# Part II

# Matching of plane curves

Often two-dimensional shapes are given by the planar curves forming their boundaries.

**Definition 3.9 (Curve, Closed curve).** *Let $\mathbb{S}^0$ denote the unit interval $[0,1]$, and $\mathbb{S}^1$ denote the unit circle $\mathbb{R} \mod 2\pi$.*

1. *A continuous mapping $f : \mathbb{S}^0 \to \mathbb{R}^2$ is called a* curve. *A* polygonal curve *is a curve $P$, such that there exists an $n > 0$, and for all $0 \leq i < n$ each $P_i := P|_{[i/n,(i+1)/n]}$ is affine, i.e., $P(i/n + \lambda/n) = (1-\lambda)P(i/n) + \lambda P((i+1)/n)$ for all $\lambda \in [0,1]$. The set of all curves will be denoted by $\mathcal{K}^0$.*

2. *A continuous mapping $f : \mathbb{S}^1 \to \mathbb{R}^2$ is called a* closed curve. *A* closed polygonal curve *is a closed curve $P$, such that there exists an $n > 0$, and for all $0 \leq i < n$ each $P_i := P|_{[i/2\pi n,(i+1)/2\pi n]}$ is affine. The set of all closed curves will be denoted by $\mathcal{K}^1$.*

The problem of measuring the similarity between two shapes then directly leads to the problem of comparing two planar curves. There are several possible distance measures to assess the 'resemblance' of the shapes, and there are also different kinds of transformations that are allowed to match them. In this Part, we will focus on the *Fréchet distance* $\delta_F$ and on the Hausdorff distance $\delta_H$ (c.f., Definition 1.4 on page 7) for polygonal curves.

**Definition 3.10 (Fréchet distance, weak Fréchet distance).** *Let $P : \mathbb{S}^k \to \mathbb{R}^2$ and $Q : \mathbb{S}^k \to \mathbb{R}^2$ with $k \in \{0,1\}$ be (closed) curves. Then $\delta_F(P,Q)$ denotes the* Fréchet distance *between $P$ and $Q$, defined as*

$$\delta_F(P,Q) := \inf_{\substack{\alpha : \mathbb{S}^k \to \mathbb{S}^k \\ \beta : \mathbb{S}^k \to \mathbb{S}^k}} \max_{t \in \mathbb{S}^k} ||P(\alpha(t)) - Q(\beta(t))||,$$

*where $\alpha$ and $\beta$ range over all continuous and* increasing[b] bijections on $\mathbb{S}^k$. *If we only require the reparametrizations $\alpha$ and $\beta$ to be continuous bijections (with the additional constraint that $\alpha(0) = 0$, $\alpha(1) = 1$, $\beta(0) = 0$, and $\beta(1) = 1$ in the case of curves), we get a distance measure that is called the* weak Fréchet distance *between $P$ and $Q$, and is denoted by $\tilde{\delta}_F(P,Q)$.*

As a popular illustration of the Fréchet distance between two curves, suppose a man is walking his dog, he is walking on one curve the dog on the other. Both are allowed to control their speed, but are not allowed to go backwards. Then the Fréchet distance of the curves is the minimal length of a leash that is necessary. In the case of two closed curves, both (man and dog) are not only allowed to control their speed but also to choose their starting point on the curve, in order to minimize the length of the leash. If both are also allowed to go backwards, we get the weak Fréchet distance between the two curves.

For any two curves $P$ and $Q$ the following inequality holds:

$$\delta_H(P,Q) \leq \tilde{\delta}_F(P,Q) \leq \delta_F(P,Q),$$

but as shown in Figures 3.1 and 3.2 neither the ratio between $\delta_H$ and $\delta_F$, nor the ratio between $\tilde{\delta}_F$ and $\tilde{\delta}_F$ is bounded in general. In chapter 5 however, we will show that for a

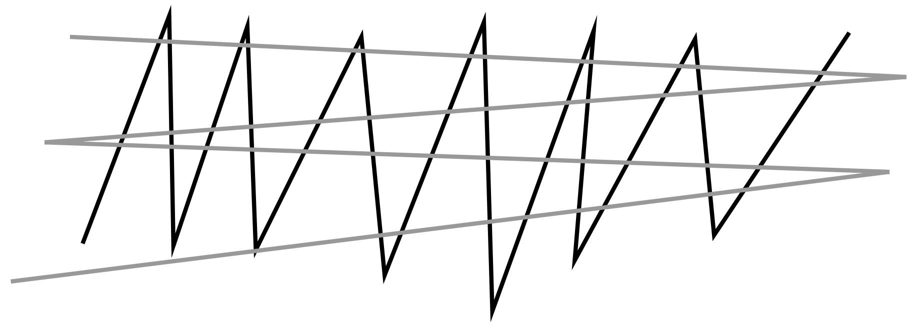

Figure 3.1: Two curves with small Hausdorff distance having a large Fréchet distance.

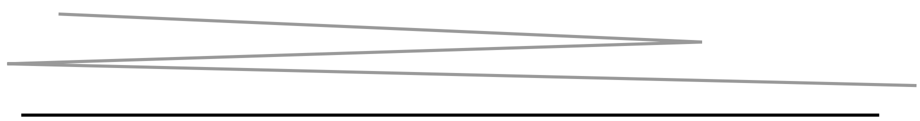

Figure 3.2: Two curves with small weak Fréchet distance distance having a large Fréchet distance.

certain class of curves there is a close relationship between the Fréchet and the Hausdorff distance (c.f. Theorem 5.3 on page 44). These curves are defined with respect to a real valued parameter $\kappa \geq 1$, which – roughly speaking – measures how much they 'resemble' a straight line; they will be called *$\kappa$-straight*. Our result gives rise to a randomized approximation algorithm that computes an upper bound on the Fréchet distance between two such curves that is off from the exact value by a multiplicative factor of $(\kappa + 1)$. The algorithm runs in $\mathcal{O}((m+n)\log^2(m+n)2^{\alpha(m+n)})$ time for given polygonal curves $P, Q$ with $m$ and $n$ vertices (c.f., Corollary 5.8 on page 49). We also provide an $\mathcal{O}(n \log^2 n)$ time algorithm to decide for any $\kappa \geq 1$, if a given polygonal curve on $n$ vertices is $\kappa$-straight (c.f. Theorem 5.9 on page 49).

In chapter 4 we will present exact and approximation algorithms to compute a *translation* which, when applied to the first curve, minimizes the (weak) Fréchet distance to the second one. To be more precise, we describe an algorithm that solves the corresponding decision problem in $\mathcal{O}\big((mn)^3(m+n)^2\big)$ ($\mathcal{O}\big((mn)^3\big)$ in case of $\tilde{\delta}_F$) time (c.f., Theorem 4.23 and Theorem 4.24 on page 38). We complement the exact solution with an $\mathcal{O}(\epsilon^{-2}mn)$ time approximation algorithm that does not necessarily compute the optimal transformation, but one that yields a Fréchet distance which differs from the optimum value by a factor of $(1+\epsilon)$ only, thus trading exactness for feasibility (c.f., Theorem 4.29 on page 40). The approximation algorithm is based on the concept of *reference points*; we conclude chapter 4

[b]A bijection $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is called *increasing*, if $\alpha|_{[0,\alpha^{-1}(0)]}$ and $\alpha|_{[\alpha^{-1}(0),2\pi]}$ are increasing (considered as functions on $[0, 2\pi]$).

with a negative result that rules out the existence of such reference points for affine maps (c.f., Theorem 4.31).

The results in this Part have partially been obtained in collaboration with Helmut Alt and Carola Wenk (chapters 4 and 5), and with Pankaj Agarwal, Rolf Klein, and Micha Sharir (chapter 5). Some of the material has already been published in [16], [17], and [1].