{\alpha,w}(\varpi_w) = \chi_v(\check{\vartheta}_u(\varpi_w))$.

let $\Gamma_{u/w} = \mathrm{Gal}(E_u/K_w)$. Then the fixer of α_u in $\Delta(\mathsf{G}'(F_v), \mathsf{T}'(F_v))$ is $\Gamma_{u/w}$. Thus
$$\ell(\vartheta_u) = [\Gamma_{u/v} : \Gamma_{u/w}] = [K_w : F_v].$$
For the the second assertion if suffices to prove that
$$\mathrm{N}_{K_w/F_v}(\check{\alpha}_w(\lambda)) = \check{\vartheta}_u(\lambda), \ \ \lambda \in F_v^\times.$$
The map ι_u induces an isomorphism
$$\iota_u^* \, : \, \mathfrak{X}_*(\mathsf{T}') \xrightarrow{\ \sim\ } \mathfrak{X}_*(\mathsf{T}'(F_v)).$$

Then ι_u^* preserves the natural pairing of roots and co-characters. The co-character $\check{\alpha}_w$ is obtained from $\check{\alpha}$ by base extension to K_w:

$$\check{\alpha} : \mathbb{G}_m/K \to \mathsf{T}'/K$$
$$\check{\alpha}_w : \mathbb{G}_m/K_w \to \mathsf{T}'/K_w$$

We have for $\lambda \in F_v^*$

$$c = \mathrm{N}_{K_w/F_v}(\check{\alpha}_w(\lambda))$$
$$= \prod_{\sigma \in \Gamma_{u/v}} \sigma(\check{\alpha}_w(\sigma^{-1}(\lambda)))$$
$$= \prod_{\sigma \in \Gamma_{u/v}/\Gamma_{u/w}} \sigma \circ (\check{\alpha}_w(\lambda))$$

Here we regard $\check{\alpha}_w \in \mathfrak{X}_*(\mathsf{T}'(F_v))$ and $\sigma \circ$ is the natural action of $\Gamma_{u/v}$ on $\mathfrak{X}_*(\mathsf{T}'(F_v))$. Since c is fixed by $\Gamma_{u/v}$, we have $c \in \mathfrak{X}_*(\mathsf{S}_v(F_v))$. In fact, the morphism $\mathbb{G}_m \xrightarrow{c} \mathsf{T}'(F_v)$ is defined over F_v. Let $\tilde{\mathsf{S}}(F_v)$ be the image of c in $\mathsf{T}'(F_v)$. Then $\tilde{\mathsf{S}}(F_v) \cdot \mathsf{S}_v(F_v) = \mathsf{S}_v(F_v)$, since S'_v is a maximal split F_v-torus in $\mathsf{T}'(F_v)$. Thus $\tilde{\mathsf{S}}(F_v) \subset \mathsf{S}_v$ and we have $\mathbb{G}_m \xrightarrow{c} \mathsf{S}'_v$.

It will suffice to prove that $c = \check{\vartheta}_u$, as elements of $\mathfrak{X}_*(\mathsf{S}'_v(F_v))$:

$$(c, \gamma)_{\mathsf{S}'_v} = (\check{\vartheta}_u, \gamma)_{\mathsf{S}'_v}$$

for all $\gamma \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$. For this it suffices to prove that

   − $(c, \vartheta_u)_{\mathsf{S}'_v} = 1$;
   − $(c, \gamma)_{\mathsf{S}'_v} = 0$ for all $\gamma \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$ with $\gamma \neq \vartheta_u$.

We have

$$(c, \vartheta_v)_{\mathsf{S}'_v} = (c, \alpha_u)_{\mathsf{T}'(F_v)} = \sum_{\sigma \in \Gamma_{u/v}/\Gamma_{u/w}} (\sigma \circ \check{\alpha}_w, \alpha_u)_{\mathsf{T}'(F_v)}.$$

Next we have $\check{\alpha}_w = \check{\alpha}_u$ in $\mathfrak{X}_*(\mathsf{T}'(F_v))$. Here

$$\check{\alpha}_w : \mathbb{G}_m/E_u \to \mathsf{T}'/E_u$$

is obtained by base extension from

$$\check{\alpha} : \mathbb{G}_m/E \to \mathsf{T}'/E,$$

and

$$\alpha_u : \mathsf{T}'/E_u \to \mathbb{G}_m/E_u$$

is obtained by base extension from

$$\alpha : \mathsf{T}'/E \to \mathbb{G}_m/E.$$

Since extending the scalars preserves the natural pairing between roots and co-roots, we have the claim. $\qquad\qquad\square$

2.9. Infinite products. — The next step is to express an infinite product of the form

$$(2.7) \qquad \prod_{v \in \mathrm{Val}(F)} \prod_{\vartheta \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))} (1 - \chi_v(\check{\vartheta}(\varpi_w))q_v^{-\ell_v(\vartheta)s_\vartheta})^{-1}$$

in terms of Hecke L-functions; this will be important in the regularization of height integrals. The above expression, as written, has an infinite number of complex variables. In practice, the number of variables is finite, since the complex numbers s_ϑ are Galois invariant in the following sense.

Consider the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{X}^*(\mathsf{T}') & \xrightarrow{\;\;\iota_v^*\;\;} & \mathfrak{X}^*(\mathsf{T}(F_v)) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_v} \\
\mathfrak{X}^*(\mathsf{S}') & \xrightarrow{\;\;\iota_v^*\;\;} & \mathfrak{X}^*(\mathsf{S}_v(F_v)).
\end{array}
$$

Let $\alpha \in \Delta(\mathsf{G}', \mathsf{T}')$ and let $\mathfrak{O} = \Gamma \cdot \alpha$ be the orbit of α (for the action of Γ on $\Delta(\mathsf{G}', \mathsf{T}')$). We set further $\mathfrak{O}_v = r_v(\iota_u^*(\mathfrak{O})) \subset \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$. For each $\vartheta \in \mathfrak{O}_v$ choose a $\beta \in \Delta(\mathsf{G}', \mathsf{T}')$ so that $r_u(\iota_u^*(\beta)) = \vartheta$. We now require that s_ϑ depend only on the Galois orbit of β in $\Delta(\mathsf{G}', \mathsf{T}')$, i.e., only on $\mathfrak{O}$; we will denote the common value of all such s_ϑ by $s_{\mathfrak{O}}$. With the assumption, the number of genuine complex parameters appearing in (2.7) is the number of distinct Galois orbits $\mathfrak{O}$ in $\Delta(\mathsf{G}', \mathsf{T}')$.

Fix an orbit $\mathfrak{O}$. For any $\beta \in \mathfrak{O}$, we have a field E_β over which β is defined. We have already described how to associate to β a Hecke character $\xi_\beta(\chi)$ of E_β. The Hecke L-function $L(s, \xi_\beta(\chi))$ depends only on the Galois orbit $\mathfrak{O}$, and not on the particular β. For this reason, we denote the L-function $L(s, \xi_\beta(\chi))$ by $L(s, \xi_{\mathfrak{O}}(\chi))$. An argument similar to the proof of Theorem 4.1.3 of [5] leads to the following proposition:

PROPOSITION 2.10. — *We have*

$$(2.8) \qquad \prod_{v \in \mathrm{Val}(F)} \prod_{\vartheta \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))} (1 - \chi_v(\check{\vartheta}(\varpi_w))q_v^{-\ell_v(\vartheta)s_\vartheta}) = \prod_{\mathfrak{O}} L(s_{\mathfrak{O}}, \xi_{\mathfrak{O}}(\chi)),$$

where the latter product is over all Galois orbits in $\Delta(\mathsf{G}', \mathsf{T}')$.

3. Eisenstein series and spectral theory

3.1. Basic spectral theory. — Let B be a minimal parabolic subgroup of G. We will work only with standard parabolic subgroups and up to association. A typical parabolic subgroup is denoted by P. We denote the Levi factor of P by M_{P}, its unipotent radical by N_{P}, and the split component of the center of M_{P} by A_{P}. We let $X(\mathsf{M}_{\mathsf{P}})_{\mathbb{Q}}$ be the group of characters of M_{P} defined over $\mathbb{Q}$, and we set

$\mathfrak{a}_\mathsf{P} = \mathrm{Hom}(X(\mathsf{M}_\mathsf{P})_\mathbb{Q}, \mathbb{R})$. Clearly, $\mathfrak{a}_\mathsf{P}^* = X(\mathsf{M}_\mathsf{P})_\mathbb{Q} \otimes \mathbb{R}$. We denote by Δ_P the set of the simple roots of (P, A). We usually drop the subscript P.

For $m = (m_v)_v \in \mathsf{M}(\mathbb{A})$ define a vector $H_\mathsf{M}(m) \in \mathfrak{a}_\mathsf{P}$ by

$$(3.1) \qquad e^{\langle H_\mathsf{M}(m), \chi \rangle} = |\chi(m)| = \prod_v |\chi(m_v)|_v.$$

for all $\chi \in X(\mathsf{M})_\mathbb{Q}$. This is a homomorphism

$$(3.2) \qquad \mathsf{M}(\mathbb{A}) \longrightarrow \mathfrak{a}_\mathsf{P}.$$

We let $\mathsf{M}(\mathbb{A})^1$ be the kernel. Then

$$(3.3) \qquad \mathsf{M}(\mathbb{A}) = \mathsf{M}(\mathbb{A})^1 \times A(\mathbb{R})^0.$$

By Iwasawa decomposition, any $x \in \mathsf{G}(\mathbb{A})$ can be written as $nmak$ with $n \in \mathsf{N}(\mathbb{A}), m \in \mathsf{M}(\mathbb{A})^1, a \in A(\mathbb{R})^0, k \in \mathsf{K}$. Set $H_\mathsf{P}(x) := H_\mathsf{M}(a) \in \mathfrak{a}_\mathsf{P}$. Denote the restricted Weyl group of (G, A) by $\mathcal{W}$. The group $\mathcal{W}$ acts on $\mathfrak{a}_\mathsf{B}$ and $\mathfrak{a}_\mathsf{B}^*$. For any $\mathsf{s} \in \mathcal{W}$, fix a representative w_s in the intersection of $\mathsf{G}(\mathbb{Q})$ with the normalizer of A_B. If $\mathsf{P}_1, \mathsf{P}_2$ are parabolic subgroups, we let $\mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ be the set of distinct isomorphisms $\mathfrak{a}_1 \to \mathfrak{a}_2$ obtained by restricting elements of $\mathcal{W}$ to $\mathfrak{a}_1$. The groups $\mathsf{P}_1, \mathsf{P}_2$ are called *associated* if $\mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ is not empty. We usually think of $\mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ as a subset of $\mathcal{W}$. We let $n(\mathsf{A})$ be the number of chambers in $\mathfrak{a}$.

Set

$$(3.4) \qquad \mathfrak{a}_\mathsf{P}^+ = \{H \in \mathfrak{a}_\mathsf{P}; \alpha(H) > 0 \text{ for all } \alpha \in \Delta_\mathsf{P}\},$$

and

$$(3.5) \qquad (\mathfrak{a}_\mathsf{P}^*)^+ = \{\Lambda \in \mathfrak{a}_\mathsf{P}^*; \Lambda(\check{\alpha}) > 0 \text{ for all } \alpha \in \Delta_\mathsf{P}\}.$$

There is a vector $\rho_\mathsf{P} \in (\mathfrak{a}_\mathsf{P}^*)^+$ such that

$$(3.6) \qquad \delta_\mathsf{P}(p) = |\det Ad\, p|_{\mathfrak{n}_\mathsf{P}(\mathbb{A})}| = e^{2\rho_\mathsf{P}(H_\mathsf{P}(p))}$$

for all $p \in \mathsf{P}(\mathbb{A})$. We take the normalization of Haar measures to be as in [1].

Let $\mathsf{L}^2_{cusp}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)$ be the space of functions ϕ in $\mathsf{L}^2(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)$ such that for any parabolic $\mathsf{P}_1 \not\subseteq \mathsf{P}$ we have

$$(3.7) \qquad \int_{\mathsf{N}_1(\mathbb{Q})\cap\mathsf{M}(\mathbb{Q})\backslash\mathsf{N}_1(\mathbb{A})\cap\mathsf{M}(\mathbb{A})} \phi(nm)\, dn = 0$$

for almost all m. It is known that

$$(3.8) \qquad \mathsf{L}^2_{cusp}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1) = \bigoplus_\rho V_\rho$$

where V_ρ is the ρ-isotypic component of ρ consisting of finitely many copies of ρ (possibly zero). The pairs (M, ρ) and (M', ρ') are considered equivalent if there is $\mathsf{s} \in \mathcal{W}(\mathfrak{a}, \mathfrak{a}')$ such that the representation

$$(3.9) \qquad (\mathsf{s}\rho)(m') = \rho(w_\mathsf{s}^{-1} m' w_\mathsf{s}) \quad (m' \in \mathsf{M}'(\mathbb{A})^1)$$

is unitarily equivalent to ρ'. Let $\mathfrak{X}$ be the set of equivalence classes of such pairs. For any $X \in \mathfrak{X}$ we have a class $\mathcal{P}_X$ of associated parabolic subgroups. If P is any parabolic and $X \in \mathfrak{X}$, set

$$(3.10) \qquad \mathsf{L}^2_{cusp}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)_X = \bigoplus_{(\rho:(\mathsf{M},\rho)\in X)} V_\rho.$$

This is a closed subspace of $\mathsf{L}^2_{cusp}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)$, and empty if $\mathsf{P} \notin \mathcal{P}_X$.

Fix P and $X \in \mathfrak{X}$. Suppose there is a $\mathsf{P}_1 \in \mathcal{P}_X$ such that $\mathsf{P}_1 \subset \mathsf{P}$. Let ψ be a smooth function on $\mathsf{N}_1(\mathbb{A})\mathsf{M}_1(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ such that

$$(3.11) \qquad \Psi_a(m,k) = \psi(amk) \quad (k \in \mathsf{K}, m \in \mathsf{M}_1(\mathbb{Q})\backslash\mathsf{M}_1(\mathbb{A}), a \in \mathsf{A}_1(\mathbb{Q})\backslash\mathsf{A}_1(\mathbb{A}))$$

vanishes outside a compact subset of $\mathsf{A}_1(\mathbb{Q})\backslash\mathsf{A}_1(\mathbb{A})$, transforms under $\mathsf{K}_{\mathbb{R}}$ according to an irreducible representation, and as a function of m belongs to $\mathsf{L}^2_{cusp}(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)$. Then the function

$$(3.12) \qquad \hat{\psi}^{\mathsf{M}}(m) = \sum_{\delta \in \mathsf{P}_1(\mathbb{Q})\cap\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{Q})} \psi(\delta m) \quad (m \in \mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)$$

is square-integrable on $\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1$. Define $\mathsf{L}^2(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)_X$ to be the span of all such $\hat{\psi}^{\mathsf{M}}$. If no such P_1 exist, latter space is the zero space.

By a result of Langlands,

$$(3.13) \qquad \mathsf{L}^2(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1) = \bigoplus_X \mathsf{L}^2(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)_X.$$

For any P, let $\Pi(\mathsf{M})$ denote the set of equivalence classes of irreducible unitary representations of $\mathsf{M}(\mathbb{A})$. For $\zeta \in \mathfrak{a}^*_{\mathbb{C}}$ and $\pi \in \Pi(\mathsf{M})$ let π_ζ be the product of π with the quasi-character

$$(3.14) \qquad x \mapsto e^{\zeta H_{\mathsf{P}}(x)} \quad (x \in \mathsf{G}(\mathbb{A})).$$

If $\zeta \in i\mathfrak{a}^*$, π_ζ is again unitary. This means that $\Pi(\mathsf{M})$ is a differentiable manifold which carries an action of $i\mathfrak{a}^*$. We use this action to define a measure $d\pi$ on $\Pi(\mathsf{M})$.

For $\pi \in \Pi(\mathsf{M})$ we let $\mathcal{H}^0_{\mathsf{P}}(\pi)$ be the space of smooth functions

$$(3.15) \qquad \phi : \mathsf{N}(\mathbb{A})\mathsf{M}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}) \to \mathbb{C}$$

satisfying

1. ϕ is right K-finite;
2. for every $x \in \mathsf{G}(\mathbb{A})$ the function

$$m \mapsto \phi(mx) \quad (m \in \mathsf{M}(\mathbb{A}))$$

is a matrix coefficient of π;
3. $\|\phi\|^2 = \int_{\mathsf{K}} \int_{\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1} |\phi(mk)|^2 \, dm \, dk < +\infty$.

Let $\mathcal{H}_{\mathsf{P}}(\pi)$ be the completion. If $\phi \in \mathcal{H}_{\mathsf{P}}(\pi)$ and $\zeta \in \mathfrak{a}^*_{\mathbb{C}}$ set

$$(3.16) \qquad \phi_\zeta(x) = \phi(x)e^{\zeta(H_{\mathsf{P}}(x))} \quad (x \in \mathsf{G}(\mathbb{A}))$$

and

$$(3.17) \qquad (I_{\mathsf{P}}(\pi_\zeta, y)\phi_\zeta)(x) = \phi_\zeta(xy)\delta_{\mathsf{P}}(xy)^{\frac{1}{2}}\delta_{\mathsf{P}}(x)^{-\frac{1}{2}}.$$

Then $I_{\mathsf{P}}(\pi_\zeta)$ is a unitary representation if $\zeta \in i\mathfrak{a}^*$.

Given $X \in \mathfrak{X}$, let $\mathcal{H}_{\mathsf{P}}(\pi)_X$ be the closed subspace of $\mathcal{H}_{\mathsf{P}}(\pi)$ consisting of those ϕ such that for all x the function $m \mapsto \phi(mx)$ belongs to $\mathsf{L}^2(\mathsf{M}(\mathbb{Q})\backslash\mathsf{M}(\mathbb{A})^1)_X$. Then

$$(3.18) \qquad \mathcal{H}_{\mathsf{P}}(\pi) = \bigoplus_X \mathcal{H}_{\mathsf{P}}(\pi)_X.$$

Let K_0 be an open-compact subgroup of $\mathsf{G}(\mathbb{A}_f)$ and W an equivalence class of irreducible representations of $\mathsf{K}_\mathbb{R}$. Let $\mathcal{H}_{\mathsf{P}}(\pi)_{X,\mathsf{K}_0}$ be the subspace of functions in $\mathcal{H}_{\mathsf{P}}(\pi)_X$ which are invariant under $\mathsf{K}_0 \cap \mathsf{K}$. Also let $\mathcal{H}_{\mathsf{P}}(\pi)_{X,\mathsf{K}_0,W}$ be the space of those functions in $\mathcal{H}_{\mathsf{P}}(\pi)_{X,\mathsf{K}_0}$ which transform under $\mathsf{K}_\mathbb{R}$ according to W. It is a theorem of Langlands that each of the spaces $\mathcal{H}_{\mathsf{P}}(\pi)_{X,\mathsf{K}_0,W}$ is finite-dimensional. We fix an orthonormal basis $\mathcal{B}(\pi)_X$ for $\mathcal{H}_{\mathsf{P}}(\pi)_X$, for each π and each X, such that for all $\zeta \in i\mathfrak{a}^*$ we have

$$(3.19) \qquad \mathcal{B}(\pi_\zeta)_X = \{\phi_\zeta : \phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X\}$$

and such that every $\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X$ belongs to one of the spaces $\mathcal{H}_{\mathsf{P}}(\pi)_{X,\mathsf{K}_0,W}$.

Suppose that $\pi \in \Pi(\mathsf{M})$, $\phi \in \mathcal{H}_{\mathsf{P}}^0(\pi)$, $\zeta \in \mathfrak{a}_\mathbb{C}^*$. For $\Re(\zeta) \in \rho_{\mathsf{P}} + (\mathfrak{a}^*)^+$ we set

$$(3.20) \qquad E(x, \phi, \zeta) = \sum_{\delta \in \mathsf{P}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{Q})} \phi_\zeta(\delta x)\delta_{\mathsf{P}}(\delta x)^{\frac{1}{2}}.$$

For $\mathsf{s} \in \mathcal{W}(\mathfrak{a}, \mathfrak{a}')$, we define the global intertwining operator

$$(3.21) \qquad \begin{aligned} &(M(\mathsf{s}, \pi, \zeta)\phi_\zeta)(x) \\ &= \int_{\mathsf{N}'(\mathbb{A}) \cap w_\mathsf{s}\mathsf{N}(\mathbb{A})w_\mathsf{s}^{-1}\backslash\mathsf{N}'(\mathbb{A})} \phi_\zeta(w_\mathsf{s}^{-1}nx)\delta_{\mathsf{P}}(w_\mathsf{s}^{-1}nx)^{\frac{1}{2}}\delta_{\mathsf{P}}(x)^{-\frac{1}{2}} \, dn. \end{aligned}$$

Both $E(x, \phi, \zeta)$ and $M(\mathsf{s}, \pi_\zeta)\phi_\zeta$ can be analytically continued to meromorphic functions in ζ to $\mathfrak{a}_\mathbb{C}^*$. For $\zeta \in i\mathfrak{a}^*$, $E(x, \phi, \zeta)$ is a smooth function of x and $M(\mathsf{s}, \pi_\zeta)$ is unitary from $\mathcal{H}_{\mathsf{P}}(\pi_\zeta)$ to $\mathcal{H}_{\mathsf{P}'}(\mathsf{s}\pi_\zeta)$.

3.2. Spectral expansion. — Let f be a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. By the *spectral expansion* of f we mean

$$(3.22) \qquad \begin{aligned} S(f, x) = &\sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \\ &\int_{\Pi(\mathsf{M})} \sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi)\left(\int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \overline{E(y, \phi)}f(y)\,dy\right)d\pi. \end{aligned}$$

For $\mathsf{M} = \mathsf{G}$, the integrals are interpreted appropriately to give a discrete sum. It is an interesting problem to determine under which conditions on the function f, we have $f(x) = S(f, x)$. Also, for the application we have in mind, we need to know that the identity $f = S_f$ holds as an identity not just of L^2-functions, but of continuous

functions. In order to show that for a given f, the spectral identity holds, we just need to prove that that right hand side of the (3.22) is uniformly convergent on compact sets. This would suffice since for all pseudo-Eisenstein series θ_ϕ (II.1.10 of [26]) we have

$$(3.23) \qquad (S(f), \theta_\phi) = (f, S(\theta_\phi)),$$

and since pseudo-Eisenstein series are rapidly decreasing (Proposition II.1.10 [26]), we have $\theta_\phi = S(\theta_\phi)$ by the spectral decomposition [1]. Consequently $(f - S(f), \theta_\phi) = 0$ for all pseudo-Eisenstein series θ_ϕ. Then by the density theorem Theorem II.1.12 of [26] we get the required identity. In our applications, we use a slightly different re-arrangement of the terms of $S(f)$ as suggested by the definition of $I(S, f, x, y)$ on page 930 of [1].

Let f be a smooth function on $\mathsf{G}(\mathbb{A})$, and suppose that f is right invariant under a compact-open subgroup K of $\mathsf{G}(\mathbb{A}_f)$. Define a function on $\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ by

$$F(g) := \sum_{\gamma \in \mathsf{G}(\mathbb{Q})} f(\gamma g).$$

Suppose that f is such that the function F is convergent for all g, is smooth and bounded. By Lemma 4.1 of [1], given m, there is an n and functions $f_1 \in C_c^m(\mathsf{G}(\mathbb{R}))^{K_\mathbb{R}}$ and $f_2 \in C_c^\infty(\mathsf{G}(\mathbb{R}))^{K_\mathbb{R}}$ such that $\Delta^n * f_1 + f_2$ is the delta distribution at the identity of $\mathsf{G}(\mathbb{R})$. This implies that after taking the finite places into account, we can find functions η_1, η_2 of compact support on $\mathsf{G}(\mathbb{A})$, with appropriate differentiability at the archimedean place, such that

$$(3.24) \qquad R(\Delta^n * \eta_1 + \eta_2)F = F,$$

where R is the right regular convolution action. We get

$$(3.25) \qquad R(\eta_1)\Delta^n F + R(\eta_2)F = F.$$

Consider the following modified spectral expansion

$$(3.26) \qquad S'(F, x) = \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1}$$
$$\int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_\mathsf{P}(\pi)_X} E(x, \phi)\overline{E(y, \phi)} \right) F(y)\, dy\, d\pi.$$

We get

$$S'(F, x) = \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1}$$

$$\int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{Q}) \backslash \mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, \phi)} \right) F(y) \, dy \, d\pi$$

$$= \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{Q}) \backslash \mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, \phi)} \right)$$

$$(R(\eta_1) \Delta^n F(y) + R(\eta_2) F(y)) \, dy \, d\pi.$$

Consequently, $S'(F, x)$ will be the sum of the following two series

$$S_1(F, x) = \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{Q}) \backslash \mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, I_{\mathsf{P}}(\pi, \tilde{\eta}_1)\phi)} \right)$$

$$\Delta^n F(y) \, dy \, d\pi$$

and

$$S_2(F, x) = \sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{Q}) \backslash \mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, I_{\mathsf{P}}(\pi, \tilde{\eta}_2)\phi)} \right)$$

$$F(y) \, dy \, d\pi$$

for appropriately chosen compactly supported function $\tilde{\eta}_1, \tilde{\eta}_2$. By Lemma 4.4 of [1], there are N, r_0 and a continuous seminorm $\| \cdot \|_{r_0}$ on $C_c^{r_0}(\mathsf{G}(\mathbb{A}))$ such that if $r \geq r_0$ and η is K-finite in $C_c^r(\mathsf{G}(\mathbb{A}))$,

$$\sum_{X \in \mathfrak{X}} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \left| \sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, I_{\mathsf{P}}(\pi, \eta)\phi)} \right| d\pi$$

is bounded by $\|\eta\|_{r_0} \cdot \|x\|^N \cdot \|y\|^N$. Here we have defined the height $\| \cdot \|$ on $\mathsf{G}(\mathbb{A})$ as in [1], page 918. Since m can be taken to be arbitrarily large, all we need to verify in order to get the uniform convergence is the convergence of the following integrals

$$(3.27) \qquad \int_{\mathsf{G}(\mathbb{A})} |f(y)| \cdot \|y\|^N \, dy$$

and

$$(3.28) \qquad \int_{\mathsf{G}(\mathbb{A})} |\Delta^n f(y)| \cdot \|y\|^N \, dy.$$

We summarize this discussion:

LEMMA 3.1. — *Let n, N be as above. Assuming convergence of the integrals* (3.27) *and* (3.28) *we have*

$$(3.29) \qquad\qquad F(x) = S_1(F, x) + S_2(F, x).$$

3.3. Truncations. — Let $T \in \mathfrak{a}_B^+$. Let Δ_0 be a set of simple roots of (B, A), and $d_B(T) := \min_{\alpha \in \Delta_0}\{\alpha(\Delta)\}$. We will need those T which are *sufficiently regular*; this means that $d_B(T)$ is large. Recall that for sufficiently regular T, the Arthur truncation $\wedge^T$ acts on functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ [**3**].

LEMMA 3.2. —

1. *If ϕ is locally bounded, then $\wedge^T \phi$ is defined everywhere.*
2. *If $\phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ is locally L^1, then we have*

$$\wedge^T \wedge^T \phi(g) = \wedge^T \phi(g)$$

   for almost all g. If ϕ is locally bounded, then the identity holds for all g.
3. *Suppose $\phi_1, \phi_2 : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ are locally L^1. If ϕ_1 is of moderate growth and ϕ_2 is rapidly decreasing, then*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\wedge^T \phi_1(g)} \phi_2(g)\, dg = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\phi_1(g)} \wedge^T \phi_2(g)\, dg.$$

Observe that the lemma implies that if ϕ_1, ϕ_2 are as above, then

$$(3.30) \qquad \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\wedge^T \phi_1(g)} \phi_2(g)\, dg = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \overline{\wedge^T \phi_1(g)} \wedge^T \phi_2(g)\, dg.$$

Given P, and $\pi \in \Pi(M_P(A))$ and $\lambda \in i\mathfrak{a}_P^*$ we define an operator

$$(3.31) \qquad\qquad \Omega_{X,\pi}^T(P, \lambda) : \mathcal{H}_P^0(\pi)_X \to \mathcal{H}_P^0(\pi)_X$$

by setting

$$(3.32) \qquad (\Omega_{X,\pi}^T(P, \lambda)\pi, \pi') = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \wedge^T E(x, \phi, \lambda) \overline{\wedge^T E(x, \phi', \lambda)}\, dx,$$

for each pair of vectors $\phi, \phi' \in \mathcal{H}_\pi^0(\pi)_X$.

Next let K_0 be a subgroup of finite index in K_f. Suppose that W is a finite dimensional representation of K_∞. Given $P \supset B$, $\pi \in P(M_P(A)^1)$, let $\mathcal{H}_P^0(\pi)_X^{K_0}$ be the space of K_0-invariant functions in $\mathcal{H}_P^0(\pi)_X$, and let $\mathcal{H}_P^0(\pi)_X^{K_0, W}$ be the subspace of functions in $\mathcal{H}_P^0(\pi)_X^{K_0}$ which transform under K_∞ according to W. If a linear operator A on $\mathcal{H}_P^0(\pi)_X$ leaves any of these subspaces $\mathcal{H}_P^0(\pi)_X^{\square}$, with $\square = K_0$ or $\square = K_0, W$, invariant, we denote by A_* the restriction of A to the appropriate subspace. Choose the differential operator Δ as in [**3**]. Then Δ acts on $\mathcal{H}_P^0(\pi)_X$ through each of the representations $I_P(\pi_\zeta)$, and we denote the action by $I_P(\pi_\zeta, \Delta)$. This action leaves the two subspaces mentioned above invariant. For $\zeta \in i\mathfrak{a}_P^*$, the action of $I_P(\pi_\zeta, \Delta)$ is via a scalar greater than 1.

LEMMA 3.3 (Arthur [2]). — *There exist integers C_0, d_0, and m such that for any subgroup $\mathsf{K}_0 \subset \mathsf{K}_f$ of finite index, and any $T \in i\mathfrak{a}_\mathsf{B}^+$ with $\mathsf{d}_\mathsf{B}(T) > C_0$, the expression*

$$(3.33) \qquad \sum_{X \in \mathfrak{X}} \sum_\mathsf{P} \sum_{\pi \in \Pi(\mathsf{M}_\mathsf{P}(\mathbb{A})^1)} |n(\mathsf{A})|^{-1} \int_{i\mathfrak{a}_\mathsf{P}^*/i\mathfrak{a}_\mathsf{G}^*} \|\Omega_{X,\pi}^T(\mathsf{P},\zeta)_{\mathsf{K}_0}.I_\mathsf{P}(\pi_\zeta, \Delta^m)^{-1}\|_1 d\zeta$$

is bounded by $C_{\mathsf{K}_0}(1 + \|T\|)^{d_0}$. Here C_{K_0} is a constant which depends only on K_0, and $\|\cdot\|_1$ is the trace class norm.

3.4. A bound for Eisenstein series. — We recall a bound for Eisenstein series embedded in the proof of Proposition 2 of [25]. Fix a large compact set C of $\mathsf{G}(\mathbb{A})$ and a small compact open subgroup K of $\mathsf{G}(\mathbb{A}_f)$, and let f be a sufficiently differentiable function with support in C.

LEMMA 3.4. — *Fix a compact set C' in $\mathsf{G}(\mathbb{A})$. There are constants $c, N > 0$ depending only on the support of f such that for sufficiently regular T we have*

$$(3.34) \qquad |E(g, I(f, \pi_\lambda)\varphi, \lambda)| \le c\|g\|^N \|f\|_\infty \cdot \| \wedge^T E(\cdot, \varphi, \lambda)\|_{\mathsf{L}^2(\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A}))}.$$

This lemma combined with Lemma 3.3 together with Lemma 4.1 of [1] implies the following result.

PROPOSITION 3.5. — *For $\phi \in \mathcal{B}_\mathsf{P}(\pi)_X$ let $\Lambda(\phi)$ be defined by $\Delta \cdot \phi = \Lambda(\phi) \cdot \phi$. Then there is an $l > 0$ such that*

$$(3.35) \qquad \sum_{X \in \mathfrak{X}} \sum_\mathsf{P} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \left(\sum_{\phi \in \mathcal{B}_\mathsf{P}(\pi)_X} \Lambda(\phi)^{-m} |E(e, \phi)|^2 \right) d\pi$$

is convergent. The outermost summation is only over those classes for which $\Lambda(\phi) \ne 0$ are fixed by K_0.

4. Spherical functions and bounds for matrix coefficients

4.1. Spherical functions. — Let π be an infinite-dimensional automorphic representation of G, and suppose ϕ is a right K-finite automorphic form in the space of π. Here $\mathsf{K} = \prod_v \mathsf{K}_v$. Define a function $M(\phi, g)$ on $\mathsf{G}(\mathbb{A})$ by

$$M(\phi, g) = \frac{1}{(\text{vol } \mathsf{K})^2} \int_\mathsf{K} \int_\mathsf{K} \phi(\kappa g \kappa') \, d\kappa \, d\kappa'.$$

If π has no K-invariant vectors, then the above integral is zero, and we may assume that ϕ is right K invariant. Let V_π be a vector space on which $\mathsf{G}(\mathbb{A})$ acts via a representation which is isomorphic to the space of K-finite vectors. Let $j : \mathsf{V}_\pi \to \mathsf{H}_\pi$ be the intertwining map. We have used the same notation for a representation and its underlying representations space. By a standard result of Jacquet, Langlands, and Flath, we have

$$\mathsf{V}_\pi = \bigotimes_v \mathsf{V}_{\pi,v}.$$

Since ϕ is right K-finite, it is in the image of j. Let $w = j^{-1}(\phi)$. Then

$$w \in \mathsf{V}_\pi^{\mathsf{K}} = \left(\bigotimes_{v \notin S_\infty} \mathsf{V}_{\pi,v}^{\mathsf{K}_v} \right) \otimes \left(\bigotimes_{v \in S_\infty} \mathsf{V}_{\pi,v} \right).$$

Set $S = S_F$. We know that for $v \notin S$ we have $\dim \mathsf{V}_{\pi,v}^{\mathsf{K}_v} = 1$. Fix a non-zero element e_v in each of these spaces. This means then

$$w = (\otimes_{v \notin S} e_v) \otimes w_S,$$

where

$$w_S \in \left(\bigotimes_{v \in S \setminus S_\infty} \mathsf{V}_{\pi,v}^{\mathsf{K}_v} \right) \otimes \left(\bigotimes_{v \in S_\infty} \mathsf{V}_{\pi,v} \right).$$

Next we examine $M(\phi, g)$. Define a functional λ on V_π by

$$\lambda(\nu) = \int_{\mathsf{K}} j(\nu)(\kappa) \, d\kappa,$$

(for $\nu \in \mathsf{V}_\pi$). Then it is easily seen that

$$\lambda \in \bigotimes_{v \notin S} \widetilde{\mathsf{V}}_{\pi,v}^{\mathsf{K}_v} \otimes \mathsf{V}_S^*.$$

Here $\mathsf{V}_S = \bigotimes_{v \in S} \mathsf{V}_{\pi,v}$, V_S^* is the dual space and $\widetilde{\mathsf{V}}_\pi$ is the smooth dual of the local representation V_π. Since the smooth dual of an admissible representation is admissible, it follows that $\dim \widetilde{\mathsf{V}}_{\pi,v}^{\mathsf{K}_v} = 1$. For each $v \notin S$ choose an element ξ_v in this space in such a way that $\xi_v(e_v) = 1$. This then means that, as in the case of $w = j^{-1}(\phi)$, we have

$$\lambda = (\otimes_{v \notin S} \xi_v) \otimes \lambda_S,$$

with λ_S in the obvious space. Combining this identity with the similar identity for w we obtain

$$\lambda(v) = (\prod_{v \notin S} \xi_v(e_v)) \cdot \lambda_S(w_S)$$
$$= \lambda_S(w_S).$$

This implies that

$$\lambda_S(w_S) = \int_{\mathsf{K}} j(w)(\kappa) \, d\kappa$$
$$= \int_{\mathsf{K}^S} j(w)(\kappa) \, d\kappa.$$

Here $\mathsf{K}^S = \prod_{v \in S} \mathsf{K}_v$. Consider an embedding $\eta : \mathsf{G}(\mathbb{A}_S) \longrightarrow \mathsf{G}(\mathbb{A})$ given by

$$g \mapsto (1, 1, \ldots, 1, g).$$

Obviously,

$$\mathrm{vol}(\mathsf{K}^S)M(\phi,g) = \lambda(\pi(g)w)$$

$$= \prod_{v \notin S} \varphi_{v,\pi}(g_v) \cdot \int_{\mathsf{K}^S} \phi(\kappa\eta(g_S))\, d\kappa,$$

where $\varphi_{v,\pi}(g_v) = \xi_v(\pi_v(g_v)e_v)$ is the normalized local spherical function.

COROLLARY 4.1. — *If* $\mathsf{K} = \prod_v \mathsf{K}_v$ *is such that for each local place v, including the archimedean places, the local compact subgroups K_v satisfy the hypotheses I and II of* [35], *then*

$$(4.1) \qquad\qquad M(\phi,g) = \phi(e)\prod_v \varphi_{v,\pi}(g_v)$$

if each π has a K_v-fixed vector; otherwise it is zero.

We note that local K_v satisfying I and II of [35] exist by [19], for archimedean places, and [10, 38, 40], for non-archimedean places.

4.2. Bounds on Matrix Coefficients. — In this section we recall an important result of H. Oh that is used in estimates leading to the proof of the spectral expansion (3.22).

Let k be a non-archimedean local field of $\mathrm{char}(k) \neq 2$, and residual degree q. Let H be the group of k-rational points of a connected reductive split or quasi-split group with $\mathsf{H}/\mathsf{Z}(\mathsf{H})$ almost k-simple. Let S be a maximal k-split torus, B a minimal parabolic subgroup of H containing S and K a good maximal compact subgroup of H with Cartan decomposition $\mathsf{G} = \mathsf{K}\mathsf{S}(k)^+\mathsf{K}$. Let Φ be the set of non-multipliable roots of the relative root system $\Phi(\mathsf{H},\mathsf{S})$, and Φ^+ the set of positive roots in Φ. A subset $\mathcal{S}$ of Φ^+ is called a strongly orthogonal system of Φ if any two distinct elements α and α' of $\mathcal{S}$ are strongly orthogonal, that is, neither of $\alpha \pm \alpha'$ belongs to Φ. Define a bi-K-invariant function $\xi_{\mathcal{S}}$ on H as follows: first set

$$n_{\mathcal{S}}(g) = \frac{1}{2}\sum_{\alpha \in \mathcal{S}} \log_q |\alpha(g)|,$$

then

$$\xi_{\mathcal{S}}(g) = q^{-n_{\mathcal{S}}(g)} \prod_{\alpha \in \mathcal{S}} \left(\frac{(\log_q |\alpha(g)|)(q-1) + (q+1)}{q+1}\right).$$

THEOREM 4.2 ([28], Theorem 1.1). — *Assume that the semi-simple k-rank of H is at least 2 and let $\mathcal{S}$ be a strongly orthogonal system of Φ. Then for any unitary representation ϱ of H without an invariant vector and with K-finite unit vectors ν and ν', one has*

$$|(\varrho(g)\nu, \nu')| \leq (\dim(\mathsf{K}\nu)\dim(\mathsf{K}\nu'))^{\frac{1}{2}} \cdot \xi_{\mathcal{S}}(g),$$

for any $g \in \mathsf{H}$.

COROLLARY 4.3. — *Let $v \notin S_F$. Let φ_v be the normalized spherical function associated with an infinite dimensional unramified principal series representation of $\mathsf{G}(F_v)$. Suppose that the semi-simple rank of $\mathsf{G}(F_v)$ is at least 2. Then for each $\vartheta \in \Delta(\mathsf{G}(F_v), \mathsf{S}_v(F_v))$ we have*

$$|\varphi_v(\check{\vartheta}(\varpi_v))| < 2q_v^{-\frac{\ell_v(\vartheta)}{2}}.$$

Proof. — Fix ϑ. By definition, a singleton is a strongly orthogonal set. Set

$$\mathcal{S}_v = \{\vartheta\}.$$

Then

$$n_{\mathcal{S}_v}(\check{\vartheta}(\varpi_v)) = \frac{\ell_v(\vartheta)}{2},$$

and

$$\xi_{\mathcal{S}_p}(\check{\vartheta}(\varpi_v)) = q_v^{-\frac{\ell_v(\vartheta)}{2}} \frac{2q_v}{q_v + 1}.$$

Since ν and ν' that define the spherical function are K_v-invariant, this equation combined with Theorem 4.2 implies

$$\begin{aligned}
|\varphi_v(\check{\vartheta}(\varpi_v)) &\leq \xi_{\mathcal{S}_v}(\check{\vartheta}(\varpi_v))) \\
&\leq q^{-\frac{\ell_v(\vartheta)}{2}} \frac{2q_v}{q_v + 1} \\
&< 2q_v^{-\frac{\ell_v(\vartheta)}{2}},
\end{aligned}$$

which gives the claim. □

We also need a similar bound on spherical functions when the semi-simple rank is equal to one. In this case, local considerations do not suffice, as the trivial representation may not be isolated in the unitary dual of the local group. However, for our purposes it will suffice to obtain a bound for a restricted class of representations. Let π be an infinite-dimensional unitary irreducible automorphic representation of G. Suppose $\pi = \otimes_v \pi_v$. Extending S_F if necessary, we may assume that for $v \notin S_F$, π_v is an unramified representation.

PROPOSITION 4.4. — *For all $v \notin S_F$, the representation π_v is infinite-dimensional.*

Proof. — Realize π on a Hilbert subspace V of $\mathsf{L}^2(\mathsf{G}(F) \backslash \mathsf{G}(\mathbb{A}))$. Denote by V^∞ the subspace of V consisting of all vectors ν such that:

- ν is K_f-finite, and
- for all archimedean places v, the map $\mathsf{G}(F_v) \to \mathsf{V}$ given by $g \mapsto \pi(g)\nu$ is C^∞.

It is standard that V^∞ is $\mathsf{G}(\mathbb{A})$-stable, and dense in V (with respect to the L^2-topology). The image of V^∞ in L^2 consists of smooth functions. Suppose that for some $v \notin S_F$ the representation π_v is not infinite-dimensional. Since this representation is admissible, it must be one-dimensional. Concretely, if we pick an element $\varphi \in \mathsf{V}^\infty$, we have

$$\varphi(\gamma x_v) = \chi_v(x_v)\varphi(e), \quad \gamma \in \mathsf{G}(F), x_v \in \mathsf{G}(F_v).$$

Here $\chi_v : \mathsf{G}(F_v) \to \mathbb{C}^\times$ is a one-dimensional representation of $\mathsf{G}(F_v)$. Next, by Satz 6.1. of [24] we know that the commutator subgroup $\mathsf{G}'(\mathbb{A})$ of $\mathsf{G}(\mathbb{A})$ is contained in the closure of $\mathsf{G}(F)\mathsf{G}(F_v)$. As $\mathsf{G}'(\mathbb{A})$ has no non-trivial one-dimensional representations, we conclude that for all $\varphi \in \mathsf{V}^\infty$ we have

(4.2) $$\varphi(g') = \varphi(e),$$

for all $g' \in \mathsf{G}'(\mathbb{A})$. Since the subspace V^∞ is $\mathsf{G}(\mathbb{A})$-invariant, (4.2) must hold for all right translates of φ by elements of $\mathsf{G}(\mathbb{A})$. Hence

$$\varphi(xg) = \varphi(g), \quad x \in \mathsf{G}'(\mathbb{A}), g \in \mathsf{G}(\mathbb{A}).$$

Since $\mathsf{G}'(\mathbb{A})$ is a normal subgroup of $\mathsf{G}(\mathbb{A})$, it follows that we must also have

$$\varphi(gx) = \varphi(g), \quad x \in \mathsf{G}'(\mathbb{A}), g \in \mathsf{G}(\mathbb{A}),$$

i.e., every element of V^∞ is invariant under the restriction of the representation π to the subgroup $\mathsf{G}'(\mathbb{A})$. This implies that the representation π on V^∞ factors through $\mathsf{G}(\mathbb{A})/\mathsf{G}'(\mathbb{A})$, which is an abelian group. Since irreducible admissible representations of abelian groups are all one-dimensional, the space V^∞ must be one-dimensional. Finally, use the fact that V^∞ is dense in V to conclude that V is also one-dimensional.
□

THEOREM 4.5. — *There is an absolute constant $c > 0$ with the following property. Let $\pi = \otimes_v \pi_v$ be an infinite-dimensional irreducible automorphic representation of G; for $\mathsf{G} = \mathrm{PGL}_2$ assume that π is not the automorphic representation associated with an Eisenstein series induced from a Borel subgroup. Let v be a place with $v \notin S_F$, and φ_v the normalized spherical function of π_v. Then for all $\alpha \in \Delta(\mathsf{G}(F_v), \mathsf{S}_v(F_v))$ we have*

$$|\varphi_v(\check{\vartheta}(\varpi_v))| < q^{-c\ell_v(\vartheta)}.$$

Proof. — According to the result of Corollary 4.3, we recognize two cases:

The case where s.s. rank is equal to one. In this case, G is a form of PGL_2 or PGU_3. For the case of PGL_2, if G is not split, then it has to be a quaternion algebra, and in this case by the Jacquet-Langlands correspondence, there is an irreducible cuspidal automorphic representation π' of GL_2, such that for all $v \notin S$, we have $\pi_v = \pi'_v$ as representations of $\mathrm{GL}_2(F_v)$. Since the local representation obtained this way is unramified, there must exist a pair of unramified quasi-characters χ and χ' of

$F_v^\times$ such that $\pi_v' = \pi(\chi, \chi')$. Since this representation has trivial central character, we must have $\chi' = \chi^{-1}$. In this case, if α is the unique positive root, we have

$$t_v(\alpha) = \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix}.$$

Next if $\chi \neq 1$, by the formula of Casselman ([13]), we have

$$\varphi_v \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix} = \frac{q^{-\frac{1}{2}}}{1+q^{-1}} \{ \frac{1 - q^{-1}\chi(\varpi_v)^{-2}}{1 - \chi(\varpi_v)^{-2}} \chi(\varpi_v) + \frac{1 - q^{-1}\chi(\varpi_v)^2}{1 - \chi(\varpi_v)^2} \chi(\varpi_v)^{-1} \}$$

$$= \frac{q^{-\frac{1}{2}}}{1+q^{-1}} (\chi(\varpi_v) + \chi(\varpi_v)^{-1}).$$

The same formula holds for $\chi = 1$ by analytic continuation. Since χ is an unramified character, we have $\chi = |\cdot|^s$, for some complex number s. We now need some non-trivial estimate towards the Ramanujan conjecture. For example, by a recent result of Kim and Shahidi ([23]), we know that

$$-\frac{1}{6} \leq \Re(s) \leq \frac{1}{6}.$$

This implies that

$$\left| \varphi_v \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix} \right| \leq \frac{2q^{-\frac{1}{2}+\frac{1}{6}}}{1+q^{-1}} < 2q^{-\frac{1}{3}}.$$

When $\mathsf{G} = \mathrm{PGL}_2$ and π is cuspidal, the result follows from the same result of Kim and Shahidi. Next, let G be an inner form of the quasi-split group $\mathsf{G}' = \mathrm{PGU}(2,1)$. Suppose that G' splits over a quadratic extension E/F. By Rogawski's theorem [33], there is an automorphic cuspidal representation $\pi' = \otimes_v \pi_v'$ of $\mathrm{PGU}(2,1)$ such that for all $v \notin S$, $\pi_v = \pi_v'$ as representations of $\mathsf{G}'(F_v)$. Consider the base change of π' from $\mathrm{PGU}(2,1)/F$ to PGL_3/E, again established by Rogawski. At this point, use the result of Oh on PGL_3.

The case where the s.s. rank is larger than one. By Proposition 4.4, π_v is not one-dimensional (for $v \notin S_F$), and the assertion follows from Corollary 4.3. $\square$

Part II. GEOMETRY AND HEIGHT FUNCTIONS

5. Geometry

In this section we recall the constructions and basic geometric properties of wonderful compactifications.

5.1. Flag varieties. — An important class of varieties, homogeneous for the action of G is the class of generalized flag varieties

$$Y_I := \mathsf{P}_I \backslash \mathsf{G}.$$

The geometry of these, and their subvarieties (for example, Schubert varieties) plays an important role in different branches of algebra, e.g., representation theory and enumerative geometry.

We now recall some basic facts about these varieties. For $\lambda \in \mathfrak{X}^*(\mathsf{T}^{sc})$ we can define a line bundle L_λ on $\mathsf{B}^{sc} \backslash \mathsf{G}^{sc} = \mathsf{B} \backslash \mathsf{G}$ by

$$\mathsf{G}^{sc} \times \mathbf{G}_a / \sim, \quad \text{with} \quad (g, a) \sim (gb, \lambda^{-1}(b)a),$$

$g \in \mathsf{G}^{sc}, b \in \mathsf{B}^{sc}$ and $a \in \mathbf{G}_a$. The canonical projection

$$\pi \,:\, L_\lambda \to \mathsf{B} \backslash \mathsf{G}$$

is given by

$$\pi(g, a) = \mathsf{B}g.$$

This gives an identification of

$$\mathrm{Pic}(Y) = \mathfrak{X}^*(\mathsf{T}^{sc}).$$

Under this identification, the (closures of the) ample and the effective cones of Y correspond to the positive Weyl chamber, that is, the set of nonnegative linear combinations of ω_i. The anticanonical class is given by

$$-K_Y = 2\rho.$$

5.2. Wonderful compactifications. —

First we work over an algebraically closed field of characteristic zero.

PROPOSITION 5.1. — [15],[12] *There exists a canonical compactification of a connected adjoint group* G*: a smooth projective variety* X *such that*

- $\mathsf{G} \subset X$ *is a Zariski open subvariety and the action of* $\mathsf{G} \times \mathsf{G}$ *on* G*, by*

$$(g_1, g_2)(g) = g_1 g g_2^{-1},$$

  extends to an action of $\mathsf{G} \times \mathsf{G}$ *on* X*;*
- *The boundary* $X \setminus \mathsf{G}$ *is a union of strict normal crossings divisors* D_i *(for* $i = 1, ..., \mathsf{r}$*). For every* $I \subset [1, ..., \mathsf{r}]$ *the subvariety* $D_I = \cap_{i \in I} D_i$ *is a* $\mathsf{G} \times \mathsf{G}$*-orbit closure. All* $\mathsf{G} \times \mathsf{G}$*-orbit closures are obtained this way;*
- X *contains a unique closed* $\mathsf{G} \times \mathsf{G}$*-orbit* $Y = \mathsf{G}/\mathsf{B} \times \mathsf{G}/\mathsf{B}$*;*
- *The components* D_I *are isomorphic to fibrations over* $\mathsf{G}/\mathsf{P}_I \times \mathsf{G}/\mathsf{P}_I$ *with fibers canonical compactifications of the adjoint form of the associated Levi groups.*

We review several constructions of wonderful compactifications over algebraically closed fields (of characteristic 0), throughout G is semi-simple adjoint:

Via Hilbert schemes: Let $\mathsf{P} \subset \mathsf{G}$ be a parabolic subgroup and $Y = \mathsf{G}/\mathsf{P}$ the associated flag variety. Then X is the $\mathsf{G} \times \mathsf{G}$-orbit closure of the diagonal of the Hilbert scheme of $Y \times Y$ (see [11]).

Via representations: Let λ be a regular dominant weight of G and V_λ the irreducible representation of G with highest weight λ. We have an action of $\mathsf{G} \times \mathsf{G}$ on
$$\mathrm{End}(V_\lambda) = V_\lambda \otimes V_\lambda^*.$$
Taking the closure of the orbit through (the image of) the identity in $\mathbb{P}(\mathrm{End}(V_\lambda))$ we obtain the canonical compactification X.

Via Lie algebras: Let $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ and $\mathsf{n} = \dim(\mathsf{G}) = \dim(\mathfrak{g})$. The variety $\mathbb{L}$ of Lie subalgebras of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ can be regarded as a subvariety of the Grassmannian $\mathrm{Gr}(n, 2n)$. It contains $\mathfrak{g}$, embedded diagonally. Moreover, $\mathbb{L}$ is a projective $\mathsf{G} \times \mathsf{G}$-variety. Taking the closure of the $\mathsf{G} \times \mathsf{G}$-orbit X° through $\mathfrak{g}$ we obtain $X \subset \mathbb{L}$. Since G is an adjoint group, the adjoint representation of G on $\mathsf{G} \times \mathsf{G}$ is faithful and we may identify X° with the variety G, or more precisely $\mathsf{G} \times \mathsf{G}/\mathrm{Diag}(\mathsf{G})$.

We proceed to describe the boundary $X \setminus \mathsf{G}$ in the latter representation. Let $\mathsf{P} = \mathsf{P}_I$ (with $I \subset [1, \dots, r]$) be a *standard* parabolic subgroup of G. Choose a Levi decomposition $\mathsf{P} = \mathsf{M} \cdot \mathsf{U}$. Let $\mathfrak{L}_P$ be the set of pairs
$$(m + u, m + u'), \quad \text{with} \quad m \in \mathrm{Lie}(\mathsf{M}), \ u, u' \in \mathrm{Lie}(\mathsf{U}).$$
Then $\mathfrak{L}_\mathsf{P}$ is a subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$. Next let $\mathcal{C} = \{\mathsf{P}\}$ denote the conjugacy class of parabolic subgroups of G containing P. We note that the $\mathsf{G} \times \mathsf{G}$-orbit of $\mathfrak{L}_\mathsf{P}$ in $\mathbb{L}$ does not depend on the particular choice of M in the Levi decomposition of P; in fact, it depends only on the class $\{\mathsf{P}\}$. We denote this orbit by $D_\mathcal{C}$. When P is maximal, the orbit $D_\mathcal{C}$ is a smooth irreducible divisor in X. Moreover,
$$X \setminus \mathsf{G} = \bigcup_\mathcal{C} D_\mathcal{C} \ ,$$
the union over classes of maximal parabolics. As the classes $\mathcal{C}$ of maximal parabolics in G are in bijection with the simple roots α of T, we may write then
$$D_\mathcal{C} = D_i,$$
if $\mathcal{C}$ is the maximal parabolic that corresponds to α_i.

For $\mathsf{w} \in \mathsf{W}$ we denote by $X(\mathsf{w})$ the closure of $\mathsf{B}\mathsf{w}\mathsf{B} \subset \mathsf{G}$ in X. The $\mathsf{B} \times \mathsf{B}$-stable boundary components D_i correspond to $X(s\mathsf{w}_0 s_i)$. Every line bundle L on X restricts to the unique closed $\mathsf{G} \times \mathsf{G}$-orbit $Y = \mathsf{G}/\mathsf{B} \times \mathsf{G}/\mathsf{B}$; we get a restriction map

$\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. Recall, that in Section 5.1 we have identified the Picard group of G/B with $\mathfrak{X}^*(\mathsf{T}^{sc})$.

PROPOSITION 5.2. — *Let X be the canonical compactification of G as above.*
- *The Picard group $\mathrm{Pic}(X)$ is freely generated by the classes $[D_i]$;*
- *The image of $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(Y)$ consists of classes*

$$L(\lambda) = (\lambda, -\mathsf{w}_0\lambda) \subset \mathrm{Pic}(Y) = \mathrm{Pic}(\mathsf{G}/\mathsf{B}) \times \mathrm{Pic}(\mathsf{G}/\mathsf{B});$$

- *The (closed) cone of effective divisors is given by*

$$\Lambda_{\mathrm{eff}}(X) := \oplus_{i=1}^{\mathsf{r}} \mathbb{R}_{\geq 0}[D_i].$$

More precisely, if $\lambda \in \mathfrak{X}(\mathsf{T}^{sc})$ is a dominant weight then the line bundle $L(\lambda)$ on X has a unique (up to scalars) global section f_λ with divisor

$$\mathrm{div}(f_\lambda) = \sum_{i=1}^{\mathsf{r}} \langle \lambda, \alpha_i^\vee \rangle D_i.$$

Moreover, f_λ is an eigenvector of $\mathsf{B}^{sc} \times \mathsf{B}^{sc}$ with weight $(-\mathsf{w}_0\lambda, \lambda)$.
- *The anticanonical class is given by*

$$-K_X = L(2\rho + \sum_{i=1}^{r} \alpha_i).$$

Proof. — See [15] and [12]. □

Let L be a line bundle on X. Then L admits a G-linearization - there exists a $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$-action on $\pi : L \to X$ such that for $(g_1, g_2) \in \mathsf{G}^{sc} \times \mathsf{G}^{sc}, l \in L$ one has

$$\pi((g_1, g_2) \circ l) = (\bar{g}_1, \bar{g}_2) \circ \pi(l) ,$$

where $\bar{g}$ is the image of $g \in \mathsf{G}^{sc}$ in G. We have a representation $\rho = \rho_L$ of $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$ on the space of global sections $H^0(X, L)$.

THEOREM 5.3. — *Let $\lambda \in \mathfrak{X}^*(\mathsf{T}^{sc})$ and $L(\lambda)$ be the associated line bundle on X. Let ρ_λ be the representation of $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$ on $H^0(X, L(\lambda))$. Then:*
- *The representation ρ_λ decomposes with multiplicity one;*
- *Let γ be the dominant weight on T^{sc} (relative to B^{sc}). Let ξ_γ be the associated irreducible representation of G^{sc} and η_γ the irreducible representation of $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$ defined by*

$$\eta_\gamma = \xi_\gamma \times \xi_\gamma^* .$$

Then each irreducible component of ρ_λ is of the form η_γ for some dominant weight γ of T^{sc}. Moreover, if η_γ appears in ρ_λ, then γ has the form

$$\gamma = \lambda - \sum_{i=1}^{r} n_i \lambda_i ,$$

with $n_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, ..., r$.

– *In particular, the restriction of ρ_λ to $\mathsf{G}^{sc} \times \{1\}$ is a sum, with multiplicity, of ξ_γ's with γ as above.*

We now return to the case of nonsplit G and discuss the canonical F-structure on the wonderful compactification X of G as well as the corresponding Galois action on the boundary divisors D_α, for $\alpha \in \Delta$.

A summary can be given as follows: in the Lie algebra model, the Galois group Γ operates on $\mathrm{Gr}(n, 2n)$ in an obvious way and preserves $\mathbb{L}$ and X°. Hence Γ operates on X. Since Γ acts on the parabolic subgroups of G, it also permutes the boundary divisors

$$\sigma(D_\alpha) = D_{\sigma(\alpha)}.$$

Here if $\mathcal{C} = \{\mathsf{P}\}$, then $\sigma(\mathcal{C}) = \{\sigma(\mathsf{P})\}$.

The F-group G' is defined by a homomorphism

$$\theta : \Gamma \to \mathrm{Out}(\mathsf{G}^{sp}) \ ,$$

to the group of outer automorphisms relative to the pair $(\mathsf{B}^{sp}, \mathsf{T}^{sp})$ and $\mathsf{B}^{sp}, \mathsf{T}^{sp}$ chosen as above and both defined over F. We have then an action of Γ on Δ. Moreover,

$$\sigma(D_\alpha) = D_{\sigma(\alpha)} \quad \text{for } \alpha \in \Delta \ .$$

Finally, since G is obtained G' by inner twisting, the two actions of Γ on the classes of maximal parabolics in $\mathsf{G}(E)$ and $\mathsf{G}'(E)$ coincide.

In detail, let $\lambda = \sum_{i=1}^{r} \omega_i$ and V_λ be the corresponding representation with highest weight λ. We first regard V_λ as a module for G'. As such, it is defined over F, since $\lambda \in \mathfrak{X}^*(\mathsf{T}')_F$ (the action of Γ on $\mathfrak{X}^*(\mathsf{T}') = \mathfrak{X}^*(\mathsf{T})$ via θ simply permutes the ω_i). Let $L_{2\lambda}$ be the corresponding line bundle on $\mathsf{B}'\backslash\mathsf{G}'$. Then $W := H^0(\mathsf{B}'\backslash\mathsf{G}', L_{2\lambda})$ has an F-structure W_F. Denote by $\rho = \rho_\lambda$ the absolute representation of G on W. By the Borel-Weil theorem, we may identify $V_{2\lambda}$ with $W_F \otimes_F \bar{F}$, this gives an action of Γ on $V_{2\lambda}$:

$$\sigma(w \otimes a) = w \otimes \sigma(a), \quad \text{for } w \in W_F, a \in \bar{F}.$$

We have the twisted action $\tilde{\sigma} = c(\sigma) \cdot \sigma$ of Γ on $\mathsf{G}(\bar{F})$, the corresponding twisted action on $W \otimes W^*$ is given by

$$\tilde{\sigma} \cdot w = \rho \otimes \rho^*(c(\sigma), c(\sigma)) \cdot \sigma.$$

If we identify $\mathrm{End}(W) = W \otimes W^*$ we see that this action is

$$(5.1) \qquad\qquad \tilde{\sigma}(A) = \rho(c(\sigma))\sigma(A)\rho(c(\sigma))^{-1}.$$

Let $e_W \in \mathrm{End}(W)$ be the identity. Since W is defined over F, $\sigma(e_W) = e_W$ and similarly, $\tilde{\sigma}(e_W)e_W$. Thus e_W is rational for the twisted action.

REMARK 5.4. — The representation ρ is actually a representation of G^{sc}. However, ρ is irreducible and therefore maps Z^{sc} to scalars. It follows that (5.1) is well-defined.

We use the twisted action to define an F-structure on $\mathrm{End}(W)$. We claim then that the map

$$\mathsf{G} \ni g \mapsto \rho(g) \cdot e_W = \rho(g)$$

is F-rational. Consider

$$\begin{aligned}
\rho(c(\sigma)\sigma(g))e_W &= \rho(c(\sigma)\sigma(g)c(\sigma)^{-1})e_W \\
&= \rho(c(\sigma))\rho(\sigma(g))e_W\rho(c(\sigma))^{-1} \\
&= \rho(c(\sigma))\sigma(\rho(g)e_W)\rho(c(\sigma))^{-1}.
\end{aligned}$$

The latter equality holds since e_W is F-rational. Thus

$$\rho(c(\sigma)\sigma(g))e_W = \tilde{\sigma}(\rho(g)e_W).$$

which proves the assertion.

For $g \in \mathsf{G}$, let $\bar{\rho}(g)$ be the image of $\rho(g)$ in $\mathbb{P}(W)$. Then

$$\bar{\rho} : \mathsf{G} \to \mathbb{P}(\mathrm{End}(W))$$

is also F-rational (equivalently, $\bar{\rho}$ is Γ-equivariant for the twisted action). Thus $\bar{\rho}(\mathsf{G})$ and its closure X in $\mathbb{P}(\mathrm{End}(W))$ are Γ-stable. This gives a canonical F-structure for the variety X and on

$$Y := \bigcap_{\alpha \in \Delta} D_\alpha,$$

(the $\mathsf{G} \times \mathsf{G}$-orbit through the image of $v_{2\lambda} \otimes v_{2\lambda}^*$ in $\mathbb{P}(\mathrm{End}(W))$, where $v_{2\lambda}$ is "the" highest weight in $V_{2\lambda}$, see 5.1 of [15]). Moreover, we have an equivariant isomorphism of F-varieties

$$Y \simeq \mathsf{G}/\mathsf{B} \times \mathsf{G}/\mathsf{B}^-.$$

Indeed, the $\bar{F}$-irreducible boundary components D_α of $X \setminus \mathsf{G}$ are permuted by Γ. Each component D_α is $\mathsf{G} \times \mathsf{G}$-stable and in particular stable for the action of the diagonal $\mathrm{Diag}(\mathsf{G})$:

$$A \mapsto \rho(g)A\rho^{-1}(g).$$

From (5.1) we have

$$\tilde{\sigma}(D_\alpha) = \sigma(D_\alpha).$$

Thus Y is invariant for the *standard* action.

We now prove that the $\mathsf{G} \times \mathsf{G}$-action on X is F-rational for the twisted F-structure. For this we need to see that for $x \in X$,

$$\tilde{\sigma}((g_1, g_2))x = \tilde{\sigma}(g_1, g_2)\tilde{\sigma}(x).$$

In fact, this holds for all $y \in \mathbb{P}(\mathrm{End}(W))$. We have to show that

$$\tilde{\sigma}(\rho(g_1)A\rho(g_2)^{-1}) = \rho(\tilde{\sigma}(g_1))\tilde{\sigma}(A)\rho(\tilde{\sigma}(g_2)^{-1}).$$

The left side is

$$\rho(c(\sigma))\sigma(\rho(g_1)A\rho(g_2)^{-1})\rho(c(\sigma))^{-1},$$

and the right side

$$\rho(c(\sigma))\rho(\sigma(g_1))\rho(c(\sigma))^{-1}\rho c(\sigma)(A)\rho(c(\sigma))^{-1}\rho(c(\sigma))\rho(\sigma(g_2))^{-1}\rho(c(\sigma))^{-1} =$$

$$\rho(c(\sigma))\rho(\sigma(g_1))\sigma(A)\rho(\sigma(g_2))^{-1}\rho(c(\sigma))^{-1}.$$

The assertion follows, since σ and $\rho(g)$, $g \in \mathsf{G}$, commute.

The restriction of line bundles to the unique closed $\mathsf{G} \times \mathsf{G}$-orbit Y induces an injection

(5.2) $\mathrm{Pic}(X) \to \mathrm{Pic}(Y).$

Note that since each divisor D_α is $\mathsf{G} \times \mathsf{G}$-stable, the two actions of Γ on X give rise to the same action on $\mathrm{Pic}(X)$. Moreover, since Y is defined over F, the injection (5.2) is Γ-equivariant. Here

$$\mathrm{Pic}(Y) \simeq \mathfrak{X}(\mathsf{T}') \oplus \mathfrak{X}(\mathsf{T}')$$

as Γ-modules. The image of $\mathrm{Pic}(X)$ is exactly the set of pairs $(\lambda, -\lambda)$, where $\lambda \in \mathfrak{X}^*(\mathsf{T}')$, with boundary divisors D_α corresponding to $(\alpha, -\alpha)$. The F-irreducible boundary components are divisors

$$D_J = \sum_{\alpha \in J} D_\alpha,$$

for any Γ-stable subset $J \subset \Delta(\mathsf{G}', \mathsf{T}')$.

6. Heights

6.1. Metrizations. — Here we recall the definitions of (adelically) metrized line bundles and the associated heights.

DEFINITION 6.1. — Let X be a smooth projective algebraic variety over a number field F. A smooth adelic metrization of a line bundle L on X is a family of v-adic norms $\| \cdot \|_v$ on $L \otimes_F F_v$ for all $v \in \mathrm{Val}(F)$ such that

  – for $v \in S_\infty$ one has $\| \cdot \|_v$ is C^∞;
  – for $v \notin S_\infty$ the norm of any local section of L is locally constant in v-adic topology;
  – there exist a finite set $S \subset \mathrm{Val}(F)$, a flat projective scheme (an integral model) $\mathcal{X}$ over $\mathrm{Spec}(\mathcal{O}_S)$ with generic fiber X together with a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that for all $v \notin S$ the v-adic metric is given by the integral model.

EXAMPLE 6.2. — If L is generated by global sections (s_i) and s is a section such that $\mathsf{s}(x) \neq 0$ then

$$\|\mathsf{s}(x)\|_v := \max_i (|\frac{\mathsf{s}_i}{\mathsf{s}}(x)|_v)^{-1}.$$

This defines a v-adic metric on L, which, of course, depends on the choice of the basis (s_i). An adelic metric on L is a collection of v-adic metrics (for all v) such that there

exists an F-rational basis (s_j) of $H^0(X, L)$ with the property that for *all but finitely many* v the v-adic metric on L is defined by means of this basis.

An adelically metrized line bundle $\mathcal{L}$ induces local and global heights: for any local section s of L and any x with $\mathsf{s}(x) \neq 0$ define

$$H_{\mathsf{s},\mathcal{L},v}(x) = \|\mathsf{s}(x)\|_v^{-1}.$$

For $x \in X(F)$ the product formula ensures that the global height

$$H_{\mathcal{L}}(x) = \prod_{v \in \mathrm{Val}(F)} H_{\mathsf{s},\mathcal{L},v}(x)$$

is independent of the choice of s. We write $\mathcal{L} = (L, \|\cdot\|)$ when we want to emphasize that $L \in \mathrm{Pic}(X)$ is adelically metrized.

Let F be a number field and V a finite-dimensional vector space over F. Thus V is the set of F-rational points $V(F)$ of a linear variety V defined over F. For $v \in \mathrm{Val}(F)$ we set $V_v := V \otimes_F F_v$.

Suppose first that v is non-archimedean. Let Λ_v be an $\mathcal{O}_v$ lattice in V_v. We define the norm

$$\|\cdot\|_{\Lambda_v} = \|\cdot\|_v$$

on V_v (associated to Λ_v) as follows. Let $n = \dim V$ and let

$$\mathcal{B}_v := \{\xi_v^{(1)}, \ldots, \xi_v^{(n)}\}$$

be an $\mathcal{O}_v$-module basis of Λ_v. Let $v \in V_v$ have the form

$$v = \sum_{j=1}^{n} a_j \xi_v^{(j)} \ .$$

We set

$$\|v\|_{\Lambda_v} = \max_{1 \le j \le n} (|a_j|_v) \ .$$

We see at once that $\|\cdot\|_{\Lambda_v}$ depends only on Λ_v and not on the particular choice of a basis of $\mathcal{B}_v$.

Now suppose that v is archimedean and let

$$\mathcal{B}_v := \{\xi_v^{(1)}, \ldots, \xi_v^{(n)}\}$$

be a basis of the F_v-vector space V_v. For

$$v = \sum_{j=1}^{n} a_j \xi_v^{(j)}$$

we set

$$\|v\|_{\Lambda_v} = \big(\sum_{1 \le j \le n} |a_j|_v^2 \big)^{1/2}$$

(for v complex, we set $|z|_v^2 = z \cdot \bar{z}$).

6.2. Heights on the canonical compactification. — First we describe the situation for G^{sp}. Recall that the (classes of) irreducible boundary components D_α, $\alpha \in \Delta = \Delta(\mathsf{G}^{sp}, \mathsf{T}^{sp})$, generate $\mathrm{Pic}(X)$. Each divisor class $[D_\alpha]$ has a unique $\mathsf{G}^{sp} \times \mathsf{G}^{sp}$-stable representative - namely this component. We put $L_\alpha = \mathcal{O}_X(D_\alpha)$. This gives us a canonical splitting of the projection

$$\mathrm{Div}(X) \to \mathrm{Pic}(X),$$

which allows us to identify $\mathrm{Pic}(X)$ with the set of all integral linear combinations

$$L = \sum_{\alpha \in \Delta} s_\alpha L_\alpha,$$

and $\mathrm{Pic}(X)_\mathbb{C}$ with formal sums $\sum_{\alpha \in \Delta} s_\alpha L_\alpha$.

We fix an integral model $\mathcal{L}_\alpha$ on the line bundle L_α, for $\alpha \in \Delta$. If $\mathcal{L}'_\alpha$ is another model then the induced integral structures on $\mathcal{L}_{\alpha,v}$, resp. $\mathcal{L}'_{\alpha,v}$ coincide for almost all v. An integral model defines a height function

$$H_\alpha = H_{\mathcal{L}_\alpha} \,:\, X(F) \to \mathbb{R}_{>0}$$

as in Section 6.1. Given an $L = \sum_{\alpha \in \Delta} s_\alpha L_\alpha$, with $s_\alpha \in \mathbb{C}$, we may define a height

$$
\begin{array}{rlc}
H_L \,:\, X(F) & \to & \mathbb{C} \\
x & \mapsto & \prod_\alpha H_\alpha(x)^{s_\alpha}
\end{array}
$$

We make the above construction more precise and explicit, by defining local heights

$$H_{\alpha,v} \,:\, \mathsf{G}(F_v) \to \mathbb{R}_{>0}, \quad \text{for all} \quad \alpha.$$

For each L_α we fix the (unique, up to scalars) F-rational global section $\mathsf{s}_\alpha \in H^0(X, L_\alpha)$, which is $\mathsf{G}^{sp} \times \mathsf{G}^{sp}$-invariant and non-vanishing on G^{sp}. Using the integral structure, put, for $g_v \in \mathsf{G}^{sp}(F_v)$ and $\alpha \in \Delta$,

$$H_{\alpha,v}(g_v) := \|\mathsf{s}_\alpha(g_v)\|_v^{-1} \quad \text{and} \quad H_\alpha := \prod_v H_{\alpha,v}.$$

PROPOSITION 6.3. — *There is a lattice Λ in some representation of G^{sp} with the following property: Let K_f be the stabilizer of Λ_f in $\mathsf{G}^{sp}(\mathbb{A}_f)$. Then for almost all finite $v \in \mathrm{Val}(F)$ and for every*

$$L = \sum_{\alpha \in \Delta} n_\alpha L_\alpha \in \mathrm{Pic}(X), \quad \text{with } n_\alpha \geq 0$$

we have

$$H_{L,v}(k_1 g k_2) = H_{L,v}(g), \quad \text{for all} \ \ g \in \mathsf{G}(F_v), \ \ k_1, k_2 \in \mathsf{K}_v.$$

Proof. — If suffices to consider $L = L_\lambda$, for regular dominant weights λ. We may assume that the action of $\mathsf{G}^{sp} \times \mathsf{G}^{sp}$ on L is defined over F (passing to a multiple of L, see [27], Section 3, Prop. 1.5). Fix a lattice $\Lambda \subset H^0(X, L)_F$, it defines an $\mathcal{O}_F$-integral structure on L. $\qquad\square$

6.3. Local heights. — Let $L = L_\lambda$ be a very ample F-rational line bundle on X and fix a lattice $\Lambda \subset H^0(X, L)$. Over the (fixed) splitting field E, there is a distinguished $\mathsf{s} \in H^0(X, L)$ which is $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$-invariant and non-vanishing on $\mathsf{G}(E)$. Moreover, s can be written as a product of sections $\mathsf{s}_\alpha \in H^0(X, L_\alpha)$, $\alpha \in \Delta$, with support in the E-rational divisor in D_α. Thus, for $\sigma \in \Gamma = \Gamma_{E/F}$ we have

$$\sigma(\mathsf{s}) = c(\sigma) \cdot \mathsf{s}$$

for some $c(\sigma) \in E^\times$. By Hilbert's theorem 90, we may assume, after replacing s by a suitable multiple, that s is fixed by Γ, i.e., s is F-rational.

We have an $\mathcal{O}_E$-integral structure on L over E, induced from $\Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_E$. Over E, we have a decomposition

$$H^0(X, L_\lambda) = \oplus_\gamma (V_\gamma \otimes V_\gamma^*),$$

as $\mathsf{G}^{sc} \times \mathsf{G}^{sc}$-modules. Here the sum is over dominant γ of the form $\gamma = \lambda - \sum_{\alpha \in \Delta} m_\alpha \alpha$, with $m_\alpha \geq 0$. The following lemma is used to calculate the local height integrals.

LEMMA 6.4. — *One may choose S_F large enough, so that for $v \notin S_F$, if $g_v = k_v t_v k_v'$ with $k_v \in \mathsf{K}_v$ and $t_v \in \mathsf{S}(F_v)^+$, then*

$$H_v(g_v) = |\chi_\lambda(t_v)|_w.$$

Here χ_λ is the rational character of T associated with the dominant weight λ.

Proof. — After replacing L_λ by a positive integral multiple we can assume that λ is trivial on the center of G^{sc}. In particular, $\chi_\lambda(t_v) \in F_v^\times$. Since L_λ is $\mathsf{G} \times \mathsf{G}$-linearizable, we have an F-rational representation ϱ of $\mathsf{G} \times \mathsf{G}$ on $H^0(X, L_\lambda)$. Then for $g_v \in \mathsf{G}(F_v)$ we have

$$H_v(g_v) = \|\varrho(g_v)\|_{\Lambda_v}$$

where $\Lambda_v = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_v$.

Let $\Lambda_E \subset H^0(X, L)_E = V_E$ be an "admissible" lattice, i.e., Λ_E is homogeneous with respect to the decomposition of $V = \oplus_\gamma V_\gamma$ into weight spaces. More precisely, the representation of $\mathsf{G}^{sp} \times \mathsf{G}^{sp}$ on V has a $\mathbb{Z}$-form on $\Lambda_\mathbb{Z}$ with $\Lambda_\mathbb{Z}$ a homogeneous lattice in $V_\mathbb{Q}$. We set $\Lambda_{\mathcal{O}_E} := \Lambda_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_E$ and $\Lambda_{\mathcal{O}_F} := \Lambda_{\mathcal{O}_E} \cap V_F$. Let $\mathcal{G}$ be the group scheme structure on the double stabilizer of $\Lambda_{\mathcal{O}_F}$ and $\mathsf{K}_v = \mathcal{G}(\mathcal{O}_v)$, $\mathsf{K}_w = \mathcal{G}(\mathcal{O}_w)$. We choose S_F so that for all $v \notin S_F$ and $w \mid v$

$$\Lambda_w = \Lambda_v \otimes_{\mathcal{O}_v} \mathcal{O}_w.$$

We choose $|\cdot|_w$ so that its restriction to F_v coincides with $|\cdot|_v$. Then the restriction of $\|\cdot\|_{\Lambda_w}$ to V_v is $\|\cdot\|_{\Lambda_v}$ and, for $g \in \mathsf{G}(F_v)$

$$H_v(g_v) = \|\varrho(g_v)\|_{\Lambda_w}.$$

Write, according to Corollary 1.3,

$$g_v = a_w x_v a_w^{-1}$$

with $x_v \in \mathsf{G}'(F_v)$, $a_w \in \mathcal{G}(\mathcal{O}_w) = \mathsf{K}_w$. Then

$$H_v(g_v) = \|\varrho(a_w)\varrho(x_v)\varrho(a_w)^{-1}\|_{\Lambda_w} = \|\varrho(x_v)\|_{\Lambda_w}$$

We set next
$$\mathsf{K}'_v := \mathsf{K}_w \cap \mathsf{G}'(F_v).$$
Recall that $\mathsf{G}(F_v) \subset \mathsf{G}(E_w), \mathsf{G}'(F_v) \subset \mathsf{G}(E_w)$. By Bruhat-Tits theory, we have
$$\mathsf{G}'(F_v) = \mathsf{K}'_v \mathsf{S}'(F_v)^+ \mathsf{K}'_v.$$
Write accordingly, $x_v = k_1 t_v k_2$, with $k_1, k_2 \in \mathsf{K}'_v,\ t_v \in \mathsf{S}'(F_v)^+$. Then
$$H_v(g_v) = \|\varrho(t_v)\|_{\Lambda_w}.$$
Next choose a basis $\mathcal{B} = \{v_\mu\}$ of Λ_w consisting of weight vectors for T_v^{sp}. Thus
$$H_v(g_v) = \max_\mu |\mu(t_v)|_w$$
Each μ has the form
$$\mu = \chi_\lambda \cdot \prod_{\alpha \in \Delta} \alpha^{-m_\alpha},$$
with $m_\alpha \in \mathbb{N}$. Since $t_v \in \mathsf{S}'_v(F_v)^+$, we have $|\alpha(t_v)|_w = |\alpha(t_v)|_v \geq 1$. Hence finally
$$(6.1) \qquad\qquad H_v(g_v) = |\chi_\lambda(t_v)|_v.$$

$\square$

6.4. Local integrals representing heights. — Let D be a central simple algebra of rank m over the number field F. Also let Λ be an arbitrary lattice in D. We set, for each place v, $\mathsf{D}_v = \mathsf{D} \otimes_F F_v$, and if v is non-archimedean, $\Lambda_v = \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_v$. In this subsection, we define a family of norms $\|\cdot\|_{\Lambda_v}$ on D_v, one for each place v of F, subject to a certain compatibility condition.

 - *non-archimedean v:* Choose a basis $\{\xi_1^v, \ldots, \xi_k^v\}$ for D_v with $\xi_i^v \in \Lambda_v = \Lambda \otimes_\mathcal{O} \mathcal{O}_v$ for all i. For $g \in \mathsf{D}_v$, write $g = \sum_i c_i(g)\xi_i^v$ and set

$$\|g\|_v = \|g\|_{\Lambda_v} := \max_{i=1,\ldots,k}\{|c_i(g)|_v\}.$$

   It is easy to see that this norm is right and left $\Lambda_v^\times$-invariant and therefore independent of the choice of the basis.
 - *archimedean v:* Fix a Banach space norm $\|\cdot\|_v = \|\cdot\|_{\mathsf{D}_v}$ on the finite dimensional real (or complex) vector space $\mathsf{D}_v = \mathsf{D} \otimes_F F_v$.

Clearly, for $c \in F_v$ and $g \in \mathsf{D}_v$, we have

$$\|cg\|_v = |c|_v \cdot \|g\|_v.$$

Consequently, for $c \in F$ and $g \in \mathsf{D}$, we have

$$(6.2) \qquad\qquad \prod_v \|cg\|_v = \prod_v \|g\|_v,$$

by the product formula. This is the compatibility condition mentioned above. Define a function Ψ_v as follows:

$$\Psi_v = \begin{cases} \text{characteristic function of } \Lambda_v, & \text{for} \quad v \notin S_\infty, \\ \exp(-\pi \| \cdot \|_v^2) & \text{for} \quad v \in S_\infty. \end{cases}$$

LEMMA 6.5. — *For all $v \in \mathrm{Val}(F)$, all $g_v \in \mathsf{D}_v$ and all s with $\Re(s) > 0$ one has*

$$\int_{F_v^\times} \Psi_v(ag_v)|a|^s \, d^\times a = \zeta_{F,v}(s) \cdot \|g_v\|_v^{-s}.$$

Here $\zeta_{F,v}(s)$ is the v-local Euler factor of the zeta function of F.

For $\Phi_v \in C_c^\infty(\mathsf{D}_v)$ and $\Re(s) > 0$ we set

$$H_v(s, g; \Phi_v) := \zeta_{F,v}(s)^{-1} \int_{F_v^\times} \Phi_v(ag)|a|^s \, d^\times a,$$

where $g \in \mathsf{D}_v$. For $g = (g_v)_v \in \mathsf{D}(\mathbb{A})$ define the *global height function*:

$$H(g) = \prod_{v \in \mathrm{Val}(F)} H_v(g) = \prod_{v \in \mathrm{Val}(F)} \|g_v\|_v.$$

Similarly, if $\Phi = \otimes_v \Phi_v$ is a global Schwartz-Bruhat function on $\mathsf{D}(\mathbb{A})$, we define

$$H(s, g; \Phi) = \prod_{v \in \mathrm{Val}(F)} H_v(s, g_v; \Phi_v).$$

We extend the functional $H(s, g; \cdot)$ to $C_c^\infty(\mathsf{D}(\mathbb{A}))$ by linearity. By the product formula, both $H(g)$ and $H(s, g; \Phi)$, (for $\Phi \in C_c^\infty(\mathsf{D}(\mathbb{A}))$) are well-defined on the projective group of D.

6.5. Complexified height function. —

NOTATION 6.6. — Let $\mathcal{T}$ be the set of all Galois invariant $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta(\mathsf{G}', \mathsf{T}')}$ (c.f. 2.9 for the definition of Galois invariance). The element $\underline{0} \in \mathcal{T}$ is defined by setting all coordinates equal to zero. For $\epsilon \in \mathbb{R}$, the set $\mathcal{T}_\epsilon$ is the set of $\mathbf{s} = (s_\alpha)_\alpha \in \mathcal{T}$ such that $\Re(s_\alpha) > \kappa_\alpha + 1 + \epsilon$, for all α. Starting with an element $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta(\mathsf{G}', \mathsf{T}')}$ and $v \notin S$, we obtain a tuple $\mathbf{s}^v = (s_\vartheta^v)$ indexed by $\Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))$ by setting $s_{r_v(\iota^*(\alpha))}^v = s_\alpha$; this is well-defined. For $\mathbf{s}, \mathbf{t} \in \mathcal{T}$ and $v \notin S$, we set

$$< \mathbf{s}, \mathbf{t} >_v = \sum_{\vartheta \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))} s_\vartheta^v t_\vartheta^v.$$

When there is no danger of confusion, we write $\mathbf{s}$ for $\mathbf{s}^v$. For each subset R of $\mathbb{C}$, we set $\mathcal{T}(R)$ to be the collection of $\mathbf{s} = (s_\alpha)_\alpha$ with $s_\alpha \in R$ for all α.

We go back to Lemma 6.4. For $v \notin S_F$, we have expressed $H_v(g)$ in terms of $\chi_\lambda \in \mathfrak{X}^*(\mathsf{T})$. Write

$$\chi_\lambda = \prod_{\alpha \in \Delta(\mathsf{G}', \mathsf{T}')} \alpha^{n_\alpha}$$

and $\lambda = \sum_{\alpha \in \Delta(\mathsf{G},\mathsf{T})} n_\alpha \alpha$, $n_\alpha \in \mathbb{N}$. We know that $(n_\alpha)_{\alpha \in \Delta(\mathsf{G}',\mathsf{T}')} \in \mathcal{T}$. Now, let $\mathbf{s} := (s_\alpha)_{\alpha \in \Delta(\mathsf{G}',\mathsf{T}')} \in \mathcal{T}$. Suppose $v \notin S_F$, and $g_v \in \mathsf{G}'(F_v)$ is written as $k_v t_v k_v'$ with $t_v \in \mathsf{S}_v'(F_v)^+$ and $k_v, k_v' \in \mathsf{K}_v'$. We define

$$
(6.3) \qquad H_v(\mathbf{s}, g_v) = \prod_{\alpha \in \Delta(\mathsf{G}',\mathsf{T}')} |\alpha(t_v)|_v^{s_\alpha}.
$$

Observe that

$$
(6.4) \qquad H_v(\mathbf{s}, g_v) = \prod_{\vartheta \in \Delta(\mathsf{G}'(F_v),\mathsf{S}_v'(F_v))} |\vartheta(t_v)|_v^{\ell_v(\vartheta)s_\vartheta}.
$$

If we use Lemma 1.3 to identify the groups $\mathsf{G}(F_v)$ and $\mathsf{G}'(F_v)$, we get a complexified local height function on $\mathsf{G}(F_v)$. Let $S = S_F$ and suppose $g = (g_v)_{v \notin S} \in \mathsf{G}(\mathbb{A}_S)$. We define

$$
(6.5) \qquad H_S(\mathbf{s}, g) := \prod_{v \notin S} H_v(\mathbf{s}, g_v).
$$

6.6. Local height integrals I. — Consider the integral of the complexified local height function

$$
(6.6) \qquad \mathcal{J}_v(s) := \int_{\mathsf{G}_v} H_v(\mathbf{s}, g)^{-1} \, dg
$$

and its versions.

THEOREM 6.7. —

1. *For all $v \notin S_\infty$ the integral $\mathcal{J}_v(s)$ is a holomorphic function of $\mathbf{s}$ for $\mathbf{s} \in \mathcal{T}_{-1}$.*
2. *Let v be an archimedean valuation and ∂ any element of the universal enveloping algebra. Then*

$$
\mathcal{J}_{v,\partial}(\mathbf{s}) := \int_{\mathsf{G}(F_v)} \partial(H(\mathbf{s}, g)^{-1}) \, dg
$$

is holomorphic for $\mathbf{s} \in \mathcal{T}_{-1}$.

Proof. — We will only prove the first part; the second part is similar. Locally, every two local integral structures give rise to essentially equivalent height functions; so, we replace the local integral structure so that the resulting height function is invariant under K_v, a good maximal compact subgroup. Let $\underline{\sigma}$ be the vector consisting of the

real parts of the components of $\mathbf{s}$. The local height integral is majorized by

$$\sum_{t\in\mathsf{S}(F_v)^+}\sum_{\omega\in\Omega_v}H(\underline{\sigma},t\omega)^{-1}\mathrm{vol}\,(\mathsf{K}_v t\omega\mathsf{K}_v)$$

$$\ll\sum_{t\in\mathsf{S}(F_v)^+}H(\underline{\sigma},t)^{-1}\delta_B(t)$$

$$=\prod_{\vartheta\in\Delta(\mathsf{G}(F_v),\mathsf{S}_v(F_v))}\sum_{l=0}^{\infty}\delta_B(\check{\vartheta}(\varpi_v^l))H(\underline{\sigma},\check{\vartheta}(\varpi_v^l))^{-1}$$

$$=\prod_{\vartheta\in\Delta(\mathsf{G}(F_v),\mathsf{S}_v(F_v))}\sum_{l=0}^{\infty}q_v^{-(\sigma_\vartheta-\kappa_\vartheta)l\ell(\vartheta)}.$$

The result is now immediate. $\qquad\square$

COROLLARY 6.8. — *In the non-archimedean situation, for each $\epsilon > 0$ there is a constant $C_v(\epsilon)$, such that $|\mathcal{J}_v(\mathbf{s})| \le C_v(\epsilon)$ for all $\mathbf{s} \in \mathcal{T}_{-1+\epsilon}$. In the archimedean situation, for all $\epsilon > 0$ and all ∂ as above, there is a constant $C_v(\partial,\epsilon)$ such that $|\mathcal{J}_{v,\partial}(\mathbf{s})| \le C_v(\partial,\epsilon)$ for all $\mathbf{s} \in \mathcal{T}_{-1+\epsilon}$.*

6.7. The integral of the local height function II. — Let G be a connected algebraic group over F_v, v outside a finite set of places, and $\mathfrak{g}$ its Lie algebra of invariant vector fields. Let X be a smooth equivariant compactification of G. Denote by $D = X\backslash\mathsf{G}$ the boundary. We assume that D is a divisor with strict normal crossings. Let $\mathcal{T}_X$ be the tangent bundle of X. We have a restriction map

$$(6.7) \qquad\qquad H^0(X,\mathcal{T}_X) \to \mathcal{T}_{X,1} = \mathfrak{g}$$

obtained by evaluating a vector field at the neutral element $1 \in \mathsf{G}$. Conversely, given $\partial \in \mathfrak{g}$, there is a unique vector field ∂^X such that for any open subset U of X and for any $f \in \mathcal{O}_X(U)$, $\partial^X(f)(x) = \partial_g f(g.x)|_{g=1}$. The map $\partial \mapsto \partial^X$ is a section of the restriction map. Let $\partial_1,\ldots,\partial_n$ be a basis for $\mathfrak{g}$. Then $\delta := \partial_1^X \wedge \cdots \wedge \partial_n^X$ is a global section of the line bundle $\det\mathcal{T}_X = K_X^{-1}$. Moreover, δ does not vanish on G. Because of these considerations, if we know that K_X^{-1} is ample, Peyre's Tamagawa measure restricts to Weil's Tamagawa measure on $\mathsf{G}(F_v)$.

If μ_v is the local Weil-Tamagawa measure, for v outside of a finite set of places, we have

$$(6.8) \qquad\qquad \mu_v(\mathsf{G}(\mathcal{O}_v)) = \frac{\#\mathsf{G}(k_v)}{q_v^{\dim\mathsf{G}}}.$$

Therefore, we normalize the local measure by the appropriate factor to guarantee that $\mu(\mathsf{G}(\mathcal{O}_v)) = 1$ for almost all v.

Let A be a subset of $\Delta(\mathsf{G},\mathsf{T})$, we set

$$D_A = \bigcap_{\alpha\in A} D_\alpha$$

and

$$D_A^0 = D_A \backslash \left(\bigcup_{\substack{A \subsetneq A'}} D_{A'} \right),$$

for D_α as defined in 5.2. The following theorem is the analog of Theorem 9.1. of [14] in this situation, with the same proof (see also Theorem 3.1. of [17]).

THEOREM 6.9. —   *We have*

$$(6.9) \qquad \int_{\mathsf{G}(F_v)} H_v(\mathsf{s}, g_v)^{-1}\, dg_v = \frac{1}{\#\mathsf{G}(\mathbf{k}_v)} \sum_A \#D_A^0(\mathbf{k}_v) \prod_{\alpha \in A} \frac{q_v - 1}{q_v^{s_\alpha - \kappa_\alpha + 1} - 1}.$$

Proof. — We split the integral along residue classes modulo $\mathfrak{p}_v$. Let $\tilde{x} \in X(\mathbf{k}_v)$, $\mathbf{k}_v$ the residue field of F_v, and $A = \{\alpha \,|\, \tilde{x} \in D_\alpha\}$, so that $\tilde{x} \in D_A^0$.

We can introduce local (étale) coordinates x_α ($\alpha \in A$), and y_β ($\beta \in B$) with $\#A + \#B = \dim X$ around $\tilde{x}$ such that, locally, the divisor D_α is defined by the vanishing of x_α. Then the local Tamagawa measure identifies with the measure $\prod dx_\alpha \times \prod dy_\beta$ on $\mathfrak{p}_v^A \times \mathfrak{p}_v^B$. If dx denotes the fixed measure on $\mathsf{G}(F_v)$, one has the equality of measures on $\mathsf{G}(F_v) \cap red^{-1}(\tilde{x})$:

$$(6.10) \qquad dx = \frac{q_v^{\dim X}}{\#\mathsf{G}(\mathbf{k}_v)} H_v(\rho, x) d\mu_v = \frac{q_v^{\dim X}}{\#\mathsf{G}(\mathbf{k}_v)} \prod_{\alpha \in A} q_v^{(\kappa_\alpha + 1)v(x_\alpha)} \prod dx_\alpha \prod dy_\beta.$$

Consequently,

$$\int_{red^{-1}(\tilde{x})} H_v(\mathsf{s}, x)^{-1} dx$$

$$= \frac{q_v^{\dim X}}{\#G(\mathbf{k}_v)} \int_{\mathfrak{p}_v^A \times \mathfrak{p}_v^B} q_v^{-\sum_{\alpha \in A}(s_\alpha - \kappa_\alpha - 1)v(x_\alpha)} \prod dx_\alpha \prod dy_\beta$$

$$= \frac{q_v^{\dim X}}{q_v^{\#B} \#G(\mathbf{k}_v)} \prod_{\alpha \in A} \int_{\mathfrak{p}_v} q_v^{-(s_\alpha - \kappa_\alpha - 1)v(x_\alpha)} dx_\alpha$$

$$= \frac{1}{\#G(\mathbf{k}_v)} \prod_{\alpha \in A} \frac{q_v - 1}{q_v^{s_\alpha - \kappa_\alpha} - 1}.$$

We have used the identity

$$\int_{\mathfrak{p}_v} q_v^{-sv(x)}\, dx = \sum_{n=1}^{\infty} q_v^{-sn} \text{vol}\left(\mathfrak{p}_v^n \backslash \mathfrak{p}_v^{n+1}\right)$$

$$= \sum_{n=1}^{\infty} q_v^{-sn} q_v^{-n} \left(1 - \frac{1}{q_v}\right)$$

$$= \frac{1}{q_v} \frac{q_v - 1}{q_v^{1+s} - 1}.$$

$\square$

6.8. An application to volumes. — We start with the following lemma in the non-archimedean situation:

LEMMA 6.10. — *Let K_v be a be a good maximal compact subgroup of $(\mathsf{G}(F_v), \mathsf{S}(F_v))$ so that the Cartan decomposition $\mathsf{G} = \mathsf{K}_v \mathsf{S}(F_v)^+ \Omega \mathsf{K}_v$ holds. Normalize measures so that $vol(\mathsf{K}_v) = 1$. Then for all $ad \in \mathsf{S}(F_v)^+ \Omega$, we have*

$$(6.11) \qquad vol\left(\mathsf{K}_v ad \mathsf{K}_v\right) \ll \delta_{\mathsf{B}}(a).$$

As Ω is a finite set, this is an immediate consequence of Lemma 4.1.1 of [38]. Below we need more detailed information on the behavior of the the above volume in the quasi-split situation. The following lemma suffices for our purposes; the analogous statement for simply-connected is classical. For split groups, the lemma is contained in [21].

LEMMA 6.11. — *There exists a constant c, independent of v, such that for all $t \in \mathsf{S}(F_v)^+$, one has*

$$\mathrm{vol}(\mathsf{K}_v t \mathsf{K}_v) \le \delta_B(t)(1 + \frac{c}{q_v}).$$

Proof. — We use Theorem 6.9. The left hand side of (6.9) is obviously equal to

$$\sum_{t \in \mathsf{S}(F_v)^+} H(\mathsf{s}, t)^{-1} \mathrm{vol}(\mathsf{K}_v t \mathsf{K}_v).$$

The comparison of this expression with the right hand side of (6.9) will give an explicit formula for the volume from which our result will easily follow. Clearly the right hand side of (6.9) is equal to

$$(6.12) \qquad \sum_{t \in \mathsf{S}(F_v)^+} \frac{\#D^0_{A(t)}(\mathbf{k}_v)(q-1)^{\#A(t)}}{\#\mathsf{G}(\mathbf{k}_v)} \delta_{\mathsf{B}}(t) H(\mathsf{s}, t)^{-1}.$$

Here $A(t) = \{\delta \in \Delta(\mathsf{G}, \mathsf{S}); \delta(t) = 1\}$; we will suppress dependence on t and simply write A. Comparison gives

$$(6.13) \qquad \mathrm{vol}(\mathsf{K}_v t \mathsf{K}_v) = \frac{\#D^0_A(\mathbf{k}_v)(q-1)^{\#A(t)}}{\#\mathsf{G}(k_v)} \delta_{\mathsf{B}}(t)$$

The algebraic set D_A is the fiber variety $D_A \to \mathsf{G}/\mathsf{P}_A \times \mathsf{G}/\mathsf{P}_A$, with fibers isomorphic to $\overline{\mathsf{M}}_A$, the wonderful compactification of the adjoint group of M_A, i.e. M_A modulo its center. Here M_A is the Levi factor of the parabolic subgroup P_A. Then D^0_A has fibers M_A modulo its center. Consequently,

$$\#D^0_A(\mathbf{k}_v) = (q-1)^{-\#A} \left[\mathsf{G}(\mathbf{k}_v) : \mathsf{P}_A(\mathbf{k}_v)\right]^2 \cdot \#\mathsf{M}_A(k_v).$$

If U_A is the unipotent radical of P_A, we have

$$\frac{\#D^0_A(\mathbf{k}_v)(q-1)^{\#A}}{\#\mathsf{G}(\mathbf{k}_v)} = \left[\mathsf{G}(\mathbf{k}_v) : \mathsf{P}_A(\mathbf{k}_v)\right] (\#\mathsf{U}_A(\mathbf{k}_v))^{-1}.$$

Let W_A be the Weyl group of the Levi factor M_A. By the Bruhat decomposition we have

$$(6.14) \qquad \mathsf{G}(\mathbf{k}_v) = \bigcup_{w \in \mathsf{W}_A \backslash \mathsf{W} / \mathsf{W}_A} \mathsf{P}_A(\mathbf{k}_v) w \left(\mathsf{U}_A(\mathbf{k}_v) \cap w^{-1} \overline{\mathsf{U}}_A(\mathbf{k}_v) w \right),$$

where $\overline{\mathsf{U}}_A$ the unipotent radical opposite to U_A. It follows that

$$(6.15) \qquad \frac{[\mathsf{G}(\mathbf{k}_v) : \mathsf{P}_A(\mathbf{k}_v)]}{\#\mathsf{U}_A(\mathbf{k}_v)} = \sum_{w \in \mathsf{W}_A \backslash \mathsf{W} / \mathsf{W}_A} \frac{\# \left(\mathsf{U}_A(\mathbf{k}_v) \cap w^{-1} \overline{\mathsf{U}}_A(\mathbf{k}_v) w \right)}{\#\mathsf{U}_A(\mathbf{k}_v)}.$$

As there is only one double coset of maximal dimension, we have

$$(6.16) \qquad \frac{[\mathsf{G}(\mathbf{k}_v) : \mathsf{P}_A(\mathbf{k}_v)]}{\#\mathsf{U}_A(\mathbf{k}_v)} \leq 1 + \frac{\# (\mathsf{W}_A \backslash \mathsf{W} / \mathsf{W}_A)}{q_v}.$$

The lemma is now clear for split groups. In the quasi-split case, we need only consider Γ-stable subsets A. For the complex parameters s_α we have the extra assumption that $s_\alpha = s_{\sigma\alpha}$, for $\sigma \in \Gamma$. The proof of the lemma in the quasi-split case is similar, and we omit it. $\qquad \square$

7. Regularization

In this section, for $v \notin S_F$, with a (slight) abuse of notation we identify $\mathsf{G}(F_v)$ and $\mathsf{G}'(F_v)$; this is permissible, in light of Corollary 1.3.

7.1. Integrals from one-dimensional representations. — Let χ be a one-dimensional automorphic representation of G, and $S = S_F$. We proceed to study analytic properties of the integral

$$(7.1) \qquad \mathcal{J}_S(\mathbf{s}, \chi) = \int_{\mathsf{G}(\mathbb{A}_S)} H_S(\mathbf{s}, g)^{-1} \chi(g) \, dg$$

for Galois invariant $\mathbf{s}$.

THEOREM 7.1. — *The product*

$$(7.2) \qquad \prod_{\mathfrak{O}} L(s_{\mathfrak{O}}, \xi_{\mathfrak{O}}(\chi))^{-1} \int_{\mathsf{G}(\mathbb{A}_S)} H_S(\mathbf{s}, g)^{-1} \chi(g) \, dg$$

is holomorphic on $\mathcal{T}_{-\epsilon}$ for some $\epsilon > 0$. The product is over all Galois orbits in $\Delta(\mathsf{G}', \mathsf{T}')$.

Proof. — We have

$$(7.3) \qquad \mathcal{J}_S(\mathbf{s}, \chi) = \prod_{v \notin S} \mathcal{J}_v(\mathbf{s}, \chi_v)$$

where

$$(7.4) \qquad \mathcal{J}_v(\mathbf{s}, \chi_v) = \int_{\mathsf{G}(F_v)} H_v(\mathbf{s}, g_v)^{-1} \chi_v(g_v) \, dg_v.$$

Since G is of adjoint type, the collection of elements $\{\check{\vartheta}(\varpi_v)\}_{\vartheta \in \Delta(\mathsf{G}(F_v), \mathsf{S}_v(F_v))}$ forms a basis for the semigroup $\mathsf{S}_v(F_v)^+$. For any vector $\mathbf{a} = (a_\alpha)_\alpha \in \mathcal{T}(\mathbb{N})$, we set

$$(7.5) \qquad t_v(\mathbf{a}) = \prod_{\vartheta \in \Delta(\mathsf{G}(F_v), \mathsf{S}_v(F_v))} \check{\vartheta}(\varpi_v)^{a_\vartheta}.$$

Write

$$1 + a_v = \sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} q_v^{-\langle \mathbf{a}, \mathbf{s} \rangle_v} \chi_v(t_v(\mathbf{a})) \delta(t_v(\mathbf{a})),$$

and

$$b_v = \sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} q_v^{-\langle \mathbf{a}, \mathbf{s} \rangle_v} \chi_v(t_v(\mathbf{a}))(\mathrm{vol}\,(\mathsf{K}_v t_v(\mathbf{a}) \mathsf{K}_v) - \delta(t_v(\mathbf{a}))),$$

so that, by Cartan Decomposition, we have

$$(7.6) \qquad \mathcal{J}_v(\mathbf{s}, \chi_v) = 1 + a_v + b_v.$$

Observe that

$$1 + a_v = \sum_{\mathbf{a}} q_v^{-\langle \mathbf{a}, \mathbf{s} \rangle_v} \chi_v(t_v(\mathbf{a})) \delta(t_v(\mathbf{a}))$$

$$(7.7) \qquad = \prod_{\vartheta} \left(\sum_{a_\vartheta = 0}^{\infty} \chi_v(\check{\vartheta}(\varpi_v))^{a_\vartheta} q_v^{-(s_\vartheta - \kappa_\vartheta)a_\vartheta \ell(\vartheta)} \right)$$

$$= \prod_{\vartheta} \frac{1}{1 - \chi_v(\check{\vartheta}(\varpi_v)) q_v^{-(s_\vartheta - \kappa_\vartheta)\ell(\vartheta)}}.$$

With Proposition 2.10 in mind, we proceed as follows. Let $\sigma = (\Re(s_\alpha))_\alpha$. Observe that in the definition b_v we may assume $\mathbf{a} \neq \underline{0}$. Since for each $v \notin S$,

$$\left\{ \mathbf{a} \mid \mathbf{a} \neq \underline{0} \right\} = \bigcup_{\vartheta \in \Delta(\mathsf{G}'(F_v), \mathsf{S}'_v(F_v))} \left\{ \mathbf{a}; a_\vartheta^v \neq 0 \right\},$$

we have

$$\sum_{v \notin S} |b_v| \leq \sum_{v \notin S} \sum_{\vartheta} \sum_{a_\vartheta \neq 0} q_v^{-\langle \mathbf{a}, \sigma \rangle_v} \left| (\mathrm{vol}\,(\mathsf{K}_v t_v(\mathbf{a}) \mathsf{K}_v) - \delta(t_v(\mathbf{a}))) \right|$$

$$\ll \sum_{v \notin S} q_v^{-1} \sum_{\vartheta} \sum_{a_\vartheta \neq 0} q_v^{-\langle \mathbf{a}, \sigma \rangle_v} \delta(t_v(\mathbf{a})) \qquad \text{(by Section 6.8)}$$

$$= \sum_{v \notin S} q_v^{-1} \sum_{\vartheta} \left(\sum_{a_\vartheta = 1}^{\infty} q_v^{-(\sigma_\vartheta - \kappa_\vartheta)a_\vartheta \ell(\vartheta)} \right) \prod_{\beta \neq \vartheta} \left(\sum_{a_\beta = 0}^{\infty} q_v^{-(\sigma_\beta - \kappa_\beta)a_\beta \ell(\beta)} \right)$$

$$= \sum_{v \notin S} q_v^{-1} \sum_{\vartheta} \frac{q_v^{-(\sigma_\vartheta - \kappa_\vartheta)\ell(\vartheta)}}{\prod_{\beta}(1 - q_v^{-(\sigma_\beta - \kappa_\beta)\ell(\beta)})}$$

$$\ll \sum_{\vartheta} \sum_{v \notin S} q_v^{-\frac{3}{2}} < \infty.$$

We need to show the existence of a $C > 0$ such that $|1 + a_v| \geq C > 0$ for all v. For this

$$|1 + a_v| \geq \prod_\vartheta \frac{1}{1 + q_v^{-\sigma_\vartheta + \kappa_\vartheta}} \geq \prod_\vartheta \frac{1}{2} \geq \frac{1}{2^r},$$

with $r = |\Delta(\mathsf{G}^{sp}, \mathsf{T}^{sp})|$. In fact, $q_v^{-\sigma_k + \kappa_{\vartheta_k}} \leq q_v^{-\frac{1}{2}} < 1$.

Note that for $\mathbf{s} \in \mathcal{T}_{-\epsilon}$ the estimates are uniform, i.e., the quotient

$$\frac{\prod_{v \notin S} \mathcal{J}_v(\chi)}{\prod_{v \notin S}(1 + a_v)}$$

is holomorphic in $\mathcal{T}_{-\epsilon}$. This finishes the proof of the theorem. $\qquad\square$

7.2. Integrals from infinite-dimensional representations. — Let $\pi = \otimes_v \pi_v$ be an infinite-dimensional automorphic representation of G which is not the automorphic representation associated to an Eisenstein series when $\mathsf{G} = \mathrm{PGL}_2$ (this exceptional case was treated in [**36**]). For $v \notin S_F$, let $\varphi_{v,\pi}$ be the normalized spherical function associated to π_v and set $S = S_F$.

THEOREM 7.2. — *Let $c > 0$ be as in Theorem 4.5. The infinite product*

$$(7.8) \qquad \mathcal{J}_S(\mathbf{s}, \pi) := \prod_{v \notin S} \int_{\mathsf{G}(F_v)} \varphi_{v,\pi}(g_v) H_v(\mathbf{s}, g_v)^{-1} \, dg_v$$

is holomorphic for $\mathbf{s} \in \mathcal{T}_{-c}$.

Proof. — We use the notation of Section 7.1. Set

$$(7.9) \qquad \mathcal{J}_v(\mathbf{s}, \pi) = \int_{\mathsf{G}(F_v)} \varphi_{v,\pi}(g_v) H_v(\mathbf{s}, g_v)^{-1} \, dg_v.$$

By Cartan decomposition,

$$(7.10) \qquad \mathcal{J}_v(\mathbf{s}, \pi) = \sum_{\mathbf{a}} \varphi_{v,\pi}(t_v(\mathbf{a})) H_v(\mathbf{s}, t_v(\mathbf{a}))^{-1} \mathrm{vol}(\mathsf{K}_v t_v(\mathbf{a}) \mathsf{K}_{\mathbf{v}}).$$

Let

$$a_v = \sum_\vartheta q_v^{-s_\vartheta} \mathrm{vol}\, (\mathsf{K}_v \check{\vartheta}(\varpi_v) \mathsf{K}_v) \varphi_{v,\pi}(\check{\vartheta}(\varpi_v))$$

and

$$b_v = \mathcal{J}_v(\mathbf{s}, \pi) - 1 - a_v.$$

Fix an $\epsilon < \frac{1}{2}$. We claim that there is a set S' of places of F and a positive constant C such that $|1 + a_v| \geq C$ for all $v \notin S'$. Since $|1 + a_v| \geq 1 - |a_v|$, we only need to

show that $|a_v|$ is asymptotically bounded away from 1. To see this we use the fact that $\|\varphi_{v,\pi}\|_{\mathrm{L}^\infty} \leq 1$. Hence for v outside a growing finite set S'

$$|a_v| \leq \sum_\vartheta q_v^{-\sigma_\vartheta + \kappa_\vartheta} \to 0,$$

for $\mathbf{s} \in \mathcal{T}_{-\epsilon}$. This implies that one can choose S' such that if $v \notin S'$ and $\mathbf{s} \in \mathcal{T}_{-\epsilon}$ then $|1 + a_v| \geq C$, for a constant C that depends only on ϵ.

Next, we prove that $\sum_v b_v$ is absolutely and uniformly convergent on $\mathcal{T}_{-\epsilon}$. Denote by $b(m)_v$ the set of terms such that $\mathbf{a}^v$ has exactly m nonzero coordinates. We have

$$\sum_v |b(1)_v| \leq \sum_\vartheta \sum_{v \notin S} \sum_{a_\vartheta \geq 2} q_v^{-\sigma_\vartheta a_\vartheta + \kappa_\vartheta a_\vartheta},$$

Note that $a_\vartheta \neq 1$, because otherwise the term would appear in a_v. This implies that

$$\sum_v |b(1)_v| \leq \sum_\vartheta \sum_{v \notin S} \frac{q_v^{-2\sigma_\vartheta + 2\kappa_\vartheta}}{1 - q_v^{-\sigma_\vartheta + \kappa_\vartheta}}$$
$$\ll \sum_\vartheta \sum_{v \notin S} q_v^{-2\sigma_\vartheta + 2\kappa_\vartheta}$$
$$\ll \sum_{v \notin S} q_v^{-2 + 2\epsilon}$$
$$< \infty,$$

for $\epsilon < \frac{1}{2}$.

To continue, we fix an ordering of the elements of $\Delta(\mathsf{G}(F_v), \mathsf{S}_v(F_v))$, say $\{\vartheta_1, \ldots, \vartheta_l\}$. Next we verify the claim for $\sum_{m \geq 2} \sum_v |b(m)_v|$, where

$$b(m)_v = \sum_{1 \leq i_1 < i_2 < \ldots i_m \leq l} \sum_{a_{i_1} > 0} \sum_{a_{i_2} > 0} \cdots \sum_{a_{i_m} > 0} q_v^{-\sum_{j=1}^m a_{i_j} s_{i_j}} \operatorname{vol}(\mathsf{K}_v t_{\mathbf{a}}^{\mathbf{m}} \mathsf{K}_v) \varphi_{v,\pi}(t_{\mathbf{a}}^{\mathbf{m}}).$$

Here $t_{\mathbf{a}}^{\mathbf{m}}$ corresponds to the vector with coordinates a_{i_j} at i_j. Next

$$|b(m)_v| \leq \sum_{1 \leq i_1 < i_2 < \ldots i_m \leq l} \sum_{a_{i_1} > 0} \sum_{a_{i_2} > 0} \cdots \sum_{a_{i_m} > 0} q_v^{-\sum_{j=1}^m a_{i_j}(\sigma_{i_j} - \kappa_{\alpha_{i_j}})}$$
$$\leq \sum_{1 \leq i_1 < i_2 < \ldots i_m \leq l} \prod_{j=1}^m \frac{q_v^{-\sigma_{i_j} + \kappa_{\alpha_{i_j}}}}{1 - q_v^{-\sigma_{i_j} + \kappa_{\alpha_{i_j}}}}$$
$$\ll \sum_{1 \leq i_1 < i_2 < \ldots i_m \leq l} \prod_{j=1}^m q_v^{-\sigma_{i_j} + \kappa_{\alpha_{i_j}}}$$

$$< \sum_{1 \le i_1 < i_2 < \dots i_m \le l} q_v^{m(-1+\epsilon)}$$
$$\ll q_v^{m(-1+\epsilon)}.$$

Since $m \ge 2$ and $\epsilon < \frac{1}{2}$, the series $\sum_v q_v^{m(1-\epsilon)}$ is convergent, and we are done.

The holomorphy of (7.8) follows from the uniform convergence of

$$\Sigma_S = \sum_{v \notin S} |a_v(\mathbf{s})|$$

on compact subsets of $\mathcal{T}$, which we now establish. Fix an $\epsilon' > 0$, and suppose that $\mathbf{s} \in \mathcal{T}_{-c+\epsilon'}$. Then

$$\Sigma_S \le \sum_{v \notin S} \sum_{\vartheta} q_v^{-\sigma_\vartheta} \text{vol}\,(\mathsf{K}_v \check{\vartheta}(\varpi_v) \mathsf{K}_v) |\varphi_v(\check{\vartheta}(\varpi_v)))|$$
$$\ll \sum_{v \notin S} \sum_{\vartheta} q_v^{-\sigma_\vartheta - c} \delta(\check{\vartheta}(\varpi_v)),$$

by Lemma of 6.8 and Theorem 4.5. Finally, since by definition

$$\delta(\check{\vartheta}(\varpi_v))) = q_v^{\kappa_\vartheta \ell(\vartheta)},$$

we conclude that

$$\Sigma_S \ll \sum_{v \notin S} \sum_{\vartheta} q_v^{-\eta-1} < \infty,$$

for some $\eta > 0$. This last inequality completes the proof of the theorem. $\qquad\square$

COROLLARY 7.3 (of the proof). — *For all $\epsilon > 0$ and all compacts $K \subset \mathcal{T}_{-c+\epsilon}$ there exists a constant $C(\epsilon, K)$ such that*

$$|\mathcal{J}_S(\mathbf{s}, \pi)| \le C(\epsilon, K).$$

for all π as above, and all $\mathbf{s} \in K$.

COROLLARY 7.4. — *Let K be as in Corollary 4.1. Let ϕ be an automorphic form in the space of an automorphic representation π which is right invariant under K. Set for $\mathbf{s} \in \mathcal{T}_{\gg 0}$*

$$(7.11) \qquad \mathcal{J}(\mathbf{s}, \phi) := \int_{\mathsf{G}(\mathbb{A})} H(\mathbf{s}, g)^{-1} \phi(g) dg.$$

Then $\mathcal{J}(s, \phi)$ has an analytic continuation to a function which is holomorphic on $\mathcal{T}_{-c}$. Define $\Lambda(\phi)$ by $\Delta \cdot \phi = \Lambda(\phi) \cdot \phi$. Then for each integer $k > 0$, and all $\epsilon > 0$, and every compact subset $K \subset \mathcal{T}_{-c+\epsilon}$, there exists a constant $C = C(\epsilon, K, k)$, independent of ϕ, such that

$$(7.12) \qquad |\mathcal{J}(\mathbf{s}, \phi)| \le C \Lambda(\phi)^{-k} |\phi(e)|$$

for all $\mathbf{s} \in K$.

Proof. — Combine Corollary 7.3, Corollary 4.1, Theorem 6.7 and its corollary. $\qquad\square$

REMARK 7.5. — The value of the spherical function at the element $\check{\vartheta}(\varpi_v)$, for $\vartheta = r_v(\iota^*(\alpha))$ is related to the trace of the dominant weight ω_α of the L-group applied to the Langlands class of π_v, and that the Euler product $\mathcal{J}_S(\mathbf{s}, \pi)$ is regularized by a product of $L_S(s_\alpha, \pi, \omega_\alpha)$. Additional information about these L-functions would lead to better error terms in asymptotics of rational points.

8. Height zeta function

8.1. The zeta function. — The main tool in the study of distribution properties of rational points is the height zeta function, defined on $\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})$ by

$$\mathcal{Z}(\mathbf{s}, g) := \sum_{\gamma \in \mathsf{G}(F)} H(\mathbf{s}, \gamma g)^{-1}$$

PROPOSITION 8.1. — *The series defining $\mathcal{Z}(\mathbf{s}, g)$ converges absolutely to a holomorphic function for $\mathbf{s} \in \mathcal{T}_{\gg 0}$. In its region of convergence*

$$\mathcal{Z}(\mathbf{s}, g) \in C^\infty(\mathsf{G}(F)\backslash\mathsf{G}(\mathbb{A})).$$

Furthermore, $\mathcal{Z}(\mathbf{s}, g)$ and all of its group derivatives are in L^2.

Proof. — It suffices to prove the absolute convergence of $\mathcal{Z}(\mathbf{s}, g)$ for $\Re(s)$ contained in some open cone. This is a general fact (see Proposition 4.4 in [14]): since X is projective the cone generated by ample classes is open in $\mathrm{Pic}(X)_{\mathbb{R}}$. Fix some ample classes L_j generating $\mathrm{Pic}(X)$. The restriction $\mathcal{Z}(sL_j, 1)$ converges for $\Re(s) > \sigma_j$, for some $\sigma_j > 0$. Now use the exponential property of heights. The proof of the last statement is identical to the proof of Proposition 2.3. of [36]. $\qquad\square$

PROPOSITION 8.2. — *The function $F(g) = \mathcal{Z}(\mathbf{s}, g)$ satisfies the conditions of Lemma 3.1, and hence has a spectral expansion.*

Proof. — Obvious from the proof of Theorem 7.1. $\qquad\square$

By Lemma 3.1, the zeta function has an expansion of the form

$$(8.1) \qquad\qquad \mathcal{Z}(\mathbf{s}, g) = S_1(\mathcal{Z}(\mathbf{s}, \cdot), g) + S_2(\mathcal{Z}(\mathbf{s}, \cdot), g).$$

Since the two sides are continuous functions of g, we may set $g = e$ to get

$$(8.2) \qquad\qquad \mathcal{Z}(\mathbf{s}) = S_1(\mathcal{Z}(\mathbf{s}, \cdot), e) + S_2(\mathcal{Z}(\mathbf{s}, \cdot), e).$$

We use this expansion to determine the analytic behavior of the height zeta function. The idea is to separate out the contribution of one-dimensional representations. Since the definition of S_1 involves the Laplace operator, the contribution of one-dimensional representations to this term will cancel (we will see momentarily that because of the uniform convergence of the inner sum the Δ^n can be moved to the Eisenstein series).

It remains to treat the contribution to S_2. Since both sums are of the shape considered in (3.26), without restriction of generality, we set

$$
(8.3) \quad S^\flat(\mathbf{s}) = \sum_{X \in \mathfrak{X}}^{\flat} \sum_{\mathsf{P}} n(\mathsf{A})^{-1}
$$
$$
\int_{\Pi(\mathsf{M})} \int_{\mathsf{G}(\mathbb{A})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(x, \phi) \overline{E(y, \phi)} \right) H(\mathbf{s}, y)^{-1} \, dy \, d\pi,
$$

where the symbol $\flat$ indicates that the summation is over those classes which do not correspond to one-dimensional representations. The fact that the innermost sum is uniformly convergent for y in compact sets is included in the first half of the proof of Lemma 4.4 of [1]. (Note that here too one needs to use Lemma 4.1 of [1]). Therefore, we may interchange the innermost summation with the integral over $\mathsf{G}(\mathbb{A})$ to obtain

$$
(8.4) \quad S^\flat(\mathbf{s}) = \sum_{X \in \mathfrak{X}}^{\flat} \sum_{\mathsf{P}} n(\mathsf{A})^{-1}
$$
$$
\int_{\Pi(\mathsf{M})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(e, \phi) \int_{\mathsf{G}(\mathbb{A})} \overline{E(y, \phi)} H(\mathbf{s}, y)^{-1} \, dy \right) d\pi,
$$

THEOREM 8.3. — *The function $S^\flat$ has an analytic continuation to a function which is holomorphic on $\mathcal{T}_{-c}$.*

Proof. — For simplicity we assume that the height function is invariant under right and left translation by the compact subgroup K as in Corollary 4.1. Keeping notation of Corollary 7.4, we write

$$
(8.5) \quad S^\flat(\mathbf{s}) = \sum_{X \in \mathfrak{X}}^{\flat} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} E(e, \phi) \mathcal{J}(\mathbf{s}, E(\phi, \cdot)) \right) d\pi.
$$

We now use the analytic continuation and bounds established in Corollary 7.4 to obtain the analytic continuation of $S^\flat$. Let K be a compact subset of $\mathcal{T}_{-c}$. Then there is an $\epsilon > 0$ such that $K \subset \mathcal{T}_{-c+\epsilon}$. By Corollary 7.4, we know that for $\mathbf{s} \in K$ and all k the expression

$$
(8.6) \quad \sum_{X \in \mathfrak{X}}^{\flat} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} |E(e, \phi)| \cdot |\mathcal{J}(\mathbf{s}, E(\phi, \cdot))| \right) d\pi
$$

is bounded by

$$
(8.7) \quad C(\epsilon, K, k) \sum_{X \in \mathfrak{X}}^{\flat} \sum_{\mathsf{P}} n(\mathsf{A})^{-1} \int_{\Pi(\mathsf{M})} \left(\sum_{\phi \in \mathcal{B}_{\mathsf{P}}(\pi)_X} \Lambda(\phi)^{-k} |E(e, \phi)|^2 \right) d\pi.
$$

The convergence of the last expression is a consequence of Proposition 3.5. This establishes Theorem 8.3. □

REMARK 8.4. — In the anisotropic situation, the desired analytic properties follow from the analytic continuation of the spectral zeta function of the Laplace operator on the corresponding compact quotient.

8.2. — Now we let $\underline{a} = (a_\alpha) \in \mathcal{T}_\mathbb{N}$, and for $s \in \mathbb{C}$, we set

$$(8.8) \qquad \mathcal{Z}_{\underline{a}}(s) = \mathcal{Z}(s(a_1, a_2, \ldots, a_r))$$

as a function of one complex variable. We need to determine the right most pole of $\mathcal{Z}_{\underline{a}}(s)$. Set

$$(8.9) \qquad \sigma(\underline{a}) = \max_i \frac{1 + \kappa_\alpha}{a_\alpha},$$

and let $S(\underline{a})$ be the set of α, modulo Galois action, for which the maximum is achieved, and $m(\underline{a}) = \#S(\underline{a})$. The theorem implies that $\mathcal{Z}_{\underline{a}}(s)$ has no pole for $\Re(s) > \sigma(\underline{a})$. The order of pole of

$$(8.10) \qquad \int_{\mathsf{G}(\mathbb{A})} H(s\underline{a}, g)^{-1} \, dg$$

at $s = \sigma(\underline{a})$ is equal to $m(\underline{a})$. Therefore, we need those automorphic characters χ such that $\xi_\alpha(\chi) = 1$ for all $\alpha \in S(\underline{a})$. Clearly, we are interested only in those automorphic χ which satisfy

$$(8.11) \qquad \int_{\mathsf{G}(\mathbb{A})} H(s, g)^{-1} \chi(g) \, dg \neq 0$$

for some s in the domain of absolute convergence. This implies that χ is right, and in this case also left, invariant under the compact open subgroup K of $\mathsf{G}(\mathbb{A}_f)$. Let $X(\underline{a})$ be the collection of all such characters. By Lemma 3.1 of [20], the set $X(\underline{a})$ is finite. The proof of the above theorem shows that

THEOREM 8.5. — *The complex function $\mathcal{Z}_{\underline{a}}(s)$ has a meromorphic continuation to $\Re(s) > \sigma(\underline{a}) - \epsilon$, $\epsilon > 0$, with an isolated pole at $\sigma(\underline{a})$ of order $m(\underline{a})$. Furthermore,*

$$\lim_{s \to \sigma(\underline{a})} (s - \sigma(\underline{a}))^{m(\underline{a})} \mathcal{Z}_{\underline{a}}(s) = \lim_{s \to \sigma(\underline{a})} (s - \sigma(\underline{a}))^{m(\underline{a})} \sum_{\chi \in X(\underline{a})} \int_{\mathsf{G}(\mathbb{A})} H(s, g)^{-1} \chi(g) \, dg.$$

The limit is a positive real number.

Proof. — If we set

$$(8.12) \qquad \mathsf{G}_{\underline{a}} = \bigcap_{\chi \in X(\underline{a})} \ker(\chi),$$

then the above limit is equal to

$$(8.13) \qquad \lim_{s \to \sigma(\underline{a})} (s - \sigma(\underline{a}))^{m(\underline{a})} \int_{\mathsf{G}_{\underline{a}}} H(s, g)^{-1} \, dg.$$

Again by Lemma 3.1. of [20], we know that the group $\mathsf{G}_{\underline{a}}$ has finite index in $\mathsf{G}(\mathbb{A})$. This easily implies that the above limit is a positive number. $\qquad\square$

A special case of particular interest is when $a_\alpha = \kappa_\alpha + 1$, for all α. Because of its relevance to the anticanonical class of the wonderful compactification (c.f. Proposition 5.2), we denote this $\underline{a}$ by $\underline{\kappa}$. In this case, $\sigma(\underline{\kappa}) = 1$, and $m(\underline{\kappa})$ is equal to the number of distinct Galois orbits in $\Delta(\mathsf{G},\mathsf{T})$.

PROPOSITION 8.6. — *Suppose χ is an automorphic character such that $\xi_\alpha(\chi) = 1$ for all α. Then $\chi \equiv 1$.*

Proof. — This easily follows from Proposition 2.9, Cartan decomposition, and the weak approximation. $\qquad\square$

The proposition shows that $X(\underline{a}_{-K}) = \{1\}$.

THEOREM 8.7. — *The complex function $\mathcal{Z}_{\underline{\kappa}}(s)$ has a meromorphic continuation to $\Re(s) > 1 - \epsilon$ for some $\epsilon > 0$ with an isolated pole of order equal to the number of Galois orbits in $\Delta(\mathsf{G},\mathsf{T})$ at $s = 1$. Furthermore,*

$$\lim_{s \to \sigma(\underline{\kappa})} (s - \sigma(\underline{\kappa}))^{m(\underline{\kappa})} \mathcal{Z}_{\underline{\kappa}}(s) = \lim_{s \to \sigma(\underline{\kappa})} (s - \sigma(\underline{\kappa}))^{m(\underline{\kappa})} \int_{\mathsf{G}(\mathbb{A})} H(s\underline{\kappa}, g)^{-1}\, dg.$$

8.3. Examples. —

EXAMPLE 8.8 (PGL_n). — Let F_v be a local field, $\mathsf{G} = \mathrm{PGL}_n$ and χ_v a one-dimensional representation of $\mathsf{G}(F_v)$. Then χ_v has the form

$$\chi_v(g) = \xi_v(\det(g)),$$

where ξ_v is a character of F_v^* whose order divides n. In the global situation,

$$\chi(g) = \xi(\det(g)), \quad g \in \mathsf{G}(\mathbb{A}),$$

with ξ of order dividing n, and $\xi|_{F^*} = 1$. Let $\alpha_1, \ldots, \alpha_{n-1}$ be the simple roots, with the convention that

(8.14) $$\alpha_i(\mathrm{diag}\,(a_1, \ldots, a_n)) = a_i/a_{i+1},$$

and

(8.15) $$\check{\alpha}_i(t) = \begin{pmatrix} tI_i & \\ & I_{n-i}. \end{pmatrix}$$

It is not hard to see that

(8.16) $$2\rho = \sum_{i=1}^{n-1} i(n-i)\alpha_i.$$

Let $\underline{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$. We define $\sigma(\underline{a})$, $S(\underline{a})$, $m(\underline{a})$, and $X(\underline{a})$ as above. Suppose $\chi \in X(\underline{a})$. Then if $\chi = \xi \circ \det$, we must have

(8.17) $$\xi^i = 1$$

for all $i \in S(\underline{a})$. In particular, if we set

(8.18) $$d(\underline{a}) = \gcd(n, \gcd_{i \in S(\underline{a})} i),$$

then we obtain that $\chi \in X(\underline{a})$, if and only if $\chi^{d(\underline{a})} = 1$.

EXAMPLE 8.9 (PGL$_4$). — Here

$$2\rho(\mathrm{diag}(a,b,c,d)) = c^3 bc^{-1} d^{-3} = \alpha^3 \beta^4 \gamma^3,$$

if $\Delta = \{\alpha = a/b, \beta = b/c, \gamma = c/d\}$. Thus $\underline{a}_{-K} = (4,5,4)$. For $\underline{a} = (1,1,1)$ we have

$$\sigma(\underline{a}) = \max(4,5,4) = 5 \quad \text{and} \quad S(\underline{a}) = \{\beta\}.$$

We have

$$\check{\beta}(\varpi) = \mathrm{diag}(\varpi, \varpi, 1, 1) \quad \text{and} \quad \chi_v(\mathrm{diag}(\varpi, \varpi, 1, 1)) = \xi_v(\varpi^2).$$

Let ξ have order two. By our analysis, $\chi = \xi \circ \det$ contributes to the asymptotic constant.

EXAMPLE 8.10 (PGU$_3$). — Let E/F be a quadratic extension and $\mathrm{GU}_3 \subset \mathrm{GL}_3(E)$ the set of all g such that

$$^{T} g^{\sigma} S g = \lambda S, \quad \text{where} \quad S = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let $\mathsf{G} = \mathrm{PGU}_3$ and write $\lambda = \nu(g)$ for the similitude norm of $g \in \mathsf{G}$. Then

$$\mathsf{G}^{sc} = \mathrm{SU}_3, \mathsf{Z} = \mathsf{Z}^{sc} = \{\zeta \cdot I_3, \; \zeta^3 = 1\}.$$

The Galois action on G^{sc} is given by

$$\tilde{\rho}(g) = S^T g \rho S^{-1}$$

if $\rho \in \Gamma_{\bar{F}/F}$ restricts to $\sigma \in \Gamma_{E/F}$ and $\tilde{\rho}(g) = g^{\rho}$ if $\rho \in \Gamma_{\bar{F}/E}$.

We claim that $j : \mathsf{T}^{sc}(F_v) \to \mathsf{T}(F_v)$ is not surjective.

We have

$$\mathsf{T}^{sc}(F_v) = \{\mathrm{diag}(a, \epsilon, a) \; a \in E_w^*\}$$

and $aa^{-\sigma}\epsilon = 1$, with $\mathrm{N}(\epsilon) = 1$, whereas

$$\mathsf{T}(F_v) = \{\mathrm{diag}(a, b, \lambda a^{-\sigma}), \quad \text{modulo scalars }\}$$

with $\lambda = b^{\sigma} b$. We can take $t \in \mathsf{T}(F_v)$ to have the form

$$\mathrm{diag}(a, 1, a^{-\sigma}).$$

Finally,

$$j(\mathrm{diag}(a, 1, a^{-\sigma})) = \mathrm{diag}(a\epsilon^{-1}, 1, a^{-\sigma}\epsilon^{\sigma}) = \mathrm{diag}(u, 1, u^{-\sigma}),$$

modulo scalars. One-dimensional automorphic representations of $\mathsf{G}(\mathbb{A})$ are given by

$$\chi(g) = \xi(\nu(g)).$$

We need to consider the range of ν.

If v remains prime in E, then

$$^T g_v^\sigma S g_v = \lambda_v \cdot S$$

which implies $\lambda_v^3 = \mathrm{N}_{w/v}(\det(g_v))$. Thus $\lambda_v \in \mathrm{N}_{w/v}(E_w^*)$. If v splits in E, then there is no condition on λ_v. Thus

$$\nu \, : \, \mathsf{G}(\mathbb{A}) \to \mathrm{N}_{E/F}(\mathbb{A}_E^*).$$

Also $\nu(\alpha_v \cdot I_3) = \mathrm{N}_{W/V}(\alpha_v)$ if v remains prime in E.

We may take for the adjoint group U_3/Z, where $\mathsf{Z} = \mathrm{diag}(\epsilon, \epsilon, \epsilon)$ and $\mathrm{N}_{E/F}(\epsilon) = 1$. We have $\chi(g) = \xi(\det(g))$, $g \in \mathsf{G}(\mathbb{A})$. Here $\xi \in \mathbb{N}^1(\mathbb{A})/\mathbb{N}^1(F)$, $\xi^3 = 1$. By Hasse's theorem,

$$\begin{array}{ccc}
\mathbb{G}_m(\mathbb{A}_E) & \to & \mathbb{N}^1(\mathbb{A}) \\
\mathbb{G}_m(E) & \to & \mathbb{N}^1(E) \\
x & \mapsto & x \cdot x^{-\sigma}
\end{array}$$

are surjective. We may view ξ as a character η of $\mathbb{G}_m(\mathbb{A}_E)/\mathbb{G}_m(E)$ with $\eta^3 = 1$. The map

$$j \, : \, \mathsf{T}^{sc}(F_v) \to \mathsf{T}(F_v)$$

is not surjective, since $\mathsf{T}(F_v)$ is the set of all $\mathrm{diag}(a, 1, a^{-\sigma})$ and $\mathsf{T}^{sc}(F_v)$ the set of $\mathrm{diag}(a, \epsilon, a^{-\sigma})$, with $aa^{-\sigma}\epsilon = 1$.

We wish to define

$$L(s, \chi) = L_E(s, \eta).$$

This leads to the necessity of introducing a norm map. In this example we have

$$\begin{array}{ccc}
\mathrm{N}_{w/v} \, : \, \mathsf{T}(E_w) & \to & \mathsf{T}(F_v) \\
t & \mapsto & t \cdot \tilde{\sigma}(t)
\end{array}$$

with $\tilde{\sigma}(t) = S t^\sigma S^{-1}$. Thus

$$\mathrm{N}_{w/v}(\mathrm{diag}(a, b, c) = \mathrm{diag}(ac^{-\sigma}, bb^{-\sigma}, ca^{-\sigma}).$$

Note that

$$1 \to \mathrm{SU}_v/\mu_3 \to \mathsf{U}_3/\mathsf{Z} \xrightarrow{\det} \mathbb{N}_v^1/(\mathbb{N}_v^1)^3 \to 1$$

is exact.

9. Manin's conjecture

In this section we apply Theorem 8.7 to obtain a result regarding the number of rational points of bounded height on wonderful compactifications of the group G.

We first recall a standard Tauberian theorem.

THEOREM 9.1. — *Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers such that for any $B > 0$ only finitely many $c_n < B$. Let f be a function with values in $\mathbb{R}_{>0}$ such that*

$$Z(s, f) = \sum_n \frac{f(n)}{c_n^s}$$

is absolutely convergent for $\Re(s) > a > 0$ and has a representation

$$Z(s, f) = \frac{1}{(s-a)^b} g(s) + h(s)$$

with $g(s)$ and $h(s)$ holomorphic for $\Re(s) \geq a$, $g(a) \neq 0$ and $b \in \mathbb{N}$. Then

$$\sum_{n \,:\, c_n \leq B} f(n) = \frac{g(a)}{a(b-1)!} B^a (\log B)^{b-1} (1 + o(1)),$$

as $B \to \infty$.

THEOREM 9.2. — *Let X be the compactification of a semi-simple group G of adjoint type over F as in Section 5 and $\mathcal{L} = (L, \|\cdot\|_v)$ an adelically metrized line bundle such that its class $[L] \in \mathrm{Pic}(X)$ is contained in the interior of the cone of effective divisors $\Lambda_{\mathrm{eff}}(X)$. Then*

$$N(\mathsf{G}, \mathcal{L}, B) := \#\{x \in \mathsf{G}(F) \,|\, H_{\mathcal{L}}(x) \leq B\} = \Theta(\mathcal{L}) B^{a(L)} \log(B)^{b(L)-1} (1 + o(1))$$

as $B \to \infty$. Here

$$a(L) = \inf\{a \mid a[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}$$

(where K_X is the canonical line bundle of X) and $b(L)$ is the (maximal) codimension of the face of $\Lambda_{\mathrm{eff}}(X)$ containing $a(L)[L] + [K_X]$. Moreover, $\Theta(-\mathcal{K}_X)$ is the constant defined in [**29**].

Proof. — We combine the following facts:

- $\mathrm{Br}(X)/\mathrm{Br}(F) = 1$;
- $\overline{G(F)} = G(\mathbb{A})$;
- $\int_{G(\mathbb{A})} \omega_S = \int_{X(\mathbb{A})} \omega_S$ (where ω_S is the Tamagawa measure defined in [**29**];
- $\omega_v = \tau_v(G)^{-1} H_{-K_X}(\cdot) dg_v$, for all $v \in \mathrm{Val}(F)$.

The first two facts are in [**34**], here it is important that G is of adjoint type. For the identifications of the measures see [**29**], [**6**], Theorem 7.2 or [**7**]. □

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