Lax Algebras — A Scenic Approach

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Introduction

In this thesis we propose *scenes* as a setting for the study of *lax algebras*.

Lax algebras are a 2-categorical generalization of Eilenberg–Moore algebras for a monad. It is well-known that categories of Eilenberg–Moore algebras conveniently describe algebraic categories. Lax algebras provide a framework for studying different *topological* categories simultaneously.

The best way to illustrate this might be to recall the motivating example of [Bar70].

In [Man69] Manes showed that the category $\mathbf{Set}^{\mathsf{U}}$ of Eilenberg–Moore algebras for the ultrafilter monad $\mathsf{U} = (U, e, m)$ on the category **Set** of sets and functions is the category of compact Hausdorff topological spaces and continuous maps. Intuitively, one can understand his result as follows. Recall that in a compact Hausdorff topological space every ultrafilter has a unique convergence point. Thus each compact Hausdorff topological space with underlying set X gives rise to a function $c : UX \longrightarrow X$, which sends each ultrafilter to its convergence point. Remarkably, the functions c which arise in this way are precisely those which satisfy $c \cdot e_X = 1_X$ and $c \cdot Uc = c \cdot m_X$, that is, those which turn (X, c) into an Eilenberg–Moore algebra for U.

For an arbitrary topological space we loose existence and uniqueness of convergence points, but we still have a convergence *relation* $k \subset UX \times X$. Barr [Bar70] characterized the relations k that arise as convergence relation of a topological space as those which satisfy

$$1_X \subset k \cdot e_X \quad \text{and} \quad k \cdot \tilde{U}k \subset k \cdot m_X. \tag{I.1}$$

Here $\tilde{U}$ is an extension of the functor U to the category **Rel** of sets and relations. Of course, the inclusion (I.1) are meaningful for *any* monad U on **Set** which has an extension to **Rel**. Hence, it is now possible to define *relational algebras* for any monad on **Set** with an extension to **Rel**.

For a while, Barr's result sparked only modest interest. In [Bun74] the question of how to construct extensions of monads is addressed; [Kam74] and [Möb81] generalize Barr's definition to relations in regular categories and to relations with respect to a suitable factorization system in a category. But the development of the theory was hindered by the fact that there was a certain lack of examples: essentially the only other category which was presented as a category of lax algebras was the category **Ord** of (pre-)ordered sets and monotone functions, which stems from the identity monad on **Set**.

But ultrafilter convergence was used profitably in [RT94] to describe effective descent morphisms of topological spaces, and in [Pis99] to characterize exponentiable topological spaces. This led Clementino and Hofmann [CH03] to start a new line of research

on relational algebras.

The setting of [CH03] is no longer a category together with a notion of relation derived from it by, for instance, a factorization system. Rather, it is a category which sits inside a 2-category with thin hom-categories, that is, between any parallel pair of 1-cells there is at most one 2-cell. This generalization opened the possibility to include Lawvere's generalized metric spaces [Law73], Lowen's approach spaces [Low97], as well as quasi-uniform spaces among the examples of relational algebras in this generalized sense; or *lax algebras*, as the objects of study were subsequently called.

In [CH03] some attention is payed to structures which only need to satisfy the left hand inequality of (I.1), and to structures which do not satisfy any inequality of (I.1) at all. In general, the so-defined supercategories of the category of lax algebras have better convenience properties. It is thus possible to perform many constructions in the supercategories, and then reflect them down to the category of lax algebras in question.

Incorporating metrics and approach structures was facilitated by using the matrix construction of [BCSW83], where the matrices take values in a quantale $\mathbf{V}$. In subsequent articles, as for example, [CHT03b], [CHT04], and [CH04a], the setting is specialized to the category $\mathbf{V}$-matrices.

The setting

The work that is presented here can be seen as a continuation of [CH03]. Indeed, the setting of [CH03] is essentially equivalent to what we choose to call *pre-scenes*. This setting has the advantage that it is general enough to capture all the topological examples of lax algebras that were studied so far. At the same time our setting is so restricted that we do not have to bother with coherence-questions, cf. [CT03]. But we loose the possibility to capture (non-thin) categories and multi-categories, as in [CT03] and [CHT03b].

A pre-scene can be seen as an ordered category with a class of maps with chosen right adjoints. That is, we will work in a category $\mathbf{O}$ enriched over the cartesian closed category **Ord**. In such a category, we can speak about (right) adjoints: $f^* : Y \longrightarrow X$ is right adjoint to $f : X \longrightarrow Y$ in $\mathbf{O}$ if $1_X \leq f^* \cdot f$ and $f \cdot f^* \leq 1_Y$ hold in $\mathbf{O}$. We call $f \in \mathbf{O}$ a *map* if there exists some f^* right adjoint to f. The notion, due to Lawvere, is motivated by the following observation: in the ordered category **Rel** of sets and relations, a morphisms has a right adjoint if and only if it is the graph of a function.

For studying lax algebras, we find it convenient to introduce some more structure on pre-scenes, leading to *scenes*. These scenes, together with an appropriate notion of morphism, are the basis for our notion of lax algebras.

We are now going to briefly describe the contents of this work:

Overview

Chapter 1 This chapter is an introduction to order-enriched category theory. Since any order-enriched category is in particular a 2-category, we can use the language and tools from higher-dimensional category theory. For the reader's convenience and to settle the notation, we include all the necessary details.

After introducing basic definitions concerning order-enriched (or ordered) categories, we define different types of morphisms of ordered categories and construct the free ordered category over a category with specified relations between parallel morphisms.

We define what we call a *map* in an ordered category and demonstrate some simple properties of maps. After that, we introduce the notion of a *Beck–Chevalley square* and some *modularity laws* which are crucial to a more refined study of lax algebras.

Kock–Zöberlein monads are introduced and some examples are discussed. Then we recall distributive laws between monads and introduce quantaloids and biproducts.

We present two important classes of examples of ordered categories: categories of relations for a regular category and matrices with values in a quantale. Using these examples we illustrate the notions introduced so far.

We define *pre-scenes* and associate to any pre-scene $\mathfrak{O}$ an ordered category $\mathrm{Cat}\,\mathfrak{O}$ of $\mathfrak{O}$-*categories* and $\mathfrak{O}$-*functors*. As a first example, we obtain the category **Ord** of ordered sets and monotone functions as $\mathrm{Cat}\,\mathfrak{O}$ for a simple pre-scene $\mathfrak{O}$. Taking **Ord** as a guiding example, we develop some basic notions of $\mathfrak{O}$-categories.

Suitable monads on **Ord** give rise to pre-scenes by the Kleisli-construction. As an example, we consider an extension of the familiar filter-monad on **Set** to **Ord**. For this monad, the categories of the corresponding pre-scene are precisely the topological spaces. Then we introduce a notion of fuzzy powerset with values in a quantale $\mathbf{V}$ and show that the category of $\mathbf{V}$-matrices considered before is the Kleisli-category for the $\mathbf{V}$-valued fuzzy powerset monad.

Chapter 2 Here we introduce *scenes* as a natural environment to study lax algebras, and present some examples, like $(\mathbf{C}, \mathrm{Rel}\,\mathbf{C})$ for a regular category $\mathbf{C}$ and $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ for a quantale $\mathbf{V}$, as well as *scenic functors* or morphisms of scenes.

Lax algebras will be defined with respect to a scene $\mathcal{S}$ and an *actor* on it, that is, a scenic functor $\mathcal{S} \longrightarrow \mathcal{S}$. This actor takes the role of (the extension of) the ultrafilter functor in Barr's example.

After a discussion of *taut* [Man02] functors and the filter-functor, we describe various ways of constructing actors by extending a base-functor.

One method of construction for extensions of functors, which has not been used so far in the lax algebra-community, is the use of distributive laws to construct extensions to Kleisli categories. We give an elementary proof — which we could not find in the literature — of the correspondence between distributive laws and extensions.

In the last section of Chapter 2 we fix some useful notation concerning scenes.

Chapter 3 The central topic of this thesis is introduced: the category $\mathbf{Alg}\,T$ of lax algebras for an actor T on a scene. We discuss some examples and show that the canonical forgetful functor of a category of lax algebras is both a fibration and an opfibration, thus is a *fibering* in the sense of [Man67]. We recall some basic facts about fiberings.

Using the free ordered category construction from Chapter 1, we prove a representation theorem for fiberings: Every fibering arises as a category of lax algebras. Unfortunately, this representation theorem is of little use. We are not aware of a single fact about any fibering which can be proved with its help.

For this reason we now start to study subcategories of $\mathbf{Alg}\,T$ defined with the help of a monad structure $\mathsf{T} = (T, e, m)$ on T. This leads us to a string

$$\mathbf{Alg}_t\mathsf{T} \longhookrightarrow \mathbf{Alg}_u\mathsf{T} \longhookrightarrow \mathbf{Alg}_r\mathsf{T} \longhookrightarrow \mathbf{Alg}\,T \tag{I.2}$$

of concrete full inclusions. We present the paradigmatic example of Barr [Bar70] and its extension due to Clementino and Hofmann [CH03]: For T the suitably extended ultrafilter monad on $\mathbf{Set}$, diagram (I.2) becomes

$$\mathbf{Top} \longhookrightarrow \mathbf{PrTop} \longhookrightarrow \mathbf{PsTop} \longhookrightarrow \mathbf{URS}.$$

Here $\mathbf{Top}$ is the category of topological spaces, $\mathbf{PrTop}$ and $\mathbf{PsTop}$ are the categories of pre- and pseudo-topological spaces, respectively, and $\mathbf{URS}$ is the category of ultra-relational spaces.

After presenting some examples of lax algebras, we continue to study the embeddings in (I.2) and give conditions under which they are reflective.

We introduce strict and costrict morphisms of lax algebras, which generalize proper and open continuous functions between topological spaces.

We continue with a description of quotients in some of the categories of (I.2). This leads to some results on descent morphisms in the above categories, by methods which are an adaption of the ones used in [CH04a]. As a corollary, we obtain that proper and open surjections in $\mathbf{Top}$ are effective descent morphisms.

Then we describe how concrete functors between categories of lax algebras are induced by transformations between their actors.

Chapter 4 In this chapter we present more examples of categories of lax algebras. For a suitable monad T on $\mathbf{Ord}$, we compare the categories of the Kleisli pre-scene of T to the lax algebras for the extension of T to the scene of distributors.

We make explicit the connection between Zhang's *tower extensions* and lax algebras defined over the scene $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ for quantale $\mathbf{V}$: we show that lax algebras over $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ provide an intrinsic description of the tower extension "by $\mathbf{V}$" of algebras over $(\mathbf{Set}, \mathbf{Rel})$.

We finally present the first "non-generic" example of this chapter, namely topological spaces via filter convergence.

Using our results on tower extensions, we present (pre-)approach and probabilistic topological spaces as lax algebras via filter convergence. Following an idea of Seal [Sea05b], we show how closure spaces arise as categories for a pre-scene. Next, we discuss how algebraic and grounded closure spaces arise as lax algebras. We close this chapter with some results on (quasi-)uniform spaces.

Chapter 5 We continue with a study of exactness properties of categories of lax algebras. First, we study the representability of partial morphisms in categories of lax algebras. This can indeed be seen as a (very weak) exactness property, cf. [AHS04].

Using the strict and costrict morphisms introduced in Chapter 3, we recover the results of Brown on representability of closed- and open-embedding partial morphisms in **Top**. Finally, using results of Dyckhoff and Tholen, we use our results on partial morphisms to deduce Niefield's result on exponentiability of locally-closed embeddings in **Top**.

Chapter 6 We extend the material from [MST06] on the structure of coproducts in categories of lax algebras; coproducts are characterized using costrict morphisms. Thus, again costrict morphisms play a crucial role. Under additional completeness conditions, we construct a functor from $\mathbf{Alg}\,T$ to the category **Loc** of locales. Here we associate to every lax algebra its frame of costrict subalgebras. In case one works over **Set**, this yields a concrete, coproduct-preserving functor into **Top**.

Using the above-mentioned characterization of coproducts in categories of lax algebras, we give conditions for the extensivity of these categories.

We construct, under suitable conditions, partial products in $\mathbf{Alg}\,T$, and thus establish that $\mathbf{Alg}\,T$ is locally cartesian closed.

The chapter ends with some remarks on quasitopoi of lax algebras.

Chapter 7 We use the approach of Clementino, Giuli, and Tholen to "topology in a category" to study compactness of lax algebras. Under suitable conditions, every category of lax algebras comes with a canonical topology, given by the strict morphisms. Often this topology turns out to be what we choose to call *stable*, that is, closed under wide pullbacks. This allows us to prove (infinitary) Tychonoff and Frolík-type theorems for these lax algebras. As a corollary we obtain the well-known results about compactness of topological spaces and about 0-compactness of approach spaces. The material in this chapter is an extension of [Sch05].

We use closedness of strict embeddings under intersection to define a categorical closure operator on $\mathbf{Alg}\,T$.

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Conventions

The end of a proof will be marked by ■. In case we do not see any need for a detailed proof, either because it would consist out of straightforward calculations or it is included in the preceding discussion, ■ will directly follow the statement of the result.

For categorical terminology we refer to [AHS04].

1 Ordered categories

Convention An *ordered set* will always be a set equipped with a reflexive and transitive relation. Hence ordered set and thin small category are synonymous terms. The category of ordered sets and monotone functions will be denoted by **Ord**. We will freely use the notions and notation introduced in [Woo04], which gives a beautiful introduction to ordered sets.

Given x, y in an ordered set $(X, \leq)$ we write

$$x \approx y \quad \text{if and only if} \quad x \leq y \text{ and } y \leq x.$$

An ordered set is called *separated* (or *antisymmetric*, *skeletal*) if $x \approx y$ implies $x = y$.

An *ordered category* $\mathbf{O}$ is a locally thin 2-category. That is, the "hom-categories" $\mathbf{O}(A, B)$ are just ordered sets. Alternatively, we may view an ordered category as a category enriched over the cartesian closed category **Ord**.

Put into elementary terms, this means that an ordered category $\mathbf{O}$ is given by an "ordinary" category $\mathbf{O}$ such that for each pair (A, B) of $\mathbf{O}$-objects we have an order on the collection $\mathbf{O}(A, B)$, and these orders are compatible in the sense that $g \leq \overline{g}$ implies $h \cdot g \cdot f \leq h \cdot \overline{g} \cdot f$ for all

$$A \xrightarrow{f} B \underset{\overline{g}}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

in $\mathbf{O}$. We will refer to instances $g \leq \overline{g}$ of the orders as *2-cells*. Any ordered category has an underlying category, obtained by forgetting the 2-cells (order-relations), and also an underlying *2-graph* (see [Str96]), obtained by forgetting composition and identities.

The primordial example of an ordered category is **Rel**, the 2-category which has sets as objects and relations as morphisms. If $r : A \longrightarrow B$ and $s : B \longrightarrow C$ are morphisms in **Rel**, that is, if $r \subset A \times B$ and $s \subset B \times C$, then

$$s \cdot r := \{\, (a, c) \in A \times C \mid \exists b \in B : (a, b) \in r, (b, c) \in s \,\}.$$

2-cells are given by inclusion.

Another important example is the category **Ord** itself, where a 2-cell $f \leq g$ exists for $f, g : (X, \leq) \longrightarrow (Y, \leq)$ iff $f(x) \leq g(x)$ holds for all $x \in X$. Any topological space X has an underlying ordered set, given by the *generalization order*: $x \leq y$ iff the principal ultrafilter $\dot{x}$ converges to y, equivalently, $y \in \mathrm{cl}\{x\}$. This order is the dual of the well-known *specialization order* [GHK$^+$03, O-5.2.]. They are antisymmetric precisely when the space is T_0. Obviously, any continuous mapping is monotone with

respect to the generalization orders, and we can consider **Top** as an ordered category by pointwise generalization order.

We call a morphism $f : X \longrightarrow Y$ in an ordered category $\mathbf{O}$ *order-monic* if $m \cdot x \leq m \cdot y$ implies $x \leq y$ for each parallel pair x, y in $\mathbf{O}$. In **Ord**, a monotone function $f : (X, \leq) \longrightarrow (Y, \leq)$ is order-monic iff f is *fully-faithful*, that is, $f(x) \leq f(x') \iff x \leq x'$ holds for all $x, x' \in X$ (cf. [Woo04])

If P is any property of ordered sets, we say an ordered category $\mathbf{O}$ *locally has* P provided $\mathbf{O}(A, B)$ has P for all $\mathbf{O}$-objects A, B. So, for instance, **Rel** is locally separated while **Ord** is not locally separated.

For an ordered category $\mathbf{O}$, we can not only form the *dual* ordered category $\mathbf{O}^{\mathrm{op}}$ by "turning around the 1-cells", but also the *conjugate* ordered category $\mathbf{O}^{\mathrm{co}}$ by "turning around the 2-cells": $g \leq f$ in $\mathbf{O}^{\mathrm{co}}$ iff $f \leq g$ in $\mathbf{O}$. Of course, these can be combined, and $\mathbf{O}^{\mathrm{coop}} = \mathbf{O}^{\mathrm{opco}}$ holds for any $\mathbf{O}$.

1.1 Morphisms for ordered categories, free ordered categories

When it comes to morphisms of ordered categories, there a several notions we can consider. The following 2-categorical concepts are well-known; see for instance [Str96] or [Bor94]. Nonetheless, we repeat them here for the reader's convenience and to settle notation. We will frequently use the notion of *2-graph morphism* between (the underlying 2-graphs) of ordered categories: Given ordered categories $\mathbf{O}$ and $\mathbf{P}$, a 2-graph morphism $\mathbf{O} \xrightarrow{G} \mathbf{P}$ assigns to every $\mathbf{O}$-object A a $\mathbf{P}$-object GA, to every $\mathbf{O}$-morphism $A \xrightarrow{f} B$ a $\mathbf{P}$-morphism $GA \xrightarrow{Gf} GB$, and to every 2-cell $f \leq \overline{f}$ in $\mathbf{O}$ a 2-cell $Gf \leq G\overline{f}$ in $\mathbf{P}$. Observe that, due to the scarcity of 2-cells in $\mathbf{O}$ and $\mathbf{P}$, G already preserves all the structure on the level of 2-cells. In particular, for any pair (A, B) of $\mathbf{O}$-objects we get a restriction

$$G_{AB} : \mathbf{O}(A, B) \longrightarrow \mathbf{P}(GA, GB),$$

which is monotone. Thus whenever P is a property of monotone functions, we will say that G has *locally* P provided G_{AB} has P for all $\mathbf{O}$-objects A, B. For example, G is called *locally fully-faithful* (or *2-full*) if $Gf \leq G\overline{f}$ implies $f \leq \overline{f}$.

Definition 1.1.1 Let $\mathbf{O}$ and $\mathbf{P}$ be ordered categories and $\mathbf{O} \xrightarrow{G} \mathbf{P}$ be a 2-graph morphism. We say that G is

(1) a *2-functor* if G is a functor between the underlying categories of $\mathbf{O}$ and $\mathbf{P}$;

(2) a *pseudo-functor* if F preserves composition and identities up to $\approx$; that is, $Gg \cdot Gf \approx G(g \cdot f)$ and $1_{GA} \approx G1_A$ hold,

(3) a *lax functor* if $Gg \cdot Gf \leq G(g \cdot f)$ and $1_{GA} \leq G1_A$ hold,

(4) an *oplax functor* if $G(g \cdot f) \leq Gg \cdot Gf$ and $G1_A \leq 1_{GA}$ hold,

whenever the compositions are defined in $\mathbf{O}$. Moreover, we call G *normal* if $G1_A \approx 1_{GA}$ holds for all A.

Next we generalize natural transformations to our "relaxed" setting:

Definition 1.1.2 Let $\mathbf{O}$ and $\mathbf{P}$ be ordered categories and $F, G : \mathbf{O} \longrightarrow \mathbf{P}$ 2-graph morphisms. A *transformation* $F \xrightarrow{\lambda} G$ is given by a family $(FO \xrightarrow{\lambda_O} FO)_{O \in \mathrm{obj}\,\mathbf{O}}$ of $\mathbf{P}$-morphisms. We call such a λ a

(1) *pseudo-natural transformation* provided

$$\begin{array}{ccc} FA & \xrightarrow{\lambda_A} & GA \\ {\scriptstyle Ff}\downarrow & \approx & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\lambda_B]{} & GB \end{array}$$

that is, $Gf \cdot \lambda_A \approx \lambda_B \cdot Ff$, holds for any $A \xrightarrow{f} B$ in $\mathbf{O}$;

(2) *lax transformation* provided $Gf \cdot \lambda_A \leq \lambda_B \cdot Ff$ holds for any $A \xrightarrow{f} B$ in $\mathbf{O}$;

(3) *oplax transformation* provided $\lambda_B \cdot Ff \leq Gf \cdot \lambda_A$ holds for any $A \xrightarrow{f} B$ in $\mathbf{O}$.

We write **OrdCat** for the category of ordered categories and pseudo-functors and $\mathbf{OrdCat}_s$ for the (non-full) subcategory of **OrdCat** given by the 2-functors.

Free ordered categories The construction of free ordered categories is extremely easy. This is due to the fact that the existence of a 2-cell $f \leq g$ is a mere property of the pair (f, g).

Definition 1.1.3 A *locally related category* is a pair $(\mathbf{C}, \tau)$, consisting of a category $\mathbf{C}$ together with family of relations $(\tau_{AB})_{(A,B) \in (\mathrm{obj}\,\mathbf{C})^2}$ with $\tau_{AB} \subset \mathbf{C}(A, B)^2$ for all A, B. A morphism of locally related categories $F : (\mathbf{C}, \tau) \longrightarrow (\mathbf{C}', \tau')$ is given by a functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ which *locally preserves* τ in the sense that $f \tau_{AB} g$ implies $F(f) \tau'_{F(A)F(B)} F(g)$. We write **lrCat** for the category of (small) locally related categories and their morphisms.

We have the obvious forgetful functors

$$\begin{array}{ccc} \mathbf{OrdCat}_s & \xrightarrow{\quad U \quad} & \mathbf{lrCat} \\ & \searrow \quad \swarrow_{V} & \\ & \mathbf{Cat}. & \end{array}$$

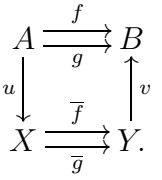
Clearly V has a left adjoint, which equips any category $\mathbf{C}$ with $\tau_{AB} = \emptyset$.

Proposition 1.1.4 *$U : \mathbf{OrdCat}_s \longrightarrow \mathbf{lrCat}$ has a concrete left adjoint O.*

Proof As the underlying category of $O(\mathbf{C}, \tau)$, simply take $\mathbf{C}$. We are going to construct the 2-cells of $O(\mathbf{C}, \tau)$ in two steps:

Step 1: For $A, B \in \operatorname{obj} \mathbf{C}$, define $\sigma_{AB} \subset (\mathbf{C}(A, B))^2$ by

$$f \sigma_{AB} g \iff \begin{cases} \exists A \xrightarrow{u} X,\ X \xrightarrow{\overline{f}} Y,\ X \xrightarrow{\overline{g}} Y,\ Y \xrightarrow{v} B : \\ f = v \cdot \overline{f} \cdot u,\ g = v \cdot \overline{g} \cdot u,\ \overline{f} \tau_{XY} \overline{g} \end{cases}$$



Clearly $\tau_{AB} \subset \sigma_{AB}$, and the composition of $\mathbf{C}$ preserves σ.

Step 2: For each (A, B), construct $\leq_{AB}$ as the reflexive and transitive reflection of σ_{AB}, that is, $\leq_{AB} = \bigcup_{n \in \mathbb{N}} \sigma_{AB}^n$.

We need to show that $O(\mathbf{C}, \tau)$, with the 2-cells given by $\leq$, is an ordered category. For this, we only need to establish the compability of $\leq$ with the composition of $\mathbf{C}$. Suppose $A \xrightarrow{f} B \underset{\overline{g}}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$ and $g \leq \overline{g}$ holds. Then there exist $g_0, g_1, \ldots, g_n \in \mathbf{C}(B, C)$ such that

$$g = g_0, \quad g_0 \sigma_{BC} g_1, \quad \ldots \quad , g_{n-1} \sigma_{BC} g_n, \quad g_n = \overline{g}$$

hold. Since the composition preserves τ', we obtain:

$$(h \cdot g_0 \cdot f) \sigma_{AD} (h \cdot g_1 \cdot f), \quad \ldots \quad , (h \cdot g_{n-1} \cdot f) \sigma_{AD} (h \cdot g_n \cdot f)$$

hence $(h \cdot g \cdot f) \leq_{AD} (h \cdot \overline{g} \cdot f)$.

That $O(\mathbf{C}, \tau)$ satisfies the required universal property is obvious. ∎

We will in general not distinguish between an ordered category and its underlying "ordinary" category and likewise for 2-functors. For instance, we will freely use the notion of concrete 2-functor between 2-categories which are concrete as "ordinary" categories.

1.2 Maps and special ordered categories

We come now to the notion of *map* in an ordered category, which was introduced by Lawvere [Law73]. It is crucial to our work.

Definition 1.2.1 A morphism $f : X \longrightarrow Y$ in an ordered category is a *map* if f has a right adjoint, that is, there exists a morphism $f^* : Y \longrightarrow X$ such that

$$1_X \leq f^* \cdot f \qquad \text{and} \qquad f \cdot f^* \leq 1_Y.$$

As usual, we write $f \dashv f^*$ in case f is a map with right adjoint f^*. If $\mathbf{O}$ is an ordered category, we will write $\operatorname{Map}\mathbf{O}$ for the class of all maps in $\mathbf{O}$.

In **Rel** a morphism $X \longrightarrow Y$ is a map iff it is the graph of a function $X \longrightarrow Y$. In **Ord**, a morphism is a map iff it has a right adjoint in the usual sense.

Clearly every pseudo-functor preserves maps, and the *duality principle for maps* holds for all ordered categories $\mathbf{O}$:

Remark 1.2.2 (Duality principle for maps) The following statements are equivalent for morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ in an ordered category $\mathbf{O}$:

(1) $f \dashv g$ in $\mathbf{O}$;

(2) $g \dashv f$ in $\mathbf{O}^{\mathrm{op}}$;

(3) $g \dashv f$ in $\mathbf{O}^{\mathrm{co}}$;

(4) $f \dashv g$ in $\mathbf{O}^{\mathrm{coop}}$. ■

Proposition 1.2.3 *The following statements are equivalent for $X \xrightarrow{l} Y$, $Y \xrightarrow{r} X$ in an ordered category $\mathbf{O}$:*

(1) $l \dashv r$;

(2) *for all $\mathbf{O}$-objects A, $\mathbf{O}(A, l) \dashv \mathbf{O}(A, r)$ holds in* **Ord**;

(3) *for all $\mathbf{O}$-objects B, $\mathbf{O}(r, B) \dashv \mathbf{O}(l, B)$ holds in* **Ord**.

By abuse of notation we will write, given any $\mathbf{O}$-morphism x, $x \cdot (\ )$ for any $\mathbf{O}(A, x)$ and $(\ ) \cdot x$ for any $\mathbf{O}(x, B)$. This abuse makes the triviality of the proof apparent:

Proof [(1) $\Rightarrow$ (2)]: $l \cdot (\ )$ and $r \cdot (\ )$ are clearly monotone. So we only need to establish that $f \leq r \cdot l \cdot f$ and $l \cdot r \cdot g \leq g$ holds whenever the composite is defined. But this follows trivially from (1).

[(2) $\Rightarrow$ (1)]: We obtain the desired inequalities by applying $r \cdot l \cdot (\ )$ and $l \cdot r \cdot (\ )$ to 1_X and 1_Y respectively.
The equivalence of (1) and (3) follows by duality. ■

Corollary 1.2.4 *If $f \in \operatorname{Map}\mathbf{O}$, then $f \cdot (\ )$ and $(\ ) \cdot f^*$ preserve suprema, $(\ ) \cdot f$ and $f^* \cdot (\ )$ preserve infima.* ■

By contrast, in the natural examples $f \cdot (\ )$ *almost never* preserves infima. Later on we will study a variety of *modularity laws*, which serve as a remedy to this inconvenience.

The adjunctions

$$f \cdot (\ ) \dashv f^* \cdot (\ ), \qquad (\ ) \cdot f^* \dashv (\ ) \cdot f,$$

for a map f in any ordered category give rise to what we choose to call *Lawvere calculus*. It is the key to our purely categorical approach to lax algebras.

Remarks 1.2.5 The usual remarks in adjunctions in a 2-category apply to maps:

(1) If $f \dashv g$ and $f \dashv g'$, then $g \approx g'$. This justifies the notation f^* for *a* right adjoint of a map.

(2) Every isomorphism is a map, with $f \dashv f^{-1}$, especially $1_X^* \approx 1_X$.

(3) The class of maps is closed under composition: $f^* \cdot g^* \approx (g \cdot f)^*$.

(4) If f and g are maps and $f \leq g$ holds, then $g^* \leq f^* \cdot f \cdot g^* \leq f^* \cdot g \cdot g^* \leq f^*$. However, we are not able to prove $f^* \leq g^*$. Consider for example adjunctions in **Ord**.

(5) On the other hand, if "$(\ )^*$ is monotone", then the order on the maps is in a way trivial. Consider:

   (i) $f \leq g$ implies $f^* \leq g^*$ for all maps f, g;

   (ii) if f and g are maps, then $f \leq g$ implies $f \approx g$.

   Then (i) and (ii) are equivalent in any ordered category: Suppose (i) holds. If $f \leq g$, then $g \leq g \cdot f^* \cdot f \leq g \cdot g^* \cdot f \leq f$, hence $f \approx g$. On the other hand, (ii) implies that if $f \leq g$, then $g \leq f$ and, by (4), $f^* \leq g^*$. ■

Definition 1.2.6 We say an ordered category has *discrete maps* provided $f \leq g$ implies $f \approx g$ for all maps f, g.

Covers and injections

Definition 1.2.7 A map $f : X \longrightarrow Y$ is called an *injection* provided $1_X \approx f^* \cdot f$, and a *cover* provided $f \cdot f^* \approx 1_Y$. We call f an *equivalence* if it is both a cover and an injection. We write $\operatorname{Inj} \mathbf{O}$ for the class of injections and $\operatorname{Cov} \mathbf{O}$ for the class of covers of $\mathbf{O}$. A source $(X \xrightarrow{f_i} X_i)_{i \in I}$ of maps is *(jointly) injecting* provided $1_X \approx \bigwedge_I f_i^* \cdot f_i$ holds; a sink $(Y_i \xrightarrow{g_i} Y)_{i \in I}$ of maps is *(jointly) covering* provided $1_Y \approx \bigvee_I g_i \cdot g_i^*$ holds.

A map f in $\mathbf{O}$ is order-monic iff f is an injection, and f is a cover iff f^* is order-monic. In particular, in **Ord**, a map $X \xrightarrow{f} Y$ is an injection iff f is fully-faithful; and f is cover iff f^* is fully-faithful. See also Lemma 1.6.3.

In **Rel**, a source is jointly injection iff it separates points in $\operatorname{Map}\mathbf{Rel} \cong \mathbf{Set}$, and a sink jointly covering iff it is jointly surjective in **Set**. In particular, a function e is surjective iff its graph is a cover. Observe that no Choice is needed for this equivalence.

The following duality principle holds:

Remark 1.2.8 (Duality principle for covering sinks and injecting sources) A source $(X \xrightarrow{f_i} Y_i)_I$ in an ordered category $\mathbf{O}$ is jointly injecting in $\mathbf{O}$ iff it is a jointly covering sink in $\mathbf{O}^{\mathrm{coop}}$. ■

In particular $\operatorname{Inj}\mathbf{O} = \operatorname{Cov}\mathbf{O}^{\mathrm{coop}}$ and $\operatorname{Cov}\mathbf{O} = \operatorname{Inj}\mathbf{O}^{\mathrm{coop}}$.

Lemma 1.2.9 *If $f \in \operatorname{Map}\mathbf{O}$, then*

(1) *$f \cdot (\ )$ is fully-faithful iff f is an injection iff $(\ ) \cdot f^*$ is fully-faithful.*

(2) *$f^* \cdot (\ )$ is fully-faithful iff f is a cover iff $(\ ) \cdot f$ is fully-faithful.* ■

Remarks 1.2.10 The following statements are true in any ordered category $\mathbf{O}$.

(1) Every isomorphism is an injection.

(2) Injections are closed under composition. More generally, jointly injecting sources are closed under composition in the following sense: Given a jointly injecting source $(X \xrightarrow{m_i} Y_i)_I$ and for each $i \in I$ a jointly injecting source $(Y_i \xrightarrow{n_{ij}} Z_{ij})_{J_i}$, the composite source $(X \xrightarrow{n_{ij} \cdot m_i} Z_{ij})_{i \in I, j \in J_i}$ is jointly injecting.

(3) If m and f are maps and $f \cdot m$ is an injection, then m is an injection.

The dual properties hold for covers and jointly covering sinks.

Proof (1) and (3) are trivial. To show (2), assume we are given sources as above. Then, using the fact that $f^* \cdot (\ ) \cdot f$ preserves infima for each map f, we obtain

$$\begin{aligned}
\bigwedge_{i \in I} \bigwedge_{j \in J_i} (n_{ij} \cdot m_i)^* \cdot n_{ij} \cdot m_i &\approx \bigwedge_{i \in I} \bigwedge_{j \in J_i} m_i^* \cdot n_{ij}^* \cdot n_{ij} \cdot m_i \\
&\approx \bigwedge_{i \in I} m_i^* \cdot \Big(\bigwedge_{j \in J_i} n_{ij}^* \cdot n_{ij} \Big) \cdot m_i \\
&\approx \bigwedge m_i^* \cdot m_i \approx 1_X
\end{aligned}$$

■

Some diagram lemmas We fix an ordered category **O**.

Lemma 1.2.11 *Let* $X \overset{\overset{f}{\longrightarrow}}{\underset{h}{\overset{\longleftarrow}{\longrightarrow}}} Y$ *be* **O***-morphisms such that* $f \dashv g \dashv h$*. Then* f *is order-monic if and only if* h *is order-monic. In this case* $f \leq h$*.* ■

There is also a well-known non-thin version of this result; see for instance [DT87, Lemma 1.3].

Lemma 1.2.12 *Suppose we are given a diagram as follows in* **O**

$$\begin{array}{ccc} X & \overset{u}{\underset{u^*}{\rightleftarrows}} & Z \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle f^*} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle g^*} \\ Y & \xrightarrow{w} & W \end{array}$$

such that $w \cdot f \approx g \cdot u$ *and* f *is a cover. Then* w *has a right adjoint* w^**, and* $u^* \cdot g^* \approx f^* \cdot w^*$ *holds.*

The "non-thin" version of this result is known as "Butler's Theorem"; see [BW00, Theorem 3.7.3].

Proof Define $w^* = f \cdot u^* \cdot g^*$. Then $w \cdot w^* = w \cdot f \cdot u^* \cdot g^* \approx g \cdot u \cdot u^* \cdot g^* \leq 1_W$ and $1_Y \approx f \cdot f^* \leq f \cdot u^* \cdot g^* \cdot g \cdot u \cdot f^* \approx w^* \cdot w \cdot f \cdot f^* \approx w^* \cdot w$ hold. The second assertion follows from the essential uniqueness of adjoints. ■

Lemma 1.2.13 *Suppose we have a diagram*

$$\begin{array}{ccc} X & \xrightarrow{m} & Z \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle h} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle g^*} \\ Y & \xrightarrow{n} & W \end{array}$$

in **O** *such that* $m \cdot h \approx g^* \cdot n$ *and* $n \cdot f \approx g \cdot m$ *holds and* m*,* n *are order-monic. Then* $f \dashv h$*.*

Proof We have $m \leq g^* \cdot g \cdot m \approx g^* \cdot n \cdot f \approx m \cdot h \cdot f$ and $n \cdot f \cdot h \approx g \cdot m \cdot h \approx g \cdot g^* \cdot n \leq n$, and the claim follows. ■

Allegories and Beck–Chevalley properties

Definition 1.2.14 Let **O** be an ordered category. A *duality* on **O** is a pseudo-functor $(\ )^\circ : \mathbf{O} \longrightarrow \mathbf{O}^{\mathrm{co}}$ which satisfies $f^{\circ\circ} \approx f$ and $A^{\circ\circ} = A$. An *involution* on **O** is a pseudo-functor $(\ )^\circ : \mathbf{O} \longrightarrow \mathbf{O}^{\mathrm{op}}$ which satisfies $f^{\circ\circ} \approx f$ and $A^\circ = A$.

Rel has an involution, given by *transposition*: $(b, a) \in r^\circ$ iff $(a, b) \in r$. **Ord** has a duality, given by the dual ordered set.

Definition 1.2.15 An ordered category **O** is an *allegory* provided

(1) all hom-sets have binary meets;

(2) it has an $\wedge$-preserving involution $(\ )^\circ$;

(3) the involution satisfies *Freyd's modularity law*, that is,

$$h \wedge g \cdot f \leq g \cdot (g^\circ \cdot h \wedge f) \tag{FML}$$

holds for all f, g, h in **O**.

Rel is, with transposition as involution, an allegory.

It is well-known ([FS90]) that in the presence of (1) and (2), (FML) is equivalent to

$$g \cdot f \wedge h \leq (g \wedge h \cdot f^\circ) \cdot f. \tag{FML$^{\mathrm{op}}$}$$

As we will see later, (FML) is quite powerful. In fact, it is almost too powerful: Many examples of interest do not satisfy it. Later on we will introduce the Frobenius laws: a weaker version of modularity which suits our purposes better.

Definition 1.2.16 We say a (not necessarily commutative) square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Z & \xrightarrow[g]{} & W \end{array}$$

an ordered category such that f and g are maps is a *Beck–Chevalley square* provided

$$\begin{array}{ccc} X & \xleftarrow{f^*} & Y \\ {\scriptstyle a}\downarrow & \approx & \downarrow{\scriptstyle b} \\ Z & \xleftarrow[g^*]{} & W \end{array}$$

holds. Often we will abbreviate Beck–Chevalley to (BC).

Remark 1.2.17 A square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow[g]{} & W \end{array}$$

of maps in an ordered category with discrete maps is a Beck–Chevalley square iff the rotated square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[k]{} & W \end{array}$$

is a Beck–Chevalley square. ∎

Lemma 1.2.18 *Suppose* $\mathbf{A}$ *is an allegory with involution* $(\ )^\circ$*. If* $f \dashv g$ *in* $\mathbf{A}$*, then* $g \approx f^\circ$*.*

Proof We have $1 \approx g \cdot f \wedge 1 \leq (g \wedge f^\circ) \cdot f \leq f^\circ \cdot f$ by ($\mathrm{FML}^{\mathrm{op}}$) and $1 \approx 1 \wedge g \cdot f \leq g \cdot (g^\circ \wedge f) \leq g \cdot g^\circ$ by (FML). This implies $f \cdot f^\circ \leq f \cdot g \cdot g^\circ \cdot f^\circ \leq 1$, and hence $f \dashv f^\circ$. ∎

Corollary 1.2.19 *Every allegory has discrete maps. If* $f \dashv g \dashv h$ *holds in an allegory, then* f *is an equivalence and* $f \approx h$ *holds.* ∎

The Frobenius laws

Definition 1.2.20 Given a pair $(\mathbf{O}, M)$, where $\mathbf{O}$ is an ordered category and M is a class of maps such that $\mathbf{O}$ is locally finitely complete, we say that $(\mathbf{O}, M)$ satisfies the

1. *Frobenius law* if for every not necessarily commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & {\scriptstyle a}\swarrow & & \searrow{\scriptstyle b} & \\ X & & \underset{f^*}{\overset{f}{\rightleftarrows}} & & Y \end{array}$$

with $f \in M$, one has

$$f \cdot (f^* \cdot b \wedge a) \approx b \wedge f \cdot a. \tag{1.1}$$

2. *coFrobenius law* if for every not necessarily commutative diagram

$$\begin{array}{ccccc} X & & \underset{f^*}{\overset{f}{\rightleftarrows}} & & Y \\ & {\scriptstyle a}\searrow & & \swarrow{\scriptstyle b} & \\ & & A & & \end{array}$$

with $f \in M$, one has

$$(a \wedge b \cdot f) \cdot f^* \approx a \cdot f^* \wedge b. \tag{1.2}$$

Remarks 1.2.21 (1) In (1.1) and (1.2) "$\leq$" holds for any map f.

(2) If $\mathbf{O}$ has an $\wedge$-preserving involution $(\ )^\circ$ such that $f^* \approx f^\circ$ holds for all $f \in M$, then the Frobenius law and the coFrobenius law are equivalent.

(3) For any allegory $\mathbf{A}$ the Frobenius law holds in $(\mathbf{A}, \operatorname{Map}\mathbf{A})$. In this instance, it is just (FML) (here we use Lemma 1.2.18). By (2), $(\mathbf{A}, \operatorname{Map}\mathbf{A})$ also satisfies the coFrobenius law. ∎

Later on we will see examples which satisfy the Frobenius law but not (FML).

1.3 Kock–Zöberlein monads

Let $\mathbf{O}$ be an ordered category, $T : \mathbf{O} \longrightarrow \mathbf{O}$ a pseudo-functor, and $e : 1_{\mathbf{O}} \longrightarrow T$ a pseudo-natural transformation. We say that (T, e) is a *Kock–Zöberlein monad* (*KZ-monad*) on $\mathbf{O}$ if for each $X \in \mathbf{O}$ there exists m_X with

$$Te_X \dashv m_X \dashv e_{TX} \tag{1.3}$$

and $m_X \cdot Te_X \approx 1_{TX}$, or equivalently, $m_X \cdot e_{TX} \approx 1_{TX}$ holds. We say that (T, e) is *co-Kock–Zöberlein* (*coKZ)* if (T, e) is Kock–Zöberlein in $\mathbf{O}^{\mathrm{co}}$. That is, for each $\mathbf{O}$-object X there exists m_X such that

$$e_{TX} \dashv m_X \dashv Te_X \tag{1.4}$$

and $m_X \cdot e_{TX} \approx 1_{TX}$, or equivalently, $m_X \cdot Te_X \approx 1_{TX}$ holds.

Proposition 1.3.1 *If (T, e) is a KZ-monad on $\mathbf{O}$ and we are given a family (m_X) such that* (1.3) *holds, then (T, e, m) is a pseudo-monad on $\mathbf{O}$.* ∎

The distinctive feature of KZ-monads (which accounts for the title of [Koc95]) is:

Proposition 1.3.2 *Let (T, e) be a KZ-monad on $\mathbf{O}$. Then a pair $(X, TX \xrightarrow{a} X)$ is a (T, e, m)-pseudo-algebra iff $a \dashv e_X$ with $a \cdot e_X \approx 1_X$ holds.* ∎

For a proof of the preceding results we refer to [Koc95]. Of course, dual results hold for coKZ-monads.

Thus the existence of an (T, e)-algebra structure on an $\mathbf{O}$-object O is essentially a property of O, and $\mathbf{O}^{(T,e)}$ is essentially a (non-full) subcategory of $\mathbf{O}$. Hence, KZ-monads generalize idempotent monads which correspond to reflective full and replete subcategories.

Submonads of KZ-monads The following Proposition, which is implicit in [RW01] (see also [MRW02]), is helpful in recognizing sub-KZ-monads.

Proposition 1.3.3 *Let (T, e, m) be a KZ-monad on an ordered category $\mathbf{O}$ such that T preserves order-monics, $S : \mathbf{O} \longrightarrow \mathbf{O}$ a pseudo-functor and $i : S \longrightarrow T$ a pointwise order-monic pseudo-natural transformation such that we have factorizations*

$$\begin{array}{ccccc} X & \overset{d_X}{\dashrightarrow} & SX & \overset{n_X}{\dashleftarrow} & SSX \\ & \searrow^{e_X} & \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle (i*i)_X} \\ & & TX & \xleftarrow{m_X} & TTX \end{array}$$

for for each X. Then $(S, (d_X), (n_X))$ is a KZ-monad on $\mathbf{O}$ and i is a monad-morphism.

Proof First observe that order-monicity of each i_X implies that (d_X) forms a pseudo-natural transformation $1_{\mathbf{O}} \longrightarrow S$. Now observe that $(i * i)_X \cdot d_{SX} \approx Ti_X \cdot i_{SX} \cdot d_{SX} \approx Ti_X \cdot e_{SX} \approx e_{TX} \cdot i_X$ and $(i * i)_X \cdot Sd_X \approx i_{TX} \cdot Si_X \cdot Sd_X \approx i_{TX} \cdot Se_X \approx Te_X \cdot i_X$ hold, hence

$$\begin{array}{ccc} SX & \underset{\overset{\longleftarrow n_X \longrightarrow}{Sd_X}}{\overset{d_{SX}}{\rightrightarrows}} & SSX \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle (i*i)_X} \\ TX & \underset{\overset{\longleftarrow m_X \longrightarrow}{Te_X}}{\overset{e_{TX}}{\rightrightarrows}} & TTX \end{array}$$

"commutes serially up to $\approx$" and Sd_X is order-monic since $Te_X \cdot i_X$ is so. By assumption $(i * i)_X \approx Ti_X \cdot i_{SX}$ is order-monic. Applying Lemma 1.2.13 twice, we see that $d_{SX} \dashv n_X \dashv Sd_X$ holds. Order-monicity of Sd_X implies $n_X \cdot Sd_X \approx 1_{SX}$. ∎

1.3.1 Downsets and upsets

For an ordered set X let $\mathbb{D}X$ denote the set of all *downsets* of X, that is,

$$\mathbb{D}X = \{\, A \subset X \mid A^{\downarrow} = A \,\},$$

ordered by $\subset$. Here $A^{\downarrow} = \{\, x \in X \mid \exists a \in A : x \leq a \,\}$ is the *down-closure* of A in X. In case X is discrete, $\mathbb{D}X$ is just the power set of X. For each ordered set X, we have a fully-faithful function $(\ )^{\downarrow} : X \longrightarrow \mathbb{D}X$, given by $x \longmapsto \{x\}^{\downarrow}$. $\mathbb{D}$ can be extended to a 2-functor on **Ord** by setting $\mathbb{D}f(A) = (f[A])^{\downarrow}$ for $X \xrightarrow{f} Y$. The family $((\ )^{\downarrow}_X : X \longrightarrow \mathbb{D}X)_X$ forms a natural transformation $1_{\mathbf{Ord}} \longrightarrow \mathbb{D}$. In fact, the following holds:

Proposition 1.3.4 *$(\mathbb{D}, (\ )^{\downarrow})$ is a Kock–Zöberlein monad on* **Ord**.

Proof $m_X : \mathbb{D}\mathbb{D}X \longrightarrow \mathbb{D}X$ is given by $\mathcal{A} \longmapsto \bigcup \mathcal{A}$. The two adjunctions are established easily. ∎

$\mathbb{D}X$ is in fact a complete lattice, with suprema given by union and infima given by intersection. Since $x \leq \bigwedge a_i \iff x \in \bigcap a_i^{\downarrow}$ holds whenever the infimum exists in X, $(\ )^{\downarrow}$ preserves all infima which might exist in X.

Remarks 1.3.5 Let f be a monotone function. The following statements hold:

(1) $\mathbb{D}f$ has a right adjoint $\mathbb{D}^*f$, given by inverse image: $\mathbb{D}^*f(B) = f^{-1}[B]$.

(2) If f is fully-faithful, then so is $\mathbb{D}f$. ■

An ordered set X "is" a $\mathbb{D}$-algebra iff $X \xrightarrow{(\,)^{\downarrow}} \mathbb{D}X$ has a left adjoint, that is, iff X is complete. Moreover, a monotone function f is a $\mathbb{D}$-algebra homomorphism iff f preserves suprema. Thus $\mathbf{Ord}^{\mathbb{D}}$ is the category $\mathbf{Sup}$ of complete ordered sets and suprema-preserving functions.

Dually, we define the set of upsets of an ordered set X by $\mathbb{U}X = (\mathbb{D}(X^{\mathrm{op}}))^{\mathrm{op}}$. That is, $\mathbb{U}X = \{\, A \subset X \mid A^{\uparrow} = A \,\}$, with $A^{\uparrow} = \{\, x \in X \mid \exists a \in A : a \leq x \}$, ordered by $\supset$. For $f : X \longrightarrow Y$, we have $\mathbb{U}f(A) = (f[A])^{\uparrow}$. Each $\mathbb{U}f$ has a *left* adjoint $\mathbb{U}^*f$, given by inverse image. Clearly $\mathbb{U}$ is a coKZ-monad on $\mathbf{Ord}$. The category $\mathbf{Ord}^{\mathbb{U}}$ is just the category of complete ordered sets and *infima*-preserving functions.

1.3.2 Finitely generated downsets

We write $\mathbb{D}_\omega X$ for the subset of $\mathbb{D}X$ given by the *finitely generated* downsets, that is,

$$\mathbb{D}_\omega X = \{\, A \in \mathbb{D}X \mid \exists B \subset X,\ B \text{ finite},\ A = B^{\downarrow} \,\}.$$

$\mathbb{D}_\omega$ is a subfunctor of $\mathbb{D}$, and since finitely generated unions of finitely generated downsets are finitely generated, Proposition 1.3.3 applies. Hence $\mathbb{D}_\omega$ is a KZ-monad on $\mathbf{Ord}$, and the natural inclusion $\mathbb{D}_\omega \longrightarrow \mathbb{D}$ forms a monad morphism.

$\mathbf{Ord}^{\mathbb{D}_\omega}$ is the category of (finite join-) semilattices and finite join preserving morphisms. Dually, we obtain the finite-generated upset-monad $\mathbb{U}_\omega$ on $\mathbf{Ord}$.

1.3.3 Non-empty downsets

Almost the same reasoning applies to the non-empty (or inhabited downsets) $\mathbb{D}_0 X$ of an ordered set X, giving us a subKZ-monad $\mathbb{D}_0$ of $\mathbb{D}$, and the corresponding subcoKZ-monad $\mathbb{U}_0$ of $\mathbb{U}$.

1.3.4 Ideals and filters

For an ordered set X let $\mathbb{I}X = \{\, A \in \mathbb{D}X \mid A \text{ is directed} \,\} \subset \mathbb{D}X$ be the set of *ideals* in X. Since $A \subset X$ directed implies $f[A]$ directed for all monotone functions $f : X \longrightarrow Y$, and B directed implies $B^{\downarrow}$ directed, $\mathbb{I}$ is a subfunctor of $\mathbb{D}_0$. Moreover, each principal downset $x^{\downarrow}$ is an ideal. Thus we obtain a factorization

$$\begin{array}{ccc} X & \xrightarrow{(\,)^{\downarrow}} & \mathbb{I}X \\ & \searrow^{(\,)^{\downarrow}} & \downarrow{\scriptstyle i_X} \\ & & \mathbb{D}X \end{array}$$

where i_X is fully-faithful. $(\ )^{\downarrow} : X \longrightarrow \mathbb{D}X$ preserves all infima and all *finite* suprema which may exist in X. The union of any ideal of ideals is an ideal, and thus, using Proposition 1.3.3, $\mathbb{I}$ is a KZ-monad on **Ord** and $i : \mathbb{I} \longrightarrow \mathbb{D}$ is a monad morphism.

Since an ordered set X is an $\mathbb{I}$-algebra precisely when $X \xrightarrow{(\ )^{\downarrow}} \mathbb{I}X$ has a left adjoint, we see that $\mathbf{Ord}^{\mathbb{I}} \cong \mathbf{Dcpo}$, the category of directed complete (pre-)ordered sets and directed suprema preserving functions; see [GHK+03].

Dually, we obtain the coKZ-monad $\mathbb{F}$ of *filters* on **Ord**. Here $\mathbb{F}X$ is given by $\{\, A \in \mathbb{U}X \mid A \text{ is filtered} \,\}$. $\mathbb{F}$ is a sub-coKZ-monad of $\mathbb{U}$.

1.3.5 Kleisli categories

Let $\mathbf{T} = (T, e, m)$ be a monad on $\mathbf{C}$. We write $\mathbf{C}_{\mathbf{T}}$ for the Kleisli category of $\mathbf{T}$, whose composition is, as usual, denoted by "$*$". If $f : X \longrightarrow Y$ is any $\mathbf{C}$-morphism we write $f^{\flat} : X \longrightarrow\!\!\!\!\circ\ Y$ for the corresponding $\mathbf{C}_{\mathbf{T}}$-morphism. We define a functor $F_{\mathbf{T}} : \mathbf{C} \longrightarrow \mathbf{C}_{\mathbf{T}}$ by $X \longmapsto X$, $g \longmapsto (e_Y \cdot g)^{\flat}$ for $g : X \longrightarrow Y$.

In case $\mathbf{T}$ is a 2-monad on an ordered category $\mathbf{O}$, $\mathbf{O}_{\mathbf{T}}$ becomes an ordered category in a natural way: 2-cells are inherited from $\mathbf{O}$, that is, $f^{\flat} \leq g^{\flat} : X \longrightarrow\!\!\!\!\circ\ Y$ iff $f \leq g$ in $\mathbf{O}$. Clearly this turns $F_{\mathbf{T}}$ into a 2-functor.

Distributors We define the ordered category **Dist** of *distributors* as follows: objects are ordered sets, and morphisms $X \nrightarrow Y$ are given by relations $r \subset X \times Y$ such that

$$x' \leq x, \quad x\, r\, y, \quad y \leq y' \qquad \text{implies} \qquad x'\, r\, y'.$$

Composition and 2-cells are inherited from **Rel**. The identity on an ordered set $(X, \leq)$ is given by $\leq$. In contrast to **Rel**, **Dist** is not self-dual.

Remark 1.3.6 $\mathbf{Ord}_{\mathbb{D}}^{\mathrm{op}} \cong \mathbf{Dist} \cong \mathbf{Ord}_{\mathbb{U}}^{\mathrm{co}}$.

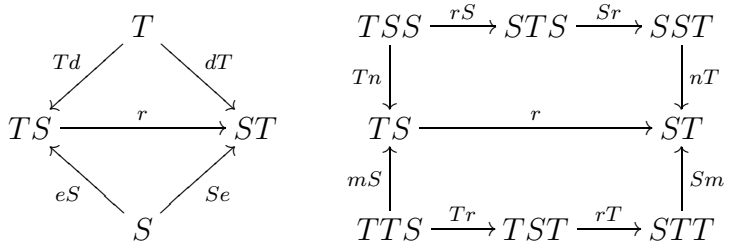
Proof The isomorphisms map objects identically and transform morphisms according to the following scheme:

$$\begin{array}{ccc} Y \xrightarrow{\check{r}} \mathbb{D}X & X \xrightarrow{r} Y & X \xrightarrow{\hat{r}} \mathbb{U}Y \\ x \in \check{r}(y) & x\, r\, y & y \in \hat{r}(x) \end{array}$$ ∎

For a monotone function $f : X \longrightarrow Y$ we have $x\, F_{\mathbb{U}} f\, y$ in **Dist** iff $f(x) \leq y$ in **Ord** and $x\, F_{\mathbb{D}} f\, y$ in **Dist** iff $y \leq f(x)$ in **Ord**. In particular, if Y is discretely ordered then $F_{\mathbb{U}} f$ is just the graph of f and $F_{\mathbb{D}} f$ is the transpose of the graph of f. But even in the more general situation of arbitrary ordered sets, we have $F_{\mathbb{U}} f \dashv F_{\mathbb{D}} f$ in **Dist**; see Section 1.6.2.

1.3.6 Distributive laws and composite monads

We recall that for monads $\mathbf{T} = (T, e, m)$ and $\mathbf{S} = (S, d, n)$ on a category $\mathbf{C}$ a *distributive law of* $\mathbf{T}$ *over* $\mathbf{S}$ is a natural transformation $TS \xrightarrow{r} ST$ such that



(1.5)

commutes. We will use the following convention from [MRW02]: since each face of (1.5) uses precisely one of the transformations e, m, d, n, we write $r[x]$ to indicate that a mere natural transformation $TS \xrightarrow{r} ST$ satisfies the equation from (1.5) involving $x \in \{ e, m, d, n \}$.

For a functor $G : \mathbf{C} \longrightarrow \mathbf{C}$ a *distributive law of* G *over* $\mathbf{T}$ is a natural transformation $GT \xrightarrow{r} TG$ which satisfies $r[e]$ and $r[m]$.

It is well-known that any distributive law r of $\mathbf{T}$ over $\mathbf{S}$ can be used to define a monad structure on the composite functor ST. Indeed, the unit of the composite monad $(\mathbf{ST})_r$ is given by $dT \cdot e$, while the multiplication is defined by $(n * m) \cdot SrT$.

A basic distributive law In [MRW02], the following distributive laws were described. Define natural transformations $r : \mathbb{UD} \longrightarrow \mathbb{DU}$ and $l : \mathbb{DU} \longrightarrow \mathbb{UD}$ as follows:

$$r_X(\mathcal{A}) = \{ U \in \mathbb{U}X \mid \forall A \in \mathcal{A}\, \exists u \in U : u \in A \},$$
$$l_X(\mathcal{B}) = \{ D \in \mathbb{D}X \mid \forall B \in \mathcal{B}\, \exists d \in D : d \in B \}.$$

In [MRW02] it was shown that r is a distributive law of $\mathbb{U}$ over $\mathbb{D}$ and l is a distributive law of $\mathbb{D}$ over $\mathbb{U}$.

Remark 1.3.7 In [MRW02] it is shown that any distributive law of $\mathbf{T}$ over $\mathbf{S}$ is unique (if it exists) provided $\mathbf{T}$ (or $\mathbf{S}$) is either a KZ- or a coKZ-monad. Thus in this case the existence of a distributive law is a property of the pair $(\mathbf{S}, \mathbf{T})$. In particular, l and r are the unique distributive laws of $\mathbb{D}$ over $\mathbb{U}$ and vice versa. ■

We are going to describe algebras for the composite monad $\mathbb{DU} = (\mathbb{DU})_r$ arising from r as above. The following discussion is due to Marmolejo, Rosebrugh, and Wood [MRW02].

Constructive complete distributivity A $\mathbb{DU}$-algebra is a $\mathbb{U}$-algebra which carries the structure of a $\mathbb{D}$-algebra such that the $\mathbb{D}$-algebra structure-morphism is a $\mathbb{U}$-morphism

(this description is due to Beck [Bec69]). Thus, an ordered set X "is" a $\mathbb{U}\mathbb{D}$-algebra iff it is complete and $\bigvee : \mathbb{D}X \longrightarrow X$ is a $\mathbb{U}$-morphism, that is, preserves arbitrary infima.

So we see that an ordered set X is a $\mathbb{U}\mathbb{D}$-algebra iff there exists $B \dashv \bigvee \dashv (\ )^{\downarrow}_{X} : X \longrightarrow \mathbb{D}X$. These ordered sets were called *constructive completely distributive* (*ccd*) in [FW90] (see also [Woo04] and the references therein). We will contrast constructive complete distributivity with (classical) complete distributivity in Section 7.2.

Observe that the adjunctions $\mathbb{D}(\ )^{\downarrow}_{X} \dashv \bigcup \dashv (\ )^{\downarrow}_{\mathbb{D}X}$ for each ordered set X which verify that $(\mathbb{D}, (\ )^{\downarrow})$ is a KZ-monad on **Ord** show that $\mathbb{D}X$ is ccd.

For any ccd ordered set X, the left adjoint $B : X \longrightarrow \mathbb{D}X$ of $\bigvee$ can be described as follows:

$$B(x) = \big\{\, v \in X \,\big|\, \forall D \in \mathbb{D}X : x \leq \bigvee D \implies v \in D \,\big\}; \tag{1.6}$$

see [FW90]. That is, B assigns to x all $v \in X$ which are *totally below* x. Thus we see the similarity between ccd ordered sets and continuous dcpos [GHK+03], which are defined using the ideal KZ-monad $\mathbb{I}$ instead of $\mathbb{D}$, see below.

Remark 1.3.8 Let X, Y be complete lattices with X being ccd. Then a function $X \xrightarrow{f} Y$ preserves suprema if and only if f is monotone and $f(x) \approx \bigvee f[B(x)]$ holds for all $x \in X$.

Proof Write $B \dashv \bigvee : \mathbb{D}X \longrightarrow X$. $\bigvee B(x) \approx x$ gives "only if". For the converse we need to establish

$$\begin{array}{ccc} \mathbb{D}X & \xrightarrow{\mathbb{D}f} & \mathbb{D}Y \\ {\scriptstyle\bigvee}\downarrow & \approx & \downarrow{\scriptstyle\bigvee} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Monotonicity of f implies $\bigvee \cdot \mathbb{D}f \leq f \cdot \bigvee$. By assumption, we have $f \approx \bigvee \cdot \mathbb{D}f \cdot B$, thus $f \cdot \bigvee \approx \bigvee \cdot \mathbb{D}f \cdot B \cdot \bigvee \leq \bigvee \cdot \mathbb{D}f$ by $B \dashv \bigvee$. ∎

Restricting the basic distributive law Let $\mathbb{U}'$ be a sub-coKZ-monad of $\mathbb{U}$. From [MRW02, 5.7] it follows that the distributive law r restricts to a distributive law $r' : \mathbb{U}'\mathbb{D} \longrightarrow \mathbb{D}\mathbb{U}'$ by

$$r'_X(\mathcal{A}) = \{\, U \in \mathbb{U}'X \mid \forall A \in \mathcal{A}\, \exists u \in U : u \in A \,\}.$$

For example, we have a distributive law $r^{\omega} : \mathbb{U}_{\omega}\mathbb{D} \longrightarrow \mathbb{D}\mathbb{U}_{\omega}$ for the finitely-generated upperset monad $\mathbb{U}_{\omega}$ and a distributive law $r^{0} : \mathbb{U}_{0}\mathbb{D} \longrightarrow \mathbb{D}\mathbb{U}_{0}$ for the non-empty upperset monad $\mathbb{U}_{0}$.

Lemma 1.3.9 *Let $\mathbb{U}_{\omega} \xrightarrow{i} \mathbb{U}$ be the inclusion. Then*

$$\begin{array}{ccc} \mathbb{U}_{\omega}\mathbb{D} & \xrightarrow{r^{\omega}} & \mathbb{D}\mathbb{U}_{\omega} \\ {\scriptstyle i_{\mathbb{D}}}\downarrow & & \downarrow{\scriptstyle \mathbb{D}i} \\ \mathbb{U}\mathbb{D} & \xrightarrow{r} & \mathbb{D}\mathbb{U} \end{array} \tag{1.7}$$

commutes. For the inclusion $\mathbb{U}_0 \xrightarrow{j} \mathbb{U}$ *we have* $r \cdot j_{\mathbb{D}} = \mathbb{D}j \cdot r^0$.

Proof We only demonstrate a proof for the first claim. The second is shown analogously.

Suppose $\mathcal{A}$ in $\mathbb{U}_\omega X$; thus there exists a finite family $(A_i)_I$ in $\mathbb{D}X$ with $\mathcal{A} = \{A_i\}^\uparrow$. We have $r_X(i_{\mathbb{D}X}(\mathcal{A})) = \{\, B \in \mathbb{U}X \mid \forall A \in \mathcal{A}\, \exists x \in B : x \in A \,\}$, and $\mathbb{D}i_X(r^\omega_X(\mathcal{A})) = \{\, B \in \mathbb{U}X \mid \exists B' \in \mathbb{U}_\omega X : B' \subset B\, \forall A \in \mathcal{A}\, \exists x \in B' : x \in A \,\}$ (recall that $\mathbb{U}X$ is ordered by "$\supset$"). Hence we trivially have $\mathbb{D}i_X(r^\omega_X(\mathcal{A})) \subset r_X(i_{\mathbb{D}X}(\mathcal{A}))$.

To show the converse inclusion, take any $B \in \mathbb{U}X$ such that for all $A \in \mathcal{A}$ there exists $x \in B$ with $x \in A$. We need to show that we can "restrict" B to a finitely generated B' with this property. For each $i \in I$ choose $x_i \in B \cap A_i$, and set $B' = \{\, x_i \mid i \in I \,\}^\uparrow \in \mathbb{U}_\omega X$. By construction we have $B' \subset B$, and for any $A \in A$ there is $i \in I$ with $x_i \in A_i \subset A$. ∎

In general we cannot restrict l to a distributive law $\mathbb{D}\mathbb{U}' \longrightarrow \mathbb{U}'\mathbb{D}$. For instance, l does not restrict to a distributive law $\mathbb{D}\mathbb{U}_\omega \longrightarrow \mathbb{U}_\omega\mathbb{D}$, cf. [MRW02, 5.7]. For the submonad $\mathbb{F}$ of $\mathbb{U}$, such a restriction exists:

Lemma 1.3.10 *l restricts to a natural transformation $l^f : \mathbb{D}\mathbb{F} \longrightarrow \mathbb{F}\mathbb{D}$, which is a distributive law. We have $l^f_X \dashv r^f_X$, with r^f_X the restriction $\mathbb{F}\mathbb{D} \longrightarrow \mathbb{D}\mathbb{F}$ of r.*

Proof We first show that l_X restricts to $\mathbb{D}\mathbb{F}X \longrightarrow \mathbb{F}\mathbb{D}X$. Take any $\mathcal{A} \in \mathbb{D}\mathbb{F}X$, and $C, D \in l_X(\mathcal{A})$. We need to show $C \cap D \in l_X(\mathcal{A})$. Take any $A \in \mathcal{A}$. We have $c \in C$, $d \in D$ with $c, d \in A$ by assumption. Since A is a filter in X, we find $b \in A$ with $b \leq c$, $b \leq d$. Down-closedness of C and D gives $b \in C \cap D$, thus $C \cap D \in l_X(\mathcal{A})$. To show $X \in l_X(\mathcal{A})$, assume that there exists $A \in \mathcal{A}$, then there exists $a \in A$ since A is a filter, and thus $x \in X$ and $X \in l_X(\mathcal{A})$.

By [MRW02, 5.7] l^f is a distributive law. $l^f_X \dashv r^f_X$ follows from Lemma 1.2.13. ∎

Filters on downsets We define the *filter on downsets* monad $\mathbf{F} = (F, e, m)$ on **Ord** as the monad constructed from $\mathbb{D}$ and $\mathbb{F}$ using the distributive law l^f. That is, $FX = \mathbb{F}\mathbb{D}X = \{\, \mathfrak{x} \subset \mathbb{D}X \mid \mathfrak{x} \text{ is filtered} \,\}$, ordered by "$\supset$", $Ff = \mathbb{F}\mathbb{D}f$, thus $B \in Ff(\mathfrak{x}) \iff f^{-1}[B] \in \mathfrak{x}$. The unit $e : 1_{\mathbf{Ord}} \longrightarrow F$ is defined by $e_X(x) = (x^\downarrow)^\uparrow$, that is, $e_X(x)$ is the familiar fixed filter $\{\, A \in \mathbb{D}X \mid x \in A \,\}$. m_X is defined as the composite

$$FFX = \mathbb{F}\mathbb{D}\mathbb{F}\mathbb{D}X \xrightarrow{\mathbb{F}(l^f_{\mathbb{D}X})} \mathbb{F}\mathbb{F}\mathbb{D}\mathbb{D}X \xrightarrow{\bigcap_{\mathbb{D}\mathbb{D}X}} \mathbb{F}\mathbb{D}\mathbb{D}X \xrightarrow{\mathbb{F}\bigcup_X} \mathbb{F}\mathbb{D}X = FX$$

Since this formula is quite unwieldy, we are now going to give a simpler description. For an ordered set X, we let $(\,)^\#_X : \mathbb{D} \longrightarrow \mathbb{D}\mathbb{F}\mathbb{D}X$ denote the diagonal of

$$\begin{array}{ccc} \mathbb{D}X & \xrightarrow{(\,)^\uparrow_{\mathbb{D}X}} & \mathbb{F}\mathbb{D}X \\ {\scriptstyle (\,)^\downarrow_{\mathbb{D}X}}\downarrow & & \downarrow{\scriptstyle (\,)^\downarrow_{\mathbb{F}\mathbb{D}X}} \\ \mathbb{D}\mathbb{D}X & \xrightarrow[\mathbb{D}(\,)^\uparrow_{\mathbb{D}X}]{} & \mathbb{D}\mathbb{F}\mathbb{D}X \end{array}$$

which commutes by naturality of $(\ )^{\downarrow}$. We have $A^{\#} = \{\, \mathfrak{x} \in FX \mid A \in \mathfrak{x} \,\}$. Observe that $(\ )^{\#}$ preserves finite intersections since both $(\ )^{\uparrow}_{\mathbb{D}X} : \mathbb{D}X \longrightarrow \mathbb{F}\mathbb{D}X$ and $(\ )^{\downarrow}_{\mathbb{F}\mathbb{D}X}$ preserve them.

Lemma 1.3.11 *For each ordered set X we have $m_X \dashv \mathbb{F}(\ )^{\#}_X$.*

Proof Note that we have $\mathbb{F}(l^f_{\mathbb{D}X}) \dashv \mathbb{F}(r^f_{\mathbb{D}X})$, $\bigwedge_{\mathbb{D}\mathbb{D}X} \dashv \mathbb{F}(\ )^{\uparrow}_{\mathbb{D}\mathbb{D}X}$, and $\mathbb{F}\bigvee_X \dashv \mathbb{F}(\ )^{\downarrow}_{\mathbb{D}X}$ since $\mathbb{F}$ is a 2-functor. By uniqueness of adjoints we just have to show $\mathbb{F}(\ )^{\#}_X = \mathbb{F}(r^f_{\mathbb{D}X}) \cdot \mathbb{F}(\ )^{\uparrow}_{\mathbb{D}\mathbb{D}X} \cdot \mathbb{F}(\ )^{\downarrow}_{\mathbb{D}X}$. But we have, using the fact that r^f is a distributive law,

$$r^f_{\mathbb{D}X} \cdot (\ )^{\uparrow}_{\mathbb{D}\mathbb{D}X} \cdot (\ )^{\downarrow}_{\mathbb{D}X} = \mathbb{D}(\ )^{\uparrow}_{\mathbb{D}X} \cdot (\ )^{\downarrow}_{\mathbb{D}X} = (\ )^{\#}_X. \qquad \blacksquare$$

Using $\mathbb{U}^*(\ )^{\#}_X \dashv \mathbb{U}(\ )^{\#}_X = \mathbb{F}(\ )^{\#}_X$ and uniqueness of adjoints, we arrive at the following description of m_X:

$$A \in m_X(\mathfrak{X}) \iff A^{\#} \in \mathfrak{X}.$$

For a discretely ordered set X, FX is just the usual set of (possibly improper) filters on the set X. Thus we can define $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ as the composite

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{discrete}} & \mathbf{Ord} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{Set} & \xleftarrow{\text{forget}} & \mathbf{Ord} \end{array} \tag{1.8}$$

Remarks 1.3.12 (1) We have $F : \mathbf{Ord} \longrightarrow \mathbf{Ord}$ as a submonad of $\mathbb{U}\mathbb{D}$, hence as a submonad of a composite of *co*variant functors. For the usual filter functor on **Set** (as on the left side of (1.8)) no analogous result holds; it is a subfunctor of P^-P^-, where P^- is the *contra*variant powerset-functor, but not of PP, cf. [MRW02, 6.4] and [Man02].

(2) For each $f : X \longrightarrow Y$ in **Ord**, Ff has a right adjoint. Indeed, one has $\mathbb{D}f \dashv \mathbb{D}^*f$, and thus $Ff = \mathbb{F}\mathbb{D}f \dashv \mathbb{F}(\mathbb{D}^*f)$ since $\mathbb{F}$ is a 2-functor. The right adjoint F^*f of Ff sends a filter $\mathfrak{y}$ to the filter generated by $\{\, f^{-1}[B] \mid B \in \mathfrak{y} \,\}$. $\blacksquare$

Algebras for F Describing the (Eilenberg–Moore) algebras for **F** is easy: recall that, by the already used result of Beck [Bec69], specifying a **F**-algebra structure on an ordered set X amounts to specifying a $\mathbb{F}$- and a $\mathbb{D}$-structure such that the $\mathbb{F}$-algebra structure-morphism is a $\mathbb{D}$-homomorphism. Since an ordered set "is" a $\mathbb{D}$-algebra iff it has arbitrary suprema and a $\mathbb{F}$-algebra iff it has filtered infima, we can thus characterize $\mathbb{F}\mathbb{D}$-algebras as those complete ordered sets X that $\bigwedge : \mathbb{F}X \longrightarrow X$ preserves suprema. Equivalently, $\bigvee : (\mathbb{F}X)^{\mathrm{op}} \longrightarrow X^{\mathrm{op}}$ preserves infima. But since $(\mathbb{F}X)^{\mathrm{op}} = \mathbb{I}(X^{\mathrm{op}})$ by definition, this is equivalent to the fact that $\bigvee : \mathbb{I}(X^{\mathrm{op}}) \longrightarrow X^{\mathrm{op}}$ preserves infima, that is, that X^{op} is a continuous lattice [GHK+03, Theorem I-1.10]. Given **F**-algebras

X and Y, a monotone function $f : X \longrightarrow Y$ is a $\mathbf{Ord}^{\mathsf{F}}$ morphism iff it preserves arbitrary suprema and filtered infima, that is, iff $f : X^{\mathrm{op}} \longrightarrow Y^{\mathrm{op}}$ is a homomorphism of continuous lattices.

Thus, we have exhibited $\mathbf{Ord}^{\mathsf{F}}$ as the category of "cocontinuous lattices". Contrast this with the well-known result of Day [Day75], who described the categories of continuous lattices as $\mathbf{Set}^{\mathsf{F}}$, where F denotes the filter-monad on $\mathbf{Set}$, cf. Section 2.3.1.

1.4 Quantaloids

Definition 1.4.1 A *quantaloid* is an (locally small) ordered category $\mathbf{Q}$ such that

1. $\mathbf{Q}$ is locally complete;
2. composition preserves suprema in each variable.

$\mathbf{Rel}$ is a quantaloid. While $\mathbf{Ord}$ fails to be a quantaloid, its non-full subcategory $\mathbf{Sup} \cong \mathbf{Ord}^{\mathbb{D}}$ is a quantaloid.

Since any suprema-preserving function between complete ordered sets has a right adjoint, we obtain:

Proposition 1.4.2 *Let $\mathbf{Q}$ be a quantaloid and $A \xrightarrow{f} B$ a $\mathbf{Q}$-morphism. For any $X \in \mathbf{Q}$ both $\mathbf{Q}(f, X)$ and $\mathbf{Q}(X, f)$ have right adjoints.* ∎

We turn the last proposition into a definition:

Definition 1.4.3 We say that an ordered category $\mathbf{O}$ *has division* if for any $f : A \longrightarrow B$ in $\mathbf{O}$ both $\mathbf{O}(f, X)$ and $\mathbf{O}(X, f)$ have right adjoints for all $X \in \mathbf{O}$.

1.4.1 Biproducts

Definition 1.4.4 Let $\mathbf{Q}$ be a quantaloid and $(A_i)_I$ be a family of objects of $\mathbf{Q}$. A *biproduct* of the family $(A_i)_I$ is given by a diagram

$$A \underset{e_i}{\overset{p_i}{\rightleftarrows}} A_i \qquad (i \in I) \tag{1.9}$$

such that

$$\bigvee_{i \in I} e_i \cdot p_i \approx 1_A, \qquad p_j \cdot e_i \approx \begin{cases} 1_{A_i} & j = i, \\ \bot & \text{else.} \end{cases}$$

$\mathbf{Rel}$ has biproducts: a biproduct of a family $(A_i)_I$ of sets is given by $\big(\coprod_I A_i \underset{e_i}{\overset{e_i^\circ}{\rightleftarrows}} A_i \big)_I$, where $(A_i \xrightarrow{e_i} \coprod_I A_i)_I$ is a coproduct in $\mathbf{Set}$.

Of course the notion of *finite biproduct* can be defined in any ordered category which is locally finitely cocomplete and in which composition preserves finite suprema; and the following remarks are, *mutatis mutandis*, applicable.

The following is immediate from the definitions:

Remark 1.4.5 If (1.9) is a biproduct of the family $(A_i)_I$, then each e_i is an injection with right adjoint p_i, and the sink $(e_i)_I$ is jointly covering. ■

Proposition 1.4.6 *If* (1.9) *is a biproduct of the family* $(A_i)_I$ *in a quantaloid, then* (A, p_i) *is a (pseudo-)product and* (A, e_i) *is a (pseudo-)coproduct of the family* (A_i).

Proof In case of products define the morphism induced by a family $(X \xrightarrow{f_i} A_i)$ by $\langle f_i \rangle = \bigvee e_i \cdot f_i$. In case of coproducts, define the morphism induced by a family $(A_i \xrightarrow{g_i} Y)$ by $[g_i] = \bigvee g_i \cdot p_i$.

To check that these morphisms are the unique (up the $\approx$) ones which make the required diagrams commute is routine. ■

Proposition 1.4.7 *The following statements are equivalent for a family* $(A_i)_I$ *of objects in a quantaloid:*

(1) $(A_i)_I$ *has a biproduct;*

(2) $(A_i)_I$ *has a (pseudo-)coproduct;*

(3) $(A_i)_I$ *has a (pseudo-)product.*

Proof By Proposition 1.4.6, (1) implies (2) and (3). To show that (2) implies (1), suppose that $(A_i \xrightarrow{e_i} A)_I$ is a coproduct diagram. Define, for $j \in I$, $p_j : A \longrightarrow A_j$ by

$$p_j \cdot e_i = \begin{cases} 1_{A_j} & j = i, \\ \bot & \text{else.} \end{cases}$$

Then $(A, e_i, p_i)_{i \in I}$ is a biproduct of $(A_i)_I$. (3) implies (1) is shown analogously. ■

Proposition 1.4.8 *Let* $\mathbf{Q}$ *be a quantaloid and let* $(A \underset{e_i}{\overset{p_i}{\rightleftarrows}} A_i)_I$ *be a biproduct in* $\mathbf{Q}$.

For a family $(A_i \xrightarrow{f_i} B)_I$ *of* $\mathbf{Q}$*-morphisms the induced morphism* $[f_i] : A \longrightarrow B$ *is a map if and only if each* f_i *is a map. Moreover,* $[f_i]$ *is a cover if and only if* (f_i) *is jointly covering.*

Corollary 1.4.9 *If* $\mathbf{Q}$ *is a quantaloid with biproducts, then* $\operatorname{Map}\mathbf{Q}$ *has coproducts.* ■

Proof Since maps compose, each f_i is a map provided $f := [f_i]$ is one.

Now suppose that for each $i \in I$ there exists f_i^* with $f_i \dashv f_i^*$. We claim that $\bigvee e_i \cdot f_i^*$ is a right adjoint of f. Indeed, we have

$$f \cdot \bigvee e_i \cdot f_i^* \approx \bigvee f \cdot e_i \cdot f_i^* \approx \bigvee f_i \cdot f_i^* \leq 1. \qquad (\dagger)$$

On the other hand, $f \approx \bigvee f_i \cdot e_i^*$ implies

$$1 \approx \bigvee e_i \cdot e_i^* \leq \bigvee e_i \cdot f_i^* \cdot f_i \cdot e_i^* \leq (\bigvee e_i \cdot f_i^*) \cdot (\bigvee f_i \cdot e_i^*) = (\bigvee e_i \cdot f_i^*) \cdot f.$$

and the first claim follows. Regarding the second claim, observe that (†) is an equivalence provided (f_i) is jointly covering. Thus f is a cover provided $(f_i)_I$ covers jointly. The converse implication follows from (the dual of) Remarks 1.2.10(2). ∎

Lemma 1.4.10 *Let $(X_i \xrightarrow{d_i} X)_I$ and $(Y_i \xrightarrow{e_i} Y)_I$ be coproducts in a quantaloid $\mathbf{Q}$, and let $(X_i \xrightarrow{a_i} Y_i)_I$ be a family of $\mathbf{Q}$-morphisms with induced morphism $X \xrightarrow{a} Y$. Then*

$$\begin{array}{ccc} X_i & \xrightarrow{d_i} & X \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle a} \\ Y_i & \xrightarrow[e_i]{} & Y \end{array}$$

is a Beck–Chevalley square for each $i \in I$.

Proof Since a is the morphism induced by $(e_i \cdot a_i)_I$, we have $a \approx \bigvee_I e_j \cdot a_j \cdot d_j^*$, thus $e_i^* \cdot a \approx \bigvee e_i^* \cdot e_j \cdot a_j \cdot d_j^* \approx a_i \cdot d_i^*$. ∎

1.5 Examples

We will now discuss the notions introduced so far for two (classes of) examples.

1.5.1 Relations in regular categories

The construction of the category of relations of a given regular category is well-known. We will only list the corresponding facts here, without giving any proofs. These may be found in the book by Freyd and Scedrov [FS90]. A category $\mathbf{C}$ is *regular* provided:

- $\mathbf{C}$ has finite limits;
- every kernel pair in $\mathbf{C}$ has a coequalizer;
- regular epimorphisms are stable under pullback in $\mathbf{C}$.

Every morphism f in a regular category factors as $f = m \cdot e$, where m is a monomorphism and e is a regular epimorphism. Every topos is regular, hence **Set** is regular. Moreover, every monadic category over **Set** is regular.

We will not notationally distinguish between a monomorphism and the subobject it represents. Given an arrow f in a regular category we write $\operatorname{Im} f$ for the image of f.

Let $\mathbf{C}$ be a well-powered regular category. The ordered category $\operatorname{Rel}\mathbf{C}$ of *relations in* $\mathbf{C}$ is defined as follows:

- objects are the objects of $\mathbf{C}$.
- morphisms $X \overset{r}{\nrightarrow} Y$ are *relations*, that is, subobjects of $X \times Y$. We write $r = \langle r_0, r_1 \rangle$.
- 2-cells are defined according to the inclusion of subobjects.
- The identity on X is given by $\langle 1_X, 1_X \rangle$.
- Composition is defined as follows: Given $X \overset{r}{\nrightarrow} Y$ and $Y \overset{s}{\nrightarrow} Z$, we first form the following pullback

$$\begin{array}{ccc} P & \xrightarrow{q} & S \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s_0} \\ R & \xrightarrow[r_1]{} & Y \end{array}$$

  and then define $s \cdot r$ by $\operatorname{Im}\langle r_0 \cdot p, s_1 \cdot q \rangle$.

Clearly, we have $\operatorname{Rel}(\mathbf{Set}) \cong \mathbf{Rel}$. $\operatorname{Rel}\mathbf{C}$ has an involution $(\ )^\circ$, defined by transposition: $\langle r_0, r_1 \rangle^\circ = \langle r_1, r_0 \rangle$. Moreover, $\operatorname{Rel}\mathbf{C}$ is locally finitely complete, $(\ )^\circ$ commutes with $\wedge$, and Freyd's modularity law holds. Hence $\operatorname{Rel}\mathbf{C}$ is an allegory.

We have an embedding $\mathbf{C} \hookrightarrow \operatorname{Rel}\mathbf{C}$ which sends $f : X \longrightarrow Y$ to its *graph* $\langle 1_X, f \rangle$, which we will also denote by f. Every graph of a $\mathbf{C}$-morphism is a map, with $f \dashv f^\circ$. Every map in $\operatorname{Rel}\mathbf{C}$ arises as the graph of a unique $\mathbf{C}$-morphism, hence $\operatorname{Map}\operatorname{Rel}\mathbf{C} \cong \mathbf{C}$. Moreover, the graph of f is

- a cover if and only if f is a regular epimorphism, and
- an injection if and only if f is a monomorphism.

For us the most fundamental fact on $\operatorname{Rel}\mathbf{C}$ is:

Lemma 1.5.1 *Every relation r can be written as $r_1 \cdot r_0^\circ$.* ∎

Hence every morphism in $\operatorname{Rel}\mathbf{C}$ can be factored as the transpose of a graph of a $\mathbf{C}$-morphisms followed by the graph of a $\mathbf{C}$-morphism.

It is well-known that every pullback in $\mathbf{C}$ is a (BC)-square in $\operatorname{Rel}\mathbf{C}$; see [FS90]. Using this fact, we can give the following characterization of (BC)-squares in $\operatorname{Rel}\mathbf{C}$:

Proposition 1.5.2 *For a not necessary commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array} \qquad (\dagger)$$

in $\mathbf{C}$ *the following statements are equivalent:*

(1) (†) *is* nearly a pullback*, that is,* (†) *commutes and the canonical morphism* κ

$$\begin{array}{ccccc} C \times_D B & & \xrightarrow{p} & & \\ & \overset{\kappa}{\searrow} & & & \\ & & A & \xrightarrow{x} & B \\ {\scriptstyle q}\searrow & & {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle f} \\ & & C & \xrightarrow[g]{} & D \end{array} \qquad (\ddagger)$$

is a regular epimorphism;

(2) (†) *is a* (BC)*-square.*

In particular, a diagram as (†) is a (BC)-square in **Set** if and only if for all $b \in B$, $c \in C$ with $f(b) = g(c)$ there exists $a \in A$ with $x(a) = b$ and $y(a) = c$.

Proof [(1) $\Rightarrow$ (2)]: Since the outer frame in (‡) is a pullback, thus a (BC)-square, we obtain $f^\circ \cdot g = p \cdot q^\circ = p \cdot \kappa \cdot \kappa^\circ \cdot q^\circ = x \cdot y^\circ$.

[(2) $\Rightarrow$ (1)]: The diagram (†) commutes since $f^\circ \cdot g = x \cdot y^\circ$ is equivalent to $g \cdot y \leq f \cdot x$ which implies $g \cdot y = f \cdot x$. Hence a comparison morphism κ as in (‡) exists. We have to show that κ is a regular epimorphism, equivalently, that $1 \leq \kappa \cdot \kappa^\circ$. First observe that $p^\circ \cdot p \wedge q^\circ \cdot q = 1$ holds; see [FS90]. This implies

$$\kappa = (p^\circ \cdot p \wedge q^\circ \cdot q) \cdot \kappa = p^\circ \cdot p \cdot \kappa \wedge q^\circ \cdot q \cdot \kappa = p^\circ \cdot x \wedge q^\circ \cdot y,$$

which in turn implies

$$\begin{aligned} 1 &\leq (g \cdot q)^\circ \cdot (g \cdot q) \wedge q^\circ \cdot q \\ &= p^\circ \cdot f^\circ \cdot q \cdot q \wedge q^\circ \cdot q \\ &= (p^\circ \cdot x \cdot y^\circ \wedge q^\circ) \cdot q \\ &= (p^\circ \cdot x \wedge q^\circ \cdot y) \cdot y^\circ \cdot q \qquad \text{by coFrobenius} \\ &= \kappa \cdot y^\circ \cdot q. \end{aligned}$$

By another application of the Frobenius law we obtain

$$1 \leq p^\circ \cdot p \wedge \kappa \cdot y^\circ \cdot q = \kappa \cdot (\kappa^\circ \cdot p^\circ \cdot p \wedge y^\circ \cdot q) = \kappa \cdot (x^\circ \cdot p \wedge y^\circ \cdot q) = \kappa \cdot \kappa^\circ,$$

and the claim is proved. ∎

More refined 2-categorical properties of $\operatorname{Rel}\mathbf{C}$ are summarized as follows:

(1) $\operatorname{Rel}\mathbf{C}$ is locally a distributive lattice, and composition distributes over finite joins if and only if $\mathbf{C}$ is a *prelogos* (in the sense of [FS90], [Joh02] calls such a category "coherent");

(2) $\operatorname{Rel}\mathbf{C}$ has division if $\mathbf{C}$ is a *logos* (in the sense of [FS90], [Joh02] calls such a category a "Heyting category");

(3) $\operatorname{Rel}\mathbf{C}$ is a quantaloid provided $\mathbf{C}$ has coproducts and is (infinitely) extensive.

Extending functors Suppose $\mathbf{C}$ and $\mathbf{C}'$ are regular categories and $F : \mathbf{C} \longrightarrow \mathbf{C}'$ is any functor. We are interested in the following questions: can F be extended to a 2-graph morphism $\operatorname{Rel}\mathbf{C} \longrightarrow \operatorname{Rel}\mathbf{C}'$, and which properties does such an extension have? These questions have already been considered by Carboni, Kelly, and Wood [CKW91]; see also [Bar70], [Kam74], and [Möb81]. We state their results without proof:

Proposition 1.5.3 ([CKW91]) *Define a 2-graph morphism* $\tilde{F} = \operatorname{Rel} F : \operatorname{Rel}\mathbf{C} \longrightarrow \operatorname{Rel}\mathbf{C}'$ *by*

$$\tilde{F}A = FA, \qquad \tilde{F}(\langle r_0, r_1\rangle) = Fr_1 \cdot (Fr_0)^\circ.$$

Then $\tilde{F}$ *commutes with* $(\ )^\circ$ *and renders*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{C}' \\ \downarrow & & \downarrow \\ \operatorname{Rel}\mathbf{C} & \xrightarrow[\tilde{F}]{} & \operatorname{Rel}\mathbf{C}' \end{array}$$

commutative. In particular, $\tilde{F}1_X = 1_{FX}$ *holds. Moreover,*

$$\tilde{F}g \cdot \tilde{F}r \leq \tilde{F}(g \cdot r) \qquad \text{and} \qquad \tilde{F}(r \cdot f) \leq \tilde{F}r \cdot \tilde{F}f$$

hold for all $r \in \operatorname{Rel}\mathbf{C}$, $f, g \in \mathbf{C}$ *such that the composites are defined.*

$\tilde{F}$ *is an oplax functor if and only if* F *preserves regular epimorphisms, and a lax functor if and only if* F nearly preserves pullbacks*, that is, for every pullback* $X \times_Z Y$ *the comparison morphism* $F(X \times_Z Y) \longrightarrow FX \times_{FZ} FY$ *is a regular epimorphism.* ∎

Nearly pullback preserving functors were introduced in [JPT+01]. Occasionally, we will need some more refined results on $\tilde{F}$ defined as above; see [Kam74] and [Möb81]:

Proposition 1.5.4 *If $m \in \operatorname{Mono}\mathbf{C}$, then $\tilde{F}(m \cdot r) = \tilde{F}m \cdot \tilde{F}r$ holds for all $r \in \operatorname{Rel}\mathbf{C}$. If f is a $\mathbf{C}$-morphism such that F preserves pullbacks along f, then $\tilde{F}(r \cdot f) = \tilde{F}r \cdot \tilde{F}f$ holds for all $r \in \operatorname{Rel}\mathbf{C}$.* ∎

The second condition is particularly useful when f is a monomorphism and F preserves inverse images.

Remark 1.5.5 ([CKW91]) The Rel-construction is normally oplax-functorial: given functors $\mathbf{C} \xrightarrow{F} \mathbf{C}' \xrightarrow{G} \mathbf{C}''$ with $\mathbf{C}$, $\mathbf{C}'$, $\mathbf{C}''$ regular, we have $\operatorname{Rel}(1_{\mathbf{C}}) = 1_{\operatorname{Rel}\mathbf{C}}$ and $\operatorname{Rel}(G \cdot F) \leq \operatorname{Rel}G \cdot \operatorname{Rel}F$. Equality holds if G preserves regular epimorphisms.

Proof We write $\tilde{F} = \operatorname{Rel}F$ and $\tilde{G} = \operatorname{Rel}G$. For a relation $r = \langle r_0, r_1\rangle$ in $\mathbf{C}$, we have

$$\widetilde{GF}r = GFr_1 \cdot (GFr_0)^\circ = \tilde{G}(Fr_1) \cdot \tilde{G}(Fr_0)^\circ \leq \tilde{G}\big(Fr_1 \cdot (Fr_0)^\circ\big) = \tilde{G}\tilde{F}(r)$$

and the inequality is an equality if G preserves regular epimorphisms. ∎

Extending natural transformations Given functors $F, G : \mathbf{C} \longrightarrow \mathbf{C}'$ with $\mathbf{C}$, $\mathbf{C}'$ regular, and a natural transformation $\lambda : F \longrightarrow G$, we can consider the family $\operatorname{Rel}\lambda = (\lambda_X : \operatorname{Rel}F(X) \longrightarrow \operatorname{Rel}G(X))_X$ of the graphs of λ. We obtain:

Proposition 1.5.6 $\operatorname{Rel}\lambda : \operatorname{Rel}F \longrightarrow \operatorname{Rel}G$ *is an oplax transformation, that is,*

$$\begin{array}{ccc} FX & \xrightarrow{\lambda_X} & GX \\ {\scriptstyle \operatorname{Rel}F\,r}\downarrow & \leq & \downarrow{\scriptstyle \operatorname{Rel}G\,r} \\ FY & \xrightarrow[\lambda_Y]{} & GY \end{array}$$

holds for all $r = \langle r_0, r_1\rangle \in \operatorname{Rel}\mathbf{C}$. Moreover, equality holds in the above diagram if the λ-naturality square of r_0 is nearly a pullback.

Proof We simply calculate: $\lambda_Y \cdot \operatorname{Rel}F(r) = \lambda_Y \cdot Fr_1 \cdot (Fr_0)^\circ = Gr_1 \cdot \lambda_X \cdot (Fr_0)^\circ \leq Gr_1 \cdot (Gr_0)^\circ \cdot \lambda_Y = \operatorname{Rel}G(r) \cdot \lambda_Y$. If the λ-naturality square of r_0 is nearly a pullback in $\mathbf{C}$, then it is a (BC)-square in $\operatorname{Rel}\mathbf{C}$ and equality holds. ∎

Recall that a natural transformation $\lambda : F \longrightarrow G$ is called *cartesian* provided

$$\begin{array}{ccc} FX & \xrightarrow{\lambda_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\lambda_Y]{} & GY \end{array} \qquad (*)$$

is a pullback for all f. We call λ *nearly cartesian* if $(*)$ is nearly a pullback for all f.

Corollary 1.5.7 $\operatorname{Rel}\lambda$ *is a natural transformation if and only if λ is nearly cartesian.*

Proof Suppose $\mathrm{Rel}\,\lambda$ is natural. By considering the naturality square of $f^\circ = \langle f, 1_Y\rangle$ we obtain $\lambda_X \cdot Ff^\circ = Gf^\circ \cdot \lambda_Y$, hence $(*)$ is a (BC)-square (in $\mathrm{Rel}\,\mathbf{C}$), and thus nearly a pullback. ■

Instead of working with relations in a regular category, we could have studied relations relative to a stable factorization system (E, M) on a finitely complete category $\mathbf{X}$, with $M \subset \mathrm{Mono}\,\mathbf{X}$ (see for instance [JW00]). Most of the results in this section hold in this more general setting.

1.5.2 V-Matrices

By a *quantale* is meant a complete lattice V equipped with a commutative, associative binary operation $\otimes$, which has a unit element k and preserves suprema in each variable, that is,

$$u \otimes \bigvee v_i \approx \bigvee u \otimes v_i$$

holds for all $u, v_i \in V$ $(i \in I)$. Observe that our terminology differs from [Ros90] in as much as we always assume quantales to be commutative and unital. Hence a quantale is a commutative monoid in the monoidal closed category **Sup**. Alternatively, we may view a quantale either as

- a one-object quantaloid, or as
- a thin symmetric monoidal closed category.

In general, we write $\mathbf{V} = (V, \otimes, k)$ to denote a quantale as above. For convenience, we will always assume V to be skeletal and *non-trivial*, that is, $k \neq \bot$.

Examples 1.5.8 (1) The most important example is the two-element chain $\mathbf{2} = (\{\bot, \top\}, \wedge, \top)$.

(2) Every frame V, in particular every complete chain and every complete Boolean algebra, is a quantale, with $\otimes = \wedge$. In this case we say that $\mathbf{V}$ *is a frame*.

(3) $\overline{\mathbb{R}}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ where we extend $+$ by setting $a + \infty = \infty = \infty + a$ for all $a \in [0,\infty]$.

(4) $([0,1], \mathbin{⊙}, 1)$, where $⊙$ is the *Łukasiewicz-conjunction*, defined by

$$a \mathbin{⊙} b = \max\{\, 0,\, a + b - 1 \,\}.$$

(5) Let $\mathbf{3}$ denote the 3-element chain. We equip it with the following multiplication

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

This quantale is useful to construct counterexamples.

Definition 1.5.9 A quantale $(V, \otimes, k)$ is called *integral* if $k = \top$ holds.

Clearly the quantales in (1)–(4) above are all integral, but the one of (5) is not integral.

Remark 1.5.10 $(V, \otimes, k)$ is integral if and only if $u \otimes v \leq u \wedge v$ holds for all $u, v \in V$.

Proof $k = \top$ implies $u \otimes v \leq u \otimes k = u$ and hence $u \otimes v \leq u \wedge v$. On the other hand we obtain $u = u \otimes k \leq u \wedge k \leq k$ for all $u \in V$, that is, $k = \top$. ∎

Matrices Given a quantale $\mathbf{V} = (V, \otimes, k)$ we define the quantaloid $\mathrm{Mat}\,\mathbf{V}$ of $\mathbf{V}$-*matrices* as follows:

- objects are sets;
- a morphism $A \mathrel{-\!\!\!\rightarrow\!\!\!\!\!\!\rightarrow} B$ is a function $A \times B \longrightarrow V$;
- the composite of $A \overset{r}{\mathrel{-\!\!\!\rightarrow\!\!\!\!\!\!\rightarrow}} B$ and $B \overset{s}{\mathrel{-\!\!\!\rightarrow\!\!\!\!\!\!\rightarrow}} C$ is given by "matrix-multiplication"

$$s \cdot r(a, c) = \bigvee_{b \in B} r(a, b) \otimes s(b, c) \tag{1.10}$$

- the identity on A is given by the diagonal-matrix

$$1_A(a, a') = \begin{cases} k & a = a', \\ \bot & \text{else}, \end{cases} \tag{1.11}$$

  where $\bot$ is the bottom element of V.

The Mat-construction was introduced in [BCSW83]. The calculations to check that $\mathrm{Mat}\,\mathbf{V}$ is indeed a quantaloid are routine. $\mathrm{Mat}\,\mathbf{V}$ has an involution $(\ )^\circ$ defined by $r^\circ(a, b) := r(b, a)$. In case $\mathbf{2} := (\{0 \to 1\}, \wedge, 1)$ we have $\mathrm{Mat}\,\mathbf{2} \cong \mathbf{Rel}$.

Remark 1.5.11 We can recover $\mathbf{V}$ from $\mathrm{Mat}\,\mathbf{V}$ as $\mathbf{V} \cong (\mathrm{Mat}\,\mathbf{V}(E, E), \cdot, 1_E)$, where E is any one-element set. ∎

A universal property

Lemma 1.5.12 Mat $\mathbf{V}$ *has biproducts. In fact, it is a free biproduct-completion of* $\mathbf{V}$*, considered as a one-object quantaloid.*

Proof See [Ros01]. ∎

Denote by $|\mathbf{V}|$ and $|\mathrm{Mat}\,\mathbf{V}|$ the underlying category of $\mathbf{V}$ and Mat $\mathbf{V}$, respectively. Then, since **Set** is the coproduct-completion of the one-morphism category $\circlearrowleft$, we obtain a functor

$$\begin{array}{ccc} \circlearrowleft & \longrightarrow & \mathbf{Set} \cong \mathrm{Fam}\,\circlearrowleft \\ \cong\downarrow & & \vdots\,(\,)_\circ \\ |\mathbf{V}| & \longrightarrow & |\mathrm{Mat}\,\mathbf{V}| \end{array} \tag{1.12}$$

where the horizontal functors are the inclusions into the respective completions and $(\ )_\circ$ is the unique (up to equivalence) coproduct-preserving functor such that (1.12) commutes.

We are now going to describe $(\ )_\circ$: Clearly $A_\circ = A$ and

$$f_\circ(a,b) = \begin{cases} k & f(a) = b, \\ \bot & \text{else,} \end{cases} \qquad \text{for } f : A \longrightarrow B \text{ in } \mathbf{Set}. \tag{1.13}$$

Lemma 1.5.13 *Each* $f_\circ$ *has a right adjoint in* Mat $\mathbf{V}$*, given by* $(f_\circ)^\circ$. ∎

In the following we will identify a function f with its $\mathbf{V}$*-graph* $f_\circ$. For the reader's convenience we note the following formulas which we will use without further reference. For functions $X \xrightarrow{f} Y$, $Z \xrightarrow{g} W$ and $\mathbf{V}$-matrices $Y \xrightarrow{a} Z$, $Z \xrightarrow{b} Y$ we have

$$\begin{aligned} a \cdot f(x,z) &= a(f(x),z), & f^\circ \cdot b(z,x) &= b(z,f(x)), \\ g \cdot a(y,w) &= \bigvee_{z \in g^{-1}\{w\}} a(y,z), & b \cdot g^\circ(w,y) &= \bigvee_{z \in g^{-1}\{w\}} b(z,y). \end{aligned}$$

Modularity laws We will now turn our attention to the question whether Mat $\mathbf{V}$ is an allegory:

Lemma 1.5.14 *The following statements are equivalent for an integral quantale* $\mathbf{V}$*:*

(1) $\otimes$ *is idempotent:* $u \otimes u = u$ *holds for all* $u \in V$*;*

(2) $\mathbf{V}$ *is a frame:* $u \otimes v = u \wedge v$ *holds for all* $u, v \in V$*;*

(3) Mat $\mathbf{V}$ *satisfies* (FML)*, hence is an allegory.*

Example 1.5.8(5) shows that integrality is necessary to obtain the above equivalence.

Proof [(1) ⇒ (2)]: By Remark 1.5.10, $u \otimes v$ is a lower bound for $\{u, v\}$. Suppose that z is another lower bound of $\{u, v\}$. Then $z = z \otimes z \leq u \otimes v$, and the claim follows.

[(2) ⇒ (3)]: This fact is well-known, see for example [FS90, 2.111].

[(3) ⇒ (1)]: Remark 1.5.10 gives $u \otimes u \leq u$. Specializing (FML) to matrices between one-element sets, with values $k = \top$ and u, we obtain $u = \top \wedge u \otimes \top \leq u \otimes (u \otimes \top \wedge \top) = u \otimes u$. ∎

Lemma 1.5.15 *The following statements are equivalent for a quantale* $\mathbf{V}$*:*

(1) V *satisfies the* frame law*, that is,* $u \wedge \bigvee v_i = \bigvee u \wedge v_i$ *holds for all* $u, v_i \in V$*,* $i \in I$*;*

(2) $(\operatorname{Mat}\mathbf{V}, \mathbf{Set})$ *satisfies the Frobenius law;*

(3) $(\operatorname{Mat}\mathbf{V}, \mathbf{Set})$ *satisfies the coFrobenius law.*

For any $\mathbf{V}$, $(\operatorname{Mat}\mathbf{V}, \operatorname{Mono}(\mathbf{Set}))$ *satisfies the Frobenius law and the coFrobenius law.*

Proof ([Sch05]) Assume that V satisfies the frame law. For $\mathbf{V}$-matrices $Z \xrightarrow{a} X$, $Z \xrightarrow{b} Y$, a function $X \xrightarrow{f} Y$, and $y \in Y$, $z \in Z$, we have

$$f \cdot (f^* \cdot b \ \wedge \ a)(z, y) = \bigvee_{x \in f^{-1}\{y\}} b(z, f(x)) \wedge a(z, x) = \bigvee_{x \in f^{-1}\{y\}} b(z, y) \wedge a(z, x),$$

and

$$(b \ \wedge \ f \cdot a)(z, y) = b(z, y) \wedge (f \cdot a(z, y)) = b(z, y) \wedge \bigvee_{x \in f^{-1}\{y\}} a(z, x)$$

which are equal by the frame law. To prove the frame law from the Frobenius law, given $u \in V$, $(v_i)_I \subset V$ set $Z = Y = \{*\}$, $b(*, *) = u$, $A = I$, $a(*, i) = v_i$ for all $i \in I$, and let $f : I \longrightarrow \{*\}$ be the unique function. The frame law is not needed in case f is injective.

The equivalence of (2) and (3) follows from 1.2.21(2). ∎

By Lemma 1.5.15, $\operatorname{Mat}\overline{\mathbb{R}}_+$ satisfies the Frobenius law, but by Lemma 1.5.14 it does not satisfy Freyd's modular law, as was first observed in [CH03].

Maps in $\operatorname{Mat}\mathbf{V}$ As we have seen, every ($\mathbf{V}$-graph of) a function is a map in $\operatorname{Mat}\mathbf{V}$. In general, the converse implication does not hold. For example, let $\mathbf{V}$ be the free Boolean algebra on one generator, so $V \cong \{\bot, a, \neg a, \top\}$, with $\otimes = \wedge$ and $k = \top$, and consider the $\mathbf{V}$-matrix $r : 2 \nrightarrow 2$ given by

$$r = \begin{pmatrix} \top & a \\ \bot & \neg a \end{pmatrix}$$

r is a map in $\operatorname{Mat}\mathbf{V}$, but clearly r is not the $\mathbf{V}$-graph of any function. In fact we have the following result [FS90, 2.14]:

Lemma 1.5.16 *Suppose* $\mathbf{V}$ *is a frame. Then* $\operatorname{Map}\operatorname{Mat}\mathbf{V} \cong \mathbf{Set}$ *if and only if* V *is connected, that is, if* $v \in V$ *is complemented (in the sense that there exists* $v' \in V$ *satisfying* $v \wedge v' = \bot$, $v \vee v' = \top$*) then* $v = \bot$ *or* $v = \top$. ∎

Changing V Let $\mathbf{V} = (V, \otimes, k)$ and $\mathbf{W} = (W, \oplus, l)$ be quantales. Every functor $\nu : V \longrightarrow W$ induces a 2-graph morphism $\operatorname{Mat}\nu : \operatorname{Mat}\mathbf{V} \longrightarrow \operatorname{Mat}\mathbf{W}$ by:

$$X \longmapsto X, \qquad r \longmapsto \nu \cdot r.$$

Proposition 1.5.17 Mat *is 2-functorial:* $\operatorname{Mat} 1_{\mathbf{V}} = 1_{\operatorname{Mat}\mathbf{V}}$, $\operatorname{Mat}\xi \cdot \operatorname{Mat}\nu = \operatorname{Mat}(\xi \cdot \nu)$, *and* $\operatorname{Mat}\nu \leq \operatorname{Mat}\nu'$ *if* $\nu \leq \nu'$. *Every* $\operatorname{Mat}\nu$ *commutes with* $(\ )^\circ$. *Moreover,*

$$\begin{array}{ccc} & \mathbf{Set} & \\ \swarrow & & \searrow \\ \operatorname{Mat}\mathbf{V} & \xrightarrow{\operatorname{Mat}\nu} & \operatorname{Mat}\mathbf{W} \end{array} \tag{1.14}$$

commutes if and onlt if $\nu(\bot_V) = \bot_W$ *and* $\nu(k) = l$ *hold. Any 2-graph morphism* $\operatorname{Mat}\mathbf{V} \xrightarrow{F} \operatorname{Mat}\mathbf{W}$ *which commutes with the* **Set***-embeddings arises in this way.*

Proof Functoriality of Mat is obvious; and clearly (1.14) commutes if $\nu(\bot_V) = \bot_W$ and $\nu(k) = l$ holds. To show the converse, consider the action of $\operatorname{Mat}\nu$ on the identity of a two-element set.

Now if $F : \operatorname{Mat}\mathbf{V} \longrightarrow \operatorname{Mat}\mathbf{W}$ commutes with the embeddings of **Set**, then F can be restricted to the endomorphism monoids of one-element sets in $\operatorname{Mat}\mathbf{V}$ and $\operatorname{Mat}\mathbf{W}$, and thus gives, by Remark 1.5.11, a functor $f : V \longrightarrow W$ such that $F = \operatorname{Mat} f$. ∎

Proposition 1.5.18 *Let* $\nu : V \longrightarrow W$ *be a functor such that* (1.14) *commutes. Then:*

(1) $\operatorname{Mat}\nu(a \cdot f) = \operatorname{Mat}\nu(a) \cdot \operatorname{Mat}\nu(f)$ *holds for all* $\mathbf{V}$*-matrices* a *and functions* f.

(2) $\operatorname{Mat}\nu(g) \cdot \operatorname{Mat}\nu(a) \leq \operatorname{Mat}\nu(g \cdot a)$ *holds for all* $\mathbf{V}$*-matrices* a *and functions* g, *and "=" holds if* g *is injective.* $\operatorname{Mat}\nu(g) \cdot \operatorname{Mat}\nu(a) = \operatorname{Mat}\nu(g \cdot a)$ *holds for arbitrary functions* g *and* $\mathbf{V}$*-matrices* a *if and only if* ν *preserves suprema.*

(3) $\operatorname{Mat}\nu$ *is a lax functor if and only if* $V \xrightarrow{\nu} W$ *preserves* $\otimes$ *laxly, that is,* $\nu u \otimes \nu v \leq \nu(u \oplus v)$ *holds.*

(4) $\operatorname{Mat}\nu$ *is a 2-functor if and only if* ν *preserves* $\otimes$ *and suprema.*

(5) $\operatorname{Mat}\nu$ *locally preserves suprema if and only if* $\nu : V \longrightarrow W$ *preserves suprema.* ∎

Embedding Rel into $\operatorname{Mat}\mathbf{V}$ For every quantale $\mathbf{V}$ we have a functor $\iota_{\mathbf{V}} : \mathbf{2} \longrightarrow \mathbf{V}$, defined by $\bot \longmapsto \bot_V$, $\top \longmapsto k_{\mathbf{V}}$. Clearly $\iota_{\mathbf{V}}$ preserves the tensor product.

Lemma 1.5.19 *For any quantale* $\mathbf{V}$ *there exists a right adjoint of* $\iota_{\mathbf{V}}$*, given by the "pessimistic" functor* $p_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{2}$*, with*

$$v \longmapsto \begin{cases} \top & k_{\mathbf{V}} \leq v, \\ \bot & \textit{else.} \end{cases}$$

A left adjoint of $\iota_{\mathbf{V}}$ *exists if and only if* $\mathbf{V}$ *is integral. In this case it is given by the "optimistic" functor* $o_{\mathbf{V}} : \mathbf{V} \longrightarrow \mathbf{2}$*, with*

$$v \longmapsto \begin{cases} \bot & v = \bot_V, \\ \top & \textit{else.} \end{cases}$$

Both $p_{\mathbf{V}}$ *and* $o_{\mathbf{V}}$ *are retractions of* $\iota_{\mathbf{V}}$*. Moreover, the following statements hold:*

- $p_{\mathbf{V}}$ *preserves* $\otimes$ *laxly.* $p_{\mathbf{V}}$ *preserves* $\otimes$ *strictly if and only if* $k_{\mathbf{V}} \leq u \otimes v$ *implies* $k_{\mathbf{V}} \leq u$ *and* $k_{\mathbf{V}} \leq v$ *for all* $u, v \in V$*.*
- $o_{\mathbf{V}}$ *preserves* $\otimes$ *oplaxly, provided it exists.* $o_{\mathbf{V}}$ *preserves* $\otimes$ *if and only if* $\bot_V$ *is* $\otimes$*-prime; that is,* $u \otimes v = \bot_V$ *implies* $u = \bot_V$ *or* $v = \bot_V$ *for all* $u, v \in V$*.* ∎

Observe that the bottom element ∞ in $\overline{\mathbb{R}}_+$ is $+$-prime. Thus we obtain 2-functors

$$\mathbf{Rel} \xrightarrow{\iota} \operatorname{Mat}\overline{\mathbb{R}}_+, \qquad \operatorname{Mat}\overline{\mathbb{R}}_+ \xrightarrow{o} \mathbf{Rel} \tag{1.15}$$

as well as a lax functor $\mathbf{Rel} \xrightarrow{p} \operatorname{Mat}\overline{\mathbb{R}}_+$.

1.6 Pre-scenes

A *pre-scene* is a diagram

$$\mathbf{S} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathbf{O} \tag{1.16}$$

where $\mathbf{O}$ is an ordered category, $\mathbf{S}$ is a mere category, $(\,)_\circ$ is a functor which is bijective on objects and $(\,)^\circ : \mathbf{S}^{\mathrm{op}} \longrightarrow \mathbf{O}$ is a (pseudo-)functor which satisfies $f_\circ \dashv f^\circ$ and $S_\circ = S^\circ$ for all $\mathbf{S}$-morphisms f and all $\mathbf{S}$-objects S. Hence $(\,)^\circ$ is determined by $(\,)_\circ$ up to equivalence, its sole purpose is to provide us with a choice of right adjoints for the morphisms of $\mathbf{S}$. We will just write S for $S_\circ$ or S°. We reserve the symbol "$\longrightarrow$" for $\mathbf{O}$-morphisms of the form $f_\circ$ and denote arbitrary $\mathbf{O}$-morphisms by "$\nrightarrow$". We will write $\mathfrak{O} = (\mathbf{S}, \mathbf{O})$ to denote a pre-scene, $(\,)_\circ$ and $(\,)^\circ$ will assumed to be understood.

Examples for pre-scenes are $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$, where we identify a function with its $\mathbf{V}$-graph, $(\mathbf{C}, \operatorname{Rel}\mathbf{C})$ for a regular category $\mathbf{C}$, and of course $(\operatorname{Map}\mathbf{O}, \mathbf{O})$ for any ordered category $\mathbf{O}$.

The author may be forgiven for not studying morphisms of pre-scenes. In Chapter 2, we will study morphisms of *scenes*, of which pre-scenes are just a special case.

1.6.1 $(\mathbf{S}, \mathbf{O})$-categories

Now we introduce $\mathfrak{O}$-categories for a pre-scene $\mathfrak{O} = (\mathbf{S}, \mathbf{O})$. This construction is related to the *monad construction* of Carboni, Kasangian, and Walters (restricted to a thin bicategory; see [CKW87], also [Ros96]). In case $\mathfrak{O} = (\mathbf{Set}, \operatorname{Mat}\mathbf{V})$, it amounts to the $\mathbf{V}$-category construction of Lawvere (for $\mathbf{V}$ thin); see [Law73].

On the other hand, $\mathfrak{O}$-categories are a special kind of lax algebras, namely the reflexive and transitive lax algebras for the identity-monad on $\mathbf{O}$. Nonetheless, we will study them separately in this section, concentrating on the facts which are special for them and are not easily stated for more general lax algebras.

Definition 1.6.1 Given a pre-scene $\mathfrak{O} = (\mathbf{S}, \mathbf{O})$ we define the ordered category $\operatorname{Cat}\mathfrak{O}$ of $\mathfrak{O}$-*categories* and $\mathfrak{O}$-*functors* as follows:

- objects are pairs (X, a), with X an $\mathbf{S}$-object and a an $\mathbf{O}$-morphism $X \nrightarrow X$ which is *reflexive* and *transitive*, that is, which satisfies $1_X \leq a$ and $a \cdot a \leq a$;
- morphism $(X, a) \longrightarrow (Y, b)$ are $\mathfrak{O}$-*functors*, that is, $\mathbf{S}$-morphisms $f : X \longrightarrow Y$ which satisfy

$$\begin{array}{ccc} X & \xrightarrow{f_\circ} & Y \\ {\scriptstyle a}\downarrow & \leq & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f_\circ]{} & Y \end{array} \tag{1.17}$$

- for $\mathfrak{O}$-functors $f, g : (X, a) \longrightarrow (Y, b)$, a 2-cell from f to g exists if and only if $g_\circ \leq b \cdot f_\circ$. In this case we write $f \Rightarrow g$. If $f \Rightarrow g$ and $g \Rightarrow f$, we write $f \mathrel{\widetilde{\Rightarrow}} g$.

Examples

- In case $\mathfrak{O} = (\mathbf{Set}, \mathbf{Rel})$, we have $\operatorname{Cat}\mathfrak{O} \cong \mathbf{Ord}$, as is seen easily.
- In case $\mathfrak{O} = (\mathbf{E}, \operatorname{Rel}\mathbf{E})$ for an (elementary) topos $\mathbf{E}$, we obtain the category $\mathbf{Ord}(\mathbf{E})$ of ordered sets in $\mathbf{E}$; see [Woo04].
- Let $\mathbf{Mon}$ denote the regular category of monoids and monoid homomorphisms. Then $\mathbf{C} = \operatorname{Cat}(\mathbf{Mon}, \operatorname{Rel}(\mathbf{Mon}))$ is the category of *ordered monoids* and *monotone monoid-homomorphisms*. Indeed, a $\mathbf{C}$-object is given by a monoid M together with a submonoid $\leq$ of $M \times M$ satisfying the required reflexivity and transitivity conditions in $\operatorname{Rel}(\mathbf{Mon})$. Since pullbacks and image-factorizations in $\mathbf{Mon}$ are concrete, we see that these are equivalent to the reflexivity and transitivity of the underlying relation of $\leq$ (on the underlying set of M). The inclusion $\leq \hookrightarrow M \times M$ is a morphism in $\mathbf{Mon}$ if and only if

$$x \leq y, \quad x' \leq y' \qquad \Longrightarrow \qquad xx' \leq yy'$$

holds. A **C**-morphism is a morphism of the underlying monoids which preserves the order-relations, again by concreteness of limits and image-factorizations in **Mon**.

- In case $\mathfrak{O} = (\mathbf{Set}, \operatorname{Mat}\mathbf{V})$, we obtain the **V**-categories as defined by Lawvere [Law73] (here we think of **V** as a thin monoidal closed category). In particular, for $\mathbf{V} = \overline{\mathbb{R}}_+$, we obtain the category **Met** of *generalized metric spaces*. That is, an $\mathfrak{O}$-category is given by a set X equipped with a distance function $X \times X \xrightarrow{a} [0, \infty]$. Reflexivity of a is equivalent to $a(x, x) = 0$ for all $x \in X$, while transitivity of a is equivalent to the triangle inequality

$$a(x, z) \leq a(x, y) + a(y, z)$$

for all $x, y, z \in X$. Here we use the natural order on $[0, \infty]$. In this context (1.17) is equivalent to

$$\forall x, x' \in X : b(f(x),\ f(x')) \leq a(x, x'),$$

thus to f being a contraction. $f \Rightarrow g$ holds if and only if we have $b(f(x),\ g(x)) = 0$ for all x. Since a does not have to be symmetric, instead of thinking of $a(x, x')$ as the distance of x and x' it seems more appropriate to think of it as the amount of energy needed to transform the "system" (X, a) from state x to state x'.

If we replace $\overline{\mathbb{R}}_+$ by any frame **V**, we obtain the category of V-valued *ultrametric spaces*.

We call a $\mathfrak{O}$-category (X, a) *discrete* if $a \approx 1_X$. A morphism $(X, a) \xrightarrow{f} (Y, b)$ is called *fully-faithful* (or *initial*, *cartesian*) if $a \approx f^\circ \cdot b \cdot f_\circ$ holds. In **Ord** and more generally in $\mathbf{Ord}(\mathbf{E})$, this notion of fully-faithfulness coincides with the one already used before, namely $f : X \longrightarrow Y$ is fully-faithful if and only if

$$x \leq x' \iff f(x) \leq f(x')$$

holds for all $x, x' \in X$.

A characterization of adjoints in $\operatorname{Cat}\mathfrak{O}$

Proposition 1.6.2 *Let (X, a), (Y, b) be $\mathfrak{O}$-categories, and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ be **S**-morphisms. The following statements are equivalent:*

(1) $b \cdot f_\circ \approx g^\circ \cdot a$;

(2) $(X, a) \xrightarrow{f} (Y, b)$ *and* $(Y, b) \xrightarrow{g} (X, a)$ *are $\mathfrak{O}$-functors and $f \dashv g$ in $\operatorname{Cat}\mathfrak{O}$, that is, $1_X \Rightarrow g \cdot f$ and $f \cdot g \Rightarrow 1_Y$ hold.*

In case $\mathrm{Cat}(\mathbf{Set}, \mathrm{Mat}\,\mathbf{2}) \cong \mathbf{Ord}$, condition (1) translates to $f(x) \leq y \iff x \leq g(y)$ for all $x \in X$, $y \in Y$, that is, to the well-known adjointness condition for ordered sets.

Proof [(1) $\Rightarrow$ (2)]: The fact that (X, a), (Y, b) are $\mathfrak{O}$-categories and (1) imply

$$1_Y \leq g^\circ \cdot g_\circ \leq g^\circ \cdot a \cdot g_\circ \approx b \cdot f_\circ \cdot g_\circ \qquad (\dagger)$$

and

$$g_\circ \cdot f_\circ \leq g_\circ \cdot b \cdot f_\circ \approx g_\circ \cdot g^\circ \cdot a \leq a. \qquad (\ddagger)$$

Using (†), we obtain $g_\circ \cdot b \leq g_\circ \cdot b \cdot b \cdot f_\circ \cdot g_\circ \leq g_\circ \cdot b \cdot f_\circ \cdot g_\circ \approx g_\circ \cdot g^\circ \cdot a \cdot g_\circ \leq a \cdot g_\circ$ hence $(Y, b) \xrightarrow{g} (X, a)$ is a $\mathfrak{O}$-functor. Using (‡), we obtain $f_\circ \cdot a \leq g^\circ \cdot g_\circ \cdot f_\circ \cdot a \leq g^\circ \cdot a \cdot a \leq g^\circ \cdot a \approx b \cdot f_\circ$ hence also $(X, a) \xrightarrow{f} (Y, b)$ is a $\mathfrak{O}$-functor.

Finally, (†) is equivalent to $f \cdot g \Rightarrow 1_Y$, and (‡) is equivalent to $1_X \Rightarrow g \cdot f$.

[(2) $\Rightarrow$ (1)]: We obtain, using the conditions arising from (2):

$$\begin{aligned} b \cdot f_\circ &\leq g^\circ \cdot g_\circ \cdot b \cdot f_\circ \leq g^\circ \cdot a \cdot g_\circ \cdot f_\circ \leq g^\circ \cdot a \cdot a \\ &\leq g^\circ \cdot a \\ &\leq b \cdot f_\circ \cdot g_\circ \cdot g^\circ \cdot a \leq b \cdot f_\circ \cdot a \leq b \cdot b \cdot f_\circ \\ &\leq b \cdot f_\circ. \end{aligned}$$

■

Lemma 1.6.3 *Let* $(X, a) \underset{g}{\overset{f}{\leftrightarrows}} (Y, b)$ *be* $\mathfrak{O}$*-functors. Then:*

(1) *f is fully-faithful if and only if $g \cdot f \Rightarrow 1_X$;*

(2) *g is fully-faithful if and only if $1_Y \Rightarrow f \cdot g$.*

Proof Straightforward using Proposition 1.6.2. ■

1.6.2 $(\mathbf{S}, \mathbf{O})$-distributors

Given $\mathfrak{O}$-categories (X, a) and (Y, b), a *$\mathfrak{O}$-distributor* $r : (X, a) \nrightarrow (Y, b)$ is given by an $\mathbf{O}$-morphism $r : X \nrightarrow Y$ which satisfies

$$b \cdot r \leq r \qquad \text{and} \qquad r \cdot a \leq r.$$

We order the set of all $\mathfrak{O}$-distributors $(X, a) \nrightarrow (Y, b)$ by the order induced by the one on $\mathbf{O}(X, Y)$.

Example 1.6.4 In case $\mathfrak{O} = (\mathbf{Set}, \mathbf{Rel})$, $\mathfrak{O}$-distributors are the same as distributors in the sense of Section 1.3.5.

Lemma 1.6.5 *$\mathfrak{O}$-distributors are closed under composition and infima in $\mathbf{O}$. For every $\mathfrak{O}$-category (X,a), $a : (X,a) \nrightarrow (X,a)$ is a $\mathfrak{O}$-distributor which is the identity on (X,a). In case $\mathbf{O}$ is a quantaloid, the class of $\mathfrak{O}$-distributors is closed under suprema in $\mathbf{O}$.* ∎

Hence we have the ordered category $\mathrm{Dist}\,\mathfrak{O}$ of $\mathfrak{O}$-categories and $\mathfrak{O}$-distributors. $\mathrm{Dist}\,\mathfrak{O}$ is a quantaloid provided $\mathbf{O}$ is one. We remark that the construction of $(\mathbf{S},\mathbf{O})$-distributors depends only on $\mathbf{O}$. Distributors are well-known under the names *ideals* [CS86], *monad morphisms* [Ros96], *compatible* or *ordered relations* (for $\mathbf{O} = \mathbf{Rel}$, [Pis99], [HT06]), or *bimodules* [Law73].

We have chosen the somewhat misleading name $(\mathbf{S},\mathbf{O})$-distributor in view of the connection with $(\mathbf{S},\mathbf{O})$-functors which we now want to make explicit:

Every $\mathfrak{O}$-functor $f : (X,a) \longrightarrow (Y,b)$ gives rise to two $\mathfrak{O}$-distributors: we write

$$f_* := b \cdot f_\circ \qquad \text{and} \qquad f^* := f^\circ \cdot b. \tag{1.18}$$

Lemma 1.6.6 *If $f : (X,a) \longrightarrow (Y,b)$ is a $\mathfrak{O}$-functor, then*

(1) *$f_* : (X,a) \nrightarrow (Y,b)$ and $f^* : (Y,b) \nrightarrow (X,a)$ are $\mathfrak{O}$-distributors;*

(2) *$f_* \dashv f^*$ in $\mathrm{Dist}\,\mathfrak{O}$;*

(3) *$1^*_{(X,a)} = a = (1_{(X,a)})_*$ holds; and $(g \cdot f)_* \approx g_* \cdot f_*$ and $(g \cdot f)^* \approx f^* \cdot g^*$ hold for any $\mathfrak{O}$-functor g such that $g \cdot f$ is defined.*

If $e : (X,a) \longrightarrow (Y,b)$ is another $\mathfrak{O}$-functor, then $f \Rightarrow e$ iff $e_* \leq f_*$ iff $f^* \leq e^*$. ∎

Corollary 1.6.7 *We have pseudo-functors*

$$\begin{array}{ccc} \mathrm{Cat}\,\mathfrak{O}^{\mathrm{co}} \longrightarrow \mathrm{Dist}\,\mathfrak{O} & \qquad & \mathrm{Cat}\,\mathfrak{O}^{\mathrm{op}} \longrightarrow \mathrm{Dist}\,\mathfrak{O} \\ f \longmapsto f_* & & f \longmapsto f^*. \end{array}$$

Both $(\ )_*$ and $(\ )^*$ are fully-faithful on 2-cells. ∎

We may summarize the above results as:

Proposition 1.6.8 *For every pre-scene $\mathfrak{O}$ we obtain a pre-scene $(\mathrm{Cat}\,\mathfrak{O}, \mathrm{Dist}\,\mathfrak{O})$. Here we consider $\mathrm{Cat}\,\mathfrak{O}$ as a mere category.* ∎

For $\mathbf{S} = \mathbf{Set}$, $\mathbf{O} = \mathrm{Mat}\,\mathbf{V}$, (1.18) specializes to: $f_*(x,y) = b(f(x),y)$ and $f^*(y,x) = b(y,f(x))$. For $\mathbf{V} = \mathbf{2}$ we obtain:

$$x\, f_*\, y \iff f(x) \leq y, \qquad y\, f^*\, x \iff y \leq f(x).$$

Observe that, by Proposition 1.6.2, $f \dashv g$ in $\mathrm{Cat}\,\mathfrak{O}$ if and only if $f_* \approx g^*$. Therefore, every adjoint situation $f \dashv g$ in $\mathrm{Cat}\,\mathfrak{O}$ gives rise to an adjoint string

$$g_* \dashv g^* \approx f_* \dashv f^* \tag{1.19}$$

in $\mathrm{Dist}\,\mathfrak{O}$.

On maps in $\mathrm{Dist}\,\mathfrak{O}$**.** As we have seen before, every $\mathfrak{O}$-functor induces an adjoint situation in $\mathrm{Dist}\,\mathfrak{O}$. The question whether all adjoint situations are of this kind leads to a notion of *completeness* for $\mathfrak{O}$-categories; see [Law73] and [CS86].

Lemma 1.6.9 *Let $f : (X, a) \longrightarrow (Y, b)$ be a $\mathfrak{O}$-functor. f_* is an injection in $\mathrm{Dist}\,\mathfrak{O}$ if and only if f is fully-faithful. f_* is a cover provided f is a cover in $\mathbf{S}$.*

Proof $f^* \cdot f_* \approx f^\circ \cdot b \cdot b \cdot f_\circ \approx f^\circ \cdot b \cdot f_\circ$ holds, so $f^* \cdot f_* \approx a$ if and only if f is fully-faithful. In case f is a cover in $\mathbf{S}$, we have $f_* \cdot f^* = b \cdot f \cdot f^\circ \cdot b \approx b \cdot b \approx b$. ∎

A monotone function $f : (X, \leq) \longrightarrow (Y, \leq)$ is a cover in **Dist** if and only if f "essentially surjective", that is, if for every $y \in Y$ there exists $x \in X$ with $f(x) \approx y$. To see this, observe that $\leq_Y \subset f_* \cdot f^*$ is equivalent to $\forall y, y' (y \leq y' \implies \exists x : y \leq f(x) \leq y')$ which is clearly equivalent to $\forall y \, \exists x \, f(x) \approx y$.

Biproducts Recall that $\mathrm{Dist}\,\mathfrak{O}$ is a quantaloid provided $\mathbf{O}$ is one.

Proposition 1.6.10 *Let $\mathbf{O}$ be a quantaloid. If $\mathbf{O}$ has biproducts, so does $\mathrm{Dist}\,\mathfrak{O}$.*

Proof Given a set-indexed family $(X_i, a_i)_I$ of $\mathfrak{O}$-categories, let $\big(X \underset{e_i}{\overset{p_i}{\rightleftarrows}} X_i \big)_I$ be a biproduct in $\mathbf{O}$ and set $a = \bigvee e_i \cdot a_i \cdot p_i$. Routine calculations show that

$$\Big((X, a) \underset{e_i \cdot a_i}{\overset{a_i \cdot p_i}{\rightleftarrows}} (X_i, a_i) \Big)_I$$

is a biproduct in $\mathrm{Dist}\,\mathfrak{O}$. ∎

1.6.3 Duality for $\mathfrak{O}$-categories and $\mathfrak{O}$-groupoids

Definition 1.6.11 Let $\mathfrak{O} = (\mathbf{S}, \mathbf{O})$ be a pre-scene. An involution $(\ )^t$ on $\mathbf{O}$ is said to be *compatible* with $\mathfrak{O}$ if

$$\begin{array}{ccc} \mathbf{O} & \xrightarrow{\ (\ )^t\ } & \mathbf{O}^{\mathrm{op}} \\ & {\scriptstyle (\ )_\circ} \searrow \quad \swarrow {\scriptstyle (\ )^\circ} & \\ & \mathbf{S} & \end{array} \qquad (\approx) \tag{1.20}$$

holds. An *involutive pre-scene* is a pre-scene equipped with a compatible involution.

In this section $\mathfrak{O}$ will be an involutive pre-scene with involution denoted by $(\ )^\circ$. This is unambiguous by (1.20).

Definition 1.6.12 Let (X, a) be a $\mathfrak{O}$-category. The *dual* $\mathfrak{O}$-category $(X, a)^{\mathrm{op}}$ is defined to be (X, a°). A $\mathfrak{O}$-*groupoid* is a $\mathfrak{O}$-category (X, a) such that $a \approx a^\circ$.

Proposition 1.6.13 $(\ )^{\mathrm{op}} : (X, a) \longmapsto (X, a)^{\mathrm{op}}$, $f \longmapsto f$ *is a duality on* $\mathrm{Cat}\,\mathfrak{O}$. ∎

We denote the full subcategory of $\mathrm{Cat}\,\mathfrak{O}$ spanned by the $\mathfrak{O}$-groupoids by $\mathrm{Grpd}\,\mathfrak{O}$.

Examples

- For $\mathfrak{O} = (\mathbf{Set}, \mathbf{Rel})$, $\mathrm{Grpd}\,\mathfrak{O}$ is the category of sets equipped with an equivalence relation and relation-preserving functions.

- In case $\mathfrak{O} = (\mathbf{Set}, \mathrm{Mat}\,\overline{\mathbb{R}}_+)$, $\mathrm{Grpd}\,\mathfrak{O}$ is the category of symmetric generalized metric spaces.

  More generally, if $\mathbf{V}$ is any quantale, $\mathrm{Grpd}(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ is the category of *global* $\mathbf{V}$*-valued sets* and *extensional functions*; see [Š01, 1.2].

Proposition 1.6.14 *If* $\mathbf{O}$ *locally has binary infima, then the inclusion* $\mathrm{Grpd}\,\mathfrak{O} \longhookrightarrow \mathrm{Cat}\,\mathfrak{O}$ *has a concrete right adjoint.*

Proof Given a $\mathfrak{O}$-category (X, a), we claim that $(X, a \wedge a^\circ) \xrightarrow{1_X} (X, a)$ is a couniversal arrow. We first observe that $1_X \leq a \wedge a^\circ$ holds, and moreover

$$\begin{aligned}(a \wedge a^\circ) \cdot (a \wedge a^\circ) &\leq a \cdot a \;\wedge\; a^\circ \cdot a \;\wedge\; a \cdot a^\circ \;\wedge\; a^\circ \cdot a^\circ \\ &\leq a \cdot a \;\wedge\; (a \cdot a)^\circ \\ &\leq a \;\wedge\; a^\circ,\end{aligned}$$

that is, $(X, a \wedge a^\circ)$ is a $\mathfrak{O}$-category, clearly it is also a $\mathfrak{O}$-groupoid. Now given any morphism $(Y, b) \xrightarrow{f} (X, a)$ in $\mathrm{Cat}\,\mathfrak{O}$ such that (Y, b) is a groupoid, we have $f \cdot b \cdot f^\circ \leq a$, and thus $f \cdot b \cdot f^\circ \approx (f \cdot b \cdot f^\circ)^\circ \leq a^\circ$. Hence $(Y, b) \xrightarrow{f} (X, a \wedge a^\circ)$ is a morphism. ■

1.6.4 Kleisli pre-scenes

Let $\mathsf{T} = (T, e, m)$ be a 2-monad on $\mathbf{Ord}$ such that for each f there exists $Tf \dashv T^*f$. We define the *Kleisli pre-scene* $\mathrm{Klei}\,\mathsf{T}$

$$\mathbf{Set} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathbf{Ord}_\mathsf{T}$$

as follows: let $F_\mathsf{T}, F^*_\mathsf{T} : \mathbf{Ord} \longrightarrow \mathbf{Ord}_\mathsf{T}$ be given by $F_\mathsf{T}X = F^*_\mathsf{T}X = X$ and $F_\mathsf{T}f = Tf \cdot e_X = e_Y \cdot f$, $F^*_\mathsf{T}f = T^*f \cdot e_Y$ for $f : X \longrightarrow Y$ in $\mathbf{Ord}$. One easily checks that $F_\mathsf{T}f \dashv F^*_\mathsf{T}f$ holds in $\mathbf{Ord}_\mathsf{T}$. Indeed, we have $F_\mathsf{T}f * F^*_\mathsf{T}f = m_Y \cdot Te_Y \cdot Tf \cdot T^*f \cdot e_Y \leq e_Y$ and $e_X = m_X \cdot Te_X \cdot e_X \leq m_X \cdot T(T^*f \cdot Tf \cdot e_X) \cdot e_X = F^*_\mathsf{T}f * F_\mathsf{T}f$. We define $(\,)_\circ$ and $(\,)^\circ$ as the restrictions of F_T and F^*_T, respectively, to discrete ordered sets.

Given $(\mathrm{Klei}\,\mathsf{T})$-categories (X, a), (Y, b) a function $f : X \longrightarrow Y$ is a $(\mathrm{Klei}\,\mathsf{T})$-functor if and only if $Tf \cdot a \leq b \cdot f$ holds in $\mathbf{Ord}$.

Filters and topological spaces Let $\mathsf{F} = (F, e, m)$ be the filter-on-downsets monad on **Ord**.

Proposition 1.6.15 $\mathrm{Cat}(\mathrm{Klei}\,\mathsf{F})$ *is concretely isomorphic to* $\mathbf{Top}^{\mathrm{co}}$*, that is, to the 2-category of topological spaces and continuous maps with the pointwise specialization order as 2-cells.*

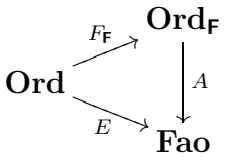
The quickest way to prove Proposition 1.6.15 is to use the description of topological spaces via neighborhoods: observe that the condition $a(x) \subset e_X(x)$ and $a(x) \subset a * a(x)$ are just the characteristic properties of neighborhoods of a point x of a topological space. This is done in [HT06]. We will use a different approach.

Define the ordered category **Fao** as follows:

- objects are ordered sets;
- a morphism $X \longrightarrow Y$ is given by a function $\mathbb{D}Y \longrightarrow \mathbb{D}X$ which preserves finite infima; composition is the usual composition of functions;
- 2-cells are defined according to pointwise order.

We have a functor $\mathbf{Ord} \xrightarrow{E} \mathbf{Fao}$ which maps objects identically and sends f to $\mathbb{D}^* f$.

Lemma 1.6.16 *The 2-functor* $A : \mathbf{Ord}_{\mathsf{F}} \longrightarrow \mathbf{Fao}$ *defined by* $AX = X$ *and* $A(X \xrightarrow{\alpha} FY) = \mathbb{D}^*\alpha \cdot (\ )^{\#}_Y$ *is an isomorphism. Moreover,*



commutes.

Proof $A(\alpha)$ preserves finite infima since both $\mathbb{D}^*\alpha$ and $(\ )^{\#}$ do so.

Note that $\mathbb{D}^*(\ )^{\uparrow}_{\mathbb{D}X} \cdot \mathbb{D}(\ )^{\uparrow}_{\mathbb{D}X} = 1_{\mathbb{D}X}$ holds. Indeed, $(\ )^{\uparrow}_{\mathbb{D}X}$ is fully-faithful, thus $\mathbb{D}(\ )^{\uparrow}_{\mathbb{D}X}$ is fully-faithful, too, and hence an injection. Moreover, we have $\mathbb{D}^*(\ )^{\downarrow}_X = \bigcup_X$ since both are right adjoints of $\mathbb{D}(\ )^{\downarrow}_X$. In particular, we have $\mathbb{D}^*(\ )^{\downarrow}_X \cdot (\ )^{\downarrow}_{\mathbb{D}X} = 1_{\mathbb{D}X}$. Using this, we obtain:

$$\mathbb{D}^*(e_Y \cdot f) \cdot (\ )^{\#}_Y = \mathbb{D}^* f \cdot \mathbb{D}^*(\ )^{\downarrow}_X \cdot \mathbb{D}^*(\ )^{\uparrow}_{\mathbb{D}X} \cdot \mathbb{D}(\ )^{\uparrow}_{\mathbb{D}X} \cdot (\ )^{\downarrow}_{\mathbb{D}X} = \mathbb{D}^* f \cdot \mathbb{D}^*(\ )^{\downarrow}_X \cdot (\ )^{\downarrow}_{\mathbb{D}X} = \mathbb{D}^* f.$$

Thus the diagram commutes and A preserves identities. To show that A preserves composition take $X \xrightarrow{\alpha} FY$, $Y \xrightarrow{\beta} FZ$ in **Ord**. For $C \in \mathbb{D}Z$ and $\mathfrak{y} \in FY$, we have

$$\begin{aligned}
\mathfrak{y} \in (m_Z \cdot F\beta)^{-1}[C^\#] &\iff (m_Z \cdot F\beta)(\mathfrak{y}) \in C^\# \\
&\iff C \in m_Z(F\beta(\mathfrak{y})) \\
&\iff C^\# \in F\beta(\mathfrak{y}) \\
&\iff \beta^{-1}[C^\#] \in \mathfrak{y} \\
&\iff \mathfrak{y} \in (\beta^{-1}[C^\#])^\#.
\end{aligned}$$

Thus, $A(\beta * \alpha)(C) = (\beta * \alpha)^{-1}[C^\#] = \alpha^{-1}[(m_Z \cdot F\beta)^{-1}[C^\#]] = \alpha^{-1}[(\beta^{-1}[C^\#])^\#] = A(\beta) \cdot A(\beta)(C)$ in **Fao**.

Finally, define A^{-1} by $A^{-1}(i)(x) = \{\, D \in \mathbb{D}Y \mid x \in i(D) \,\}$. The proof that A^{-1} is a 2-functor and is inverse to A is routine. ■

Proof (of 1.6.15) Under the isomorphism A, a lax Klei F-category (X, a) corresponds to a set X equipped with a function $i = A(a) : PX \longrightarrow PX$ which preserves finite infima. As easily seen, $1_X \leq a$ in $\mathbf{Ord}_\mathsf{F}$ is equivalent to $iB \subset B$ for all $B \subset X$, whereas transitivity of a is equivalent to $iB \subset iiB$. Thus a lax algebra is equivalent to a set equipped with a topological interior operator, thus to a topological space. The proof that a function is a Klei F-functor if and only if it is continuous is routine. ■

Fuzzy upsets

For an ordered set X, we can identify $\mathbb{U}X$ with the set of all monotone functions $X \longrightarrow \mathbf{2}$. We will generalize this to *fuzzy upsets* by considering monotone functions $X \longrightarrow \mathbf{V}$, where $\mathbf{V}$ is an arbitrary quantale. To ease the calculations we will use the presentation of monads as *Kleisli-triples*.

Kleisli-triples A Kleisli-triple on a category $\mathbf{C}$ is given by the following data

- a function $\operatorname{obj}\mathbf{C} \xrightarrow{T} \operatorname{obj}\mathbf{C}$;
- for each $X \in \operatorname{obj}\mathbf{C}$, a morphism $X \xrightarrow{e_X} TX$;
- for each $X, Y \in \operatorname{obj}\mathbf{C}$, a function $\mathbf{C}(X, TY) \xrightarrow{(\,)^\#} \mathbf{C}(TX, TY)$;

subject to the following conditions

(K1) $(e_X)^\# = 1_{TX}$;

(K2) $f^\# \cdot e_X = f$ for $f : X \longrightarrow TY$;

(K3) $g^\# \cdot f^\# = (g^\# \cdot f)^\#$ for $f : X \longrightarrow TY$, $g : Y \longrightarrow TZ$.

Every monad (T, e, m) gives rise to a Kleisli-triple $(T, e, (\ )^{\#})$ by $f^{\#} = m_Y \cdot Tf$; from a Kleisli-triple we obtain a monad by $Tf = (e_Y \cdot f)^{\#}$, $m_X = 1^{\#}_{TX}$, and these constructions are inverse to each other; see [MS04].

In case $\mathbf{C}$ is an ordered category a Kleisli-triple $(T, e, (\ )^{\#})$ on $\mathbf{C}$ defines a 2-monad precisely when $(\ )^{\#}$ is monotone.

Fuzzy upsets Fix a quantale $\mathbf{V} = (V, \otimes, k)$. We define the $\mathbf{V}$-*valued fuzzy upset Kleisli-triple* $(U_{\mathbf{V}}, e, (\ )^{\#})$ on **Ord** as follows:

- $U_{\mathbf{V}}X = \{\, X \xrightarrow{\varphi} V \mid \varphi \text{ monotone} \,\}$, ordered by $\varphi \leq \psi$ iff $\psi(x) \leq \varphi(x)$ holds for all x;
- $e_X : X \longrightarrow U_{\mathbf{V}}X$ by $e_X(x)(x') = \begin{cases} k & x \leq x', \\ \bot & \text{else;} \end{cases}$
- for $f : X \longrightarrow U_{\mathbf{V}}Y$, we define $f^{\#}(\varphi)(y) = \bigvee_{x \in X} \varphi(x) \otimes f(x)(y)$.

Clearly the required monotonicity conditions are fulfilled. Ad (K1):

$$e^{\#}_X(\varphi)(x) = \bigvee_{x' \in X} \varphi(x') \otimes e_X(x')(x) = \bigvee_{x' \leq x} \varphi(x') = \varphi(x)$$

where the last equation holds since φ is monotone. Thus $e^{\#}_X = 1_{U_{\mathbf{V}}X}$ holds. To show (K2), observe

$$\big(f^{\#} \cdot e_X(x)\big)(y) = f^{\#}(e_X(x))(y) = \bigvee_{x' \in X} e_X(x)(x') \otimes f(x')(y) = \bigvee_{x \leq x'} f(x')(y) = f(x)(y);$$

here the last equation holds since $f(\ )(y)$ is contravariantly functorial for each $y \in Y$. Finally, to show that (K3) is fulfilled, observe that

$$\begin{aligned} \big(g^{\#} \cdot f^{\#}\big)(\varphi)(z) &= g^{\#}(f^{\#}(\varphi))(z) \\ &= \bigvee_{y \in Y} f^{\#}(\varphi)(y) \otimes g(y)(z) \\ &= \bigvee_{y \in Y} \Big(\bigvee_{x \in X} \varphi(x) \otimes f(x)(y) \Big) \otimes g(y)(z); \end{aligned} \tag{$\dagger$}$$

and

$$\begin{aligned} \big(g^{\#} \cdot f\big)(\varphi)(z) &= \bigvee_{x \in X} \varphi(x) \otimes (g^{\#} \cdot f)(x)(z) \\ &= \bigvee_{x \in X} \varphi(x) \otimes \Big(\bigvee_{y \in Y} f(x)(y) \otimes g(y)(z) \Big). \end{aligned} \tag{$\ddagger$}$$

Since $\otimes$ distributes over $\bigvee$, (†) and (‡) are equal. Thus, for $f : X \longrightarrow Y$ in **Ord**, $U_{\mathbf{V}}f$ is

$$U_{\mathbf{V}}f(\varphi)(y) = (e_Y \cdot f)^{\#}(\varphi)(y) = \bigvee_{x\in X} \varphi(x) \otimes e_Y(f(x))(y) = \bigvee_{\substack{x\in X \\ f(x)\leq y}} \varphi(x).$$

Observe that for each f in **Ord**, $U_{\mathbf{V}}f$ has a left adjoint $U^*_{\mathbf{V}}f$, defined by $U^*_{\mathbf{V}}f(\psi) = \psi \cdot f$. The multiplication m is defined by $m_X(\Phi)(x) = (1_{U_{\mathbf{V}}X})^{\#}(\Phi)(x) = \bigvee_{\varphi\in U_{\mathbf{V}}X} \Phi(\varphi)\otimes\varphi(x)$.

We remark that we did not use the commutativity of $\otimes$ for describing $U_{\mathbf{V}}$. In case $\mathbf{V} = \mathbf{2}$, $U_{\mathbf{V}}$ is the upperset monad $\mathbb{U}$ on **Ord**.

Fuzzy downsets The fuzzy downsets monad $D_{\mathbf{V}}$ on **Ord** is defined by $D_{\mathbf{V}}(X) = (U_{\mathbf{V}}(X^{\mathrm{op}}))^{\mathrm{op}}$. Thus $D_{\mathbf{V}}(X)$ is the set of all monotone function $X^{\mathrm{op}} \longrightarrow V$, ordered by the pointwise order, $e_X : X \longrightarrow D_{\mathbf{V}}X$ is given by $e_X(x)(x') = k$ iff $x' \leq x$ and $\bot$ otherwise, and the extension operator $(\ )^{\#}$ for $D_{\mathbf{V}}$ is defined as the one for $U_{\mathbf{V}}$. For any f in **Ord**, $D_{\mathbf{V}}(f)$ has a *right adjoint* $D^*_{\mathbf{V}}f$.

Fuzzy powersets We define the fuzzy-powerset monad $P_{\mathbf{V}}$ for a quantale $\mathbf{V}$ to be the "trace" of $D_{\mathbf{V}}$ to **Set**, that is, $P_{\mathbf{V}}$ is defined as the composite

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{discrete}} & \mathbf{Ord} \\ {\scriptstyle P_{\mathbf{V}}}\downarrow & & \downarrow{\scriptstyle D_{\mathbf{V}}} \\ \mathbf{Set} & \xleftarrow{\text{forget}} & \mathbf{Ord}, \end{array}$$

where we define units and extension operators accordingly. Thus, $P_{\mathbf{V}}X$ is the set V^X of all functions $X \longrightarrow V$.

The following result can be seen as a straightforward generalization of the well-known fact that $\mathbf{Rel} \cong \mathbf{Set}_P = \mathbf{Set}_{P_{\mathbf{2}}}$.

Remark 1.6.17 $\mathbf{Set}_{P_{\mathbf{V}}}$ is isomorphic to the underlying category of $\mathrm{Mat}\,\mathbf{V}$ and the isomorphism commutes with the embeddings of **Set**.

Proof A $\mathbf{Set}_{P_{\mathbf{V}}}$-morphism $X \xrightarrow{a} Y$ corresponds to a function $X \times Y \xrightarrow{\overline{a}} V$ via $\overline{a}(x, y) = a(x)(y)$. Clearly we have $\overline{e_Y \cdot f} = f_\circ$. For $X \xrightarrow{a} Y \xrightarrow{b} Z$, we obtain

$$\overline{b * a}(x, z) = b^{\#}(a(x))(z) = \bigvee_{y\in Y} a(x, y) \otimes b(y, z) = \bigvee_{y\in Y} \overline{a}(x, y) \otimes \overline{b}(y, z) = \overline{b} \cdot \overline{a}(x, z). \quad \blacksquare$$

2 Scenes and actors

In Section 1.6 we have defined, for any pre-scene $(\mathbf{S}, \mathbf{O})$, a notion of $(\mathbf{S}, \mathbf{O})$-*category*: the objects are pairs (X, a), X a $\mathbf{S}$-object and $a : X \nrightarrow X$ an $\mathbf{O}$-morphisms.

Lax algebras generalize this concept: instead of considering structures given by $\mathbf{O}$-morphisms $X \nrightarrow X$, we consider structure-morphisms $TX \nrightarrow X$, where T is a special kind of 2-graph morphism. Morphisms of lax algebras are appropriately generalized $(\mathbf{S}, \mathbf{O})$-functors.

In fact it is convenient for the study of lax algebras to generalize from pre-scenes to *scenes*; this will be done in this chapter: we introduce scenes and their morphisms.

2.1 Scenes

Definition 2.1.1 A *scene* $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ is given by the following data: a diagram

$$\mathbf{S} \xrightarrow{E} \mathbf{C} \underset{(\ )_*}{\overset{(\ )^*}{\rightrightarrows}} \mathbf{D} \tag{2.1}$$

where

- $\mathbf{S}$ is a category;
- $\mathbf{C}$ and $\mathbf{D}$ are ordered categories;
- E is an embedding;
- $(\ )_*$ is a pseudo-functor $\mathbf{C}^{\mathrm{co}} \longrightarrow \mathbf{D}$, that is, $(\ )_*$ acts covariantly on 1-cells and contravariantly on 2-cells;
- $(\ )^*$ is a pseudo-functor $\mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{D}$, that is, $(\ )^*$ acts contravariantly on 1-cells and covariantly on 2-cells;

such that the following compatibility-conditions are satisfied

(1) $f_* \dashv f^*$ for all $\mathbf{C}$-morphisms f; in particular, $C_* = C^*$ for all $\mathbf{C}$-objects C;

(2) $(\ )_* \cdot E$ and $(\ )^* \cdot E$ are faithful;

(3) $(\ )_*$ (and hence $(\ )^*$) are bijective on objects;

(4) $(\ )_* \cdot E$ is locally fully-faithful, that is, $f_* \leq g_*$ implies $f \leq g$ (thus $f = g$) in $\mathbf{S}$.

While the compatibility condition (1) is essential for all results concerning scenes, the sole purpose of (2), (3), and (4) is to ease the exposition. They are not essential.

A diagram

$$\begin{array}{c} X \\ g \left(\leq\right) f \\ Y \end{array}$$

in $\mathbf{C}$ is transformed into

$$\begin{array}{c} X^* \\ g^* \left(\leq\right) f^* \\ Y^* \end{array} \qquad \begin{array}{c} X_* \\ f_* \left(\leq\right) g_* \\ Y_* \end{array}$$

We will usually not distinguish between EX and X or Ef and f. Furthermore, we will not distinguish between C, C_*, and C^*, and often simply write f instead of f_*. As before, we will denote arbitrary $\mathbf{D}$-morphisms by "$\nrightarrow$", and reserve "$\longrightarrow$" for (( )$_*$-images of) $\mathbf{S}$- and $\mathbf{C}$-morphisms.

2.1.1 Examples

The role-model of all scenes is the *distributor scene* $\mathcal{D}$, given by the following diagram

$$\mathbf{Set} \xrightarrow{\text{discrete}} \mathbf{Ord} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathbf{Dist}$$

where $\mathbf{Dist} = \mathrm{Dist}(\mathbf{Set}, \mathbf{Rel})$; and $(\,)_*$, $(\,)^*$ are as in (1.18); see Corollary 1.6.7.

V-matrices The scene of $\mathbf{V}$-matrices for a quantale $\mathbf{V}$ is given by the following diagram

$$\mathbf{Set} = \mathbf{Set} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathrm{Mat}\ \mathbf{V}.$$

Scenes whose embedding E is an isomorphism will be called *discrete*.

Relations For any regular category $\mathbf{C}$ we have a discrete scene $\mathrm{Rel}\,\mathbf{C}$, defined by

$$\mathbf{C} = \mathbf{C} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathrm{Rel}\ \mathbf{C}.$$

Distributors We write $\mathrm{Dist}\,\mathcal{S}$ for the *distributor-scene* of a scene $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$, given by

$$\mathbf{S} \xrightarrow{\text{discrete}} \mathrm{Cat}(\mathbf{S}, \mathbf{D}) \underset{(\,)*}{\overset{(\,)_*}{\rightrightarrows}} \mathrm{Dist}(\mathbf{S}, \mathbf{D}). \tag{2.2}$$

Here $(\mathbf{S}, \mathbf{D})$ is the *underlying pre-scene* of $\mathcal{S}$; see 2.2.1. If $\mathbf{V}$ is a quantale, we write $\mathrm{Cat}\,\mathbf{V}$ for $\mathrm{Cat}(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ and $\mathrm{Dist}\,\mathbf{V}$ for $\mathrm{Dist}(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ and obtain the *scene of* $\mathbf{V}$*-distributors*

$$\mathbf{Set} \xrightarrow{\text{discrete}} \mathrm{Cat}\,\mathbf{V} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathrm{Dist}\,\mathbf{V}$$

which is not discrete; see Corollary 1.6.7.

Ordered V-matrices For a quantale $\mathbf{V}$ we denote by $\mathrm{OrdMat}(\mathbf{V})$ the 1-full and 2-full subcategory of $\mathrm{Dist}(\mathbf{V})$ spanned by the $\mathbf{V}$-categories (X, a) such that a takes values in $\{\bot, k\}$. We may identify the objects of $\mathrm{OrdMat}(\mathbf{V})$ with those of $\mathbf{Ord}$. A morphism $(X, \leq) \overset{r}{\nrightarrow} (Y, \leq)$ in $\mathrm{OrdMat}(\mathbf{V})$ is given by a $\mathbf{V}$-matrix $X \overset{r}{\nrightarrow} Y$ such that

$$x \leq x',\ y' \leq y \quad \text{implies} \quad r(x', y') \leq r(x, y),$$

for all $x, x' \in X$, $y, y' \in Y$. Equivalently, r is a *monotone* function $X^{\mathrm{op}} \times Y \longrightarrow V$. This leads us to the following generalization of Remark 1.3.6:

Remark 2.1.2 $\mathbf{Ord}^{\mathrm{op}}_{D_{\mathbf{V}}} \cong \mathrm{OrdMat}(\mathbf{V}) \cong \mathbf{Ord}^{\mathrm{co}}_{U_{\mathbf{V}}}$.

Proof Since $\mathbf{Ord}$ is cartesian closed, $X^{\mathrm{op}} \times Y \xrightarrow{r} V$ corresponds to $Y \xrightarrow{\check{r}} V^{X^{\mathrm{op}}} = D_{\mathbf{V}}X$, thus to a morphism $X \xrightarrow{\check{r}} Y$ in $\mathbf{Ord}^{\mathrm{op}}_{D_{\mathbf{V}}}$. On the other hand, r corresponds to $X^{\mathrm{op}} \xrightarrow{\hat{r}} V^Y$, which corresponds to $X \xrightarrow{\hat{r}} (V^Y)^{\mathrm{op}} = U_{\mathbf{V}}Y$.

Functoriality of these transformations is shown as in Remark 1.6.17. ■

For the restrictions of $(\,)_*$ and $(\,)^*$ to $\mathrm{OrdMat}(\mathbf{V})$ we have, in view of Remark 2.1.2, $F_{U_{\mathbf{V}}} = (\,)_* : \mathbf{Ord} \longrightarrow \mathbf{Ord}_{U_{\mathbf{V}}} \cong \mathrm{OrdMat}(\mathbf{V})^{\mathrm{co}}$ and $(\,)^* = F_{D_{\mathbf{V}}} : \mathbf{Ord} \longrightarrow \mathbf{Ord}_{D_{\mathbf{V}}} \cong \mathrm{OrdMat}(\mathbf{V})^{\mathrm{op}}$.

The scene $\mathcal{O}(\mathbf{V})$ of ordered $\mathbf{V}$-matrices is defined as the following diagram

$$\mathbf{Set} \xrightarrow{\text{discrete}} \mathbf{Ord} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathrm{OrdMat}(\mathbf{V})$$

Clearly $\mathrm{OrdMat}(\mathbf{V})$ is, as a 1-full and 2-full subcategory of a quantaloid, a quantaloid. Moreover, it is closed under biproducts in $\mathrm{Dist}(\mathbf{V})$, hence has biproducts. For the two-element chain $\mathbf{2}$, $\mathcal{O}(\mathbf{2}) = \mathcal{D}$ holds.

Kleisli scenes Let $\mathbf{T} = (T, e, m)$ be a 2-monad on $\mathbf{Ord}$ such that Tf has a right adjoint T^*f for each $f \in \mathbf{Ord}$. We define the *Kleisli scene* $\mathcal{K}(\mathbf{T})$ by

$$\mathbf{Set} \xrightarrow{\text{discrete}} \mathbf{Ord} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathbf{Ord}^{\mathrm{op}}_{\mathbf{T}}$$

The pseudo-functor $(\,)^* : \mathbf{Ord} \longrightarrow \mathbf{Ord}_{\mathbf{T}}$ is defined by $X^* = X$ and $f^* = e_Y \cdot f$ for any $f : X \longrightarrow Y$ in $\mathbf{Ord}$, that is, $(\,)^*$ is just $F_{\mathbf{T}}$. The pseudo-functor $(\,)_* : \mathbf{Ord} \longrightarrow \mathbf{Ord}^{\mathrm{coop}}_{\mathbf{T}}$ is defined by $X_* = X$ and $f_* = T^*f \cdot e_Y$.

2.2 Morphisms for scenes and actors

We fix scenes $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ and $\mathcal{S}' = (\mathbf{S}', \mathbf{C}', \mathbf{D}')$.

Definition 2.2.1 A *morphism of scenes* (or *scenic functor*) $\mathcal{S} \longrightarrow \mathcal{S}'$ is a pair (T, U) where T is a pseudo-functor $\mathbf{C} \longrightarrow \mathbf{C}'$ and U is a 2-graph morphism $\mathbf{D} \longrightarrow \mathbf{D}'$ such that

(M_1)

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{(\ )_*} & \mathbf{D} \\ {\scriptstyle T}\downarrow & \approx & \downarrow{\scriptstyle U} \\ \mathbf{C}' & \xrightarrow[(\ )_*]{} & \mathbf{D}' \end{array} \tag{2.3}$$

holds, that is, $U(C_*) = (TC)_*$ and $U(f_*) \approx (Tf)_*$ hold for all $\mathbf{C}$-objects C and all $\mathbf{C}$-morphisms f; so we can think of U as an "extension" of T along $(\ )_*$.

(M_{2a}) $Uf_* \cdot Ua \leq U(f_* \cdot a)$, and

(M_{2b}) $U(a \cdot g_*) \leq Ua \cdot Ug_*$ hold for all $a \in \mathbf{D}$ and $f, g \in \mathbf{C}$.

An *actor* on $\mathcal{S}$ is a scenic functor $\mathcal{S} \longrightarrow \mathcal{S}$.

Examples

(1) $(1_{\mathbf{C}}, 1_{\mathbf{D}})$ defines a scenic functor $\mathcal{S} \longrightarrow \mathcal{S}$.

(2) Let $\mathbf{C}$, $\mathbf{C}'$ be regular categories and let $F : \mathbf{C} \longrightarrow \mathbf{C}'$ be an arbitrary functor. We obtain, by Proposition 1.5.3, a scenic functor $(\mathbf{C}, \mathrm{Rel}\ \mathbf{C}) \xrightarrow{(F, \mathrm{Rel}\ F)} (\mathbf{C}', \mathrm{Rel}\ \mathbf{C}')$.

(3) Suppose $\mathbf{V} = (V, k, \otimes)$ and $\mathbf{W} = (W, l, \oplus)$ are quantales. By Proposition 1.5.18 any functor $V \xrightarrow{\nu} W$ which satisfies $\nu(k) = l$ and $\nu(\bot_V) = \nu(\bot_W)$ defines a scenic functor $(1_{\mathbf{Set}}, \mathrm{Mat}\,\nu) : (\mathbf{Set}, \mathrm{Mat}\,\mathbf{V}) \longrightarrow (\mathbf{Set}, \mathrm{Mat}\,\mathbf{W})$.

Lemma 2.2.2 *Every scenic functor (T, U) as above preserves the adjointness relation:*

$$U(f_*) \dashv U(f^*)$$

holds for all $f \in \mathbf{C}$.

Proof Observe first that (M_1) implies $U((1_A)_*) \approx (1_{TA})_*$. Hence ($\mathrm{M}_{2a}$) implies $U(f_*) \cdot U(f^*) \leq U(f_* \cdot f^*) \leq U(1_*) \approx 1$, analogously $1 \leq U(f^*) \cdot U(f_*)$. ∎

Preservation of the adjointness relation is in a way characteristic to scenic functors among certain classes of 2-graph morphisms:

Lemma 2.2.3 *Let $T : \mathbf{C} \longrightarrow \mathbf{C}'$ be a pseudo-functor. Given a lax or oplax functor $U : \mathbf{D} \longrightarrow \mathbf{D}'$ which extends T in the sense that* (2.3) *holds, (T, U) is a scenic functor if and only if $U(f_*) \dashv U(f^*)$ holds for all $f \in \mathbf{C}$.*

Proof Assume that U is lax and preserves the adjointness relation. We only need to show (M_{2b}): $U(a \cdot g_*) \leq U(a \cdot g_*) \cdot U(g^*) \cdot U(g_*) \leq U(a \cdot g_* \cdot g^*) \cdot U(g_*) \leq Ua \cdot U(g_*)$.
The other assertions follow by an analogous argument. ■

The *canonical lax extensions* of [Sea05b] are lax functors which are not actors.

Lemma 2.2.4 *The following statements hold for any pseudo-functor $\mathbf{C} \xrightarrow{T} \mathbf{C}'$ and any 2-graph morphism $\mathbf{D} \xrightarrow{U} \mathbf{D}'$:*

(1) *If T and U fulfill* (M_{2a}) *and* (M_{2b})*, then* (M_1) *is equivalent to $U(C^*) = (TC)^*$ and $U(f^*) \approx (Tf)^*$ for all $\mathbf{C}$-objects C and all $\mathbf{C}$-morphism f.*

(2) *The following statements are equivalent:*

   (i) *(T, U) is a scenic functor $\mathcal{S} \longrightarrow \mathcal{S}'$;*

   (ii) (M_1)*, $Ua \cdot U(f^*) \leq U(a \cdot f^*)$, and $U(g^* \cdot a) \leq U(g^*) \cdot Ua$ hold;*

   (iii) (M_1)*, $Uf_* \cdot Ua \cdot U(g^*) \leq U(f_* \cdot a \cdot g^*)$, and $U(f^* \cdot a \cdot g_*) \leq U(f^*) \cdot Ua \cdot Ug_*$ hold.* ■

Thus, the concept of scenic functor could be formulated in terms of $(\ )^*$ instead of $(\ )_*$. For a class $\mathcal{F} \subset \operatorname{mor} \mathbf{C}$, we write $\mathcal{F}_* = \{ f_* \mid f \in \mathcal{F} \}$ and $\mathcal{F}^* = \{ f^* \mid f \in \mathcal{F} \}$.

Definition 2.2.5 Given $\mathcal{G} \subset \operatorname{mor} \mathbf{D}$ we say that $(T, U) : \mathcal{S} \longrightarrow \mathcal{S}'$

1. is *left $\mathcal{G}$-strict* if $U(g \cdot a) \approx Ug \cdot Ua$ holds for all $g \in \mathcal{G}$, $a \in \mathbf{D}$;
2. is *right $\mathcal{G}$-strict* if $U(b \cdot g) \approx Ub \cdot Ug$ holds for all $g \in \mathcal{G}$, $b \in \mathbf{D}$;
3. *preserves initiality* if (T, U) is left $(\operatorname{mor} \mathbf{C})^*$-strict and right $(\operatorname{mor} \mathbf{C})_*$-strict, that is, $U(g^* \cdot a \cdot f) \approx U(g^*) \cdot Ua \cdot Uf$ holds for all $f, g \in \mathbf{C}$, $a \in \mathbf{D}$;
4. *preserves finality* if (T, U) is left $(\operatorname{mor} \mathbf{C})_*$-strict and right $(\operatorname{mor} \mathbf{C})^*$-strict, that is, $U(g \cdot a \cdot f^*) \approx Ug \cdot Ua \cdot U(f^*)$ holds for all $f, g \in \mathbf{C}$, $a \in \mathbf{D}$.

Let $\mathcal{F}$ be a class of $\mathbf{C}$-morphisms. We say that an actor (T, U) on $\mathcal{S}$

5. *preserves $\mathcal{F}$-whiskers* if (T, U) is left $\mathcal{F}_*$-strict and right $(T[\mathcal{F}])_*$-strict;
6. *preserves $\mathcal{F}^*$-whiskers* if (T, U) is left $\mathcal{F}^*$-strict and right $(T[\mathcal{F}])^*$-strict.

Remark 2.2.6 Let (T, U) be an actor. If U is a lax functor, then (T, U) preserves initiality. If U is an oplax functor, then (T, U) preserves finality. ■

Examples 2.2.7 (1) Let $\mathbf{C}$, $\mathbf{C}'$ be regular categories and $F : \mathbf{C} \longrightarrow \mathbf{C}'$ be a functor. The scenic functor $(F, \mathrm{Rel}\ F) : (\mathbf{C}, \mathrm{Rel}\,\mathbf{C}) \longrightarrow (\mathbf{C}', \mathrm{Rel}\,\mathbf{C}')$ defined as in Proposition 1.5.3

- is left $(\mathrm{Mono}\,\mathbf{C})_*$-strict and right $(\mathrm{Mono}\,\mathbf{C})^*$-strict;
- is right $(\mathrm{Mono}\,\mathbf{C})_*$-strict and left $(\mathrm{Mono}\,\mathbf{C})^*$-strict provided F preserves inverse images;
- preserves finality provided F preserves regular epimorphisms;
- preserves initiality provided F nearly preserves pullbacks.

(2) Let $\mathbf{V}$ and $\mathbf{W}$ be quantales, $\nu : V \longrightarrow W$ a functor with $\nu(\bot_V) = \bot_W$ and $\nu(k_{\mathbf{V}}) = k_{\mathbf{W}}$. The scenic functor $(1_{\mathbf{Set}}, \mathrm{Mat}\,\nu) : (\mathbf{Set}, \mathrm{Mat}\,\mathbf{V}) \longrightarrow (\mathbf{Set}, \mathrm{Mat}\,\mathbf{W})$

- is left $(\mathrm{Mono}\,\mathbf{Set})_*$- and right $(\mathrm{Mono}\,\mathbf{Set})^*$-strict;
- preserves initiality;
- preserves finality if and only if ν preserves suprema.

Let $(T, U) : \mathcal{S} \longrightarrow \mathcal{S}'$ be a scenic functor. If $(\ )_* : \mathbf{C}' \longrightarrow \mathbf{D}'$ is an embedding, then T is uniquely determined by U. Moreover, a 2-graph morphism $U : \mathbf{D} \longrightarrow \mathbf{D}'$ defines a scene-morphism $\mathcal{S} \longrightarrow \mathcal{S}'$ if and only if (M_{2a}) and (M_{2b}) hold and the 2-graph morphism $\mathbf{C} \overset{U(\ )_*}{\longrightarrow} \mathbf{D}'$ factors as

$$\begin{array}{ccc} \mathbf{C} & & \\ \vdots & \searrow^{U(\ )_*} & \\ \mathbf{C}' & \longrightarrow & \mathbf{D}' \end{array} \tag{2.4}$$

with $U(1_C)_* \approx 1_{(UC)_*}$ for all $C \in \mathbf{C}$;

Remark 2.2.8 Given scenic functors $(T, U) : \mathcal{S} \longrightarrow \mathcal{S}'$ and $(T', U') : \mathcal{S}' \longrightarrow \mathcal{S}''$ their composite $\mathcal{S} \longrightarrow \mathcal{S}''$ is defined pointwise: $(T', U') \cdot (T, U) = (T'T, U'U)$. The identity $1_{\mathcal{S}}$ on a scene $\mathcal{S}$ is given by $(1_{\mathbf{C}}, 1_{\mathbf{D}})$.

Proof Clearly $(T'T, U'U)$ satisfies (M_1). To prove (M_{2a}) consider $U'U(f_*) \cdot U'Ua \approx U'(Tf)_* \cdot U'Ua \leq U'((Tf)_* \cdot Ua) \approx U'(Uf_* \cdot Ua) \leq U'U(f_* \cdot a)$; so (M_{2a}) holds. (M_{2b}) can be shown in the same manner. ∎

Definition 2.2.9 Given parallel scene-morphisms $(T, U), (K, L) : \mathcal{S} \longrightarrow \mathcal{S}'$ a *slim transformation* of scene-morphisms $(T, U) \longrightarrow (K, L)$ is a pseudo-natural transformation $T \longrightarrow K$.

Clearly an instance of the interchange law holds for scenic functors and slim transformations. We define an (illegitimate) 2-category **Scene** by the following data:

- 0-cells: scenes;
- 1-cells: scenic functors;
- 2-cells: slim transformations;

with the obvious compositions.

Definition 2.2.10 We call a slim transformation $(T,U) \xrightarrow{\lambda} (K,L)$ *lax* if the family $\big((\lambda_C)_*\big)_{C\in\mathbf{C}}$ is a lax transformation $U \longrightarrow L$. Analogously for oplax and pseudo-natural transformations.

Observe that, since the definition of scenic functor can be formulated in terms of $(\ )^*$ instead of $(\ )_*$, the above definition is somewhat arbitrary. In fact, we will encounter slim transformations $\lambda : (T,U) \longrightarrow (K,L)$ which extend along $(\ )^*$ (that is, the $(\lambda^*_C : KC \longrightarrow TC)$ form a pseudo-natural transformation $L \longrightarrow U$), while λ does *not* extend to a pseudo-natural transformation $U \longrightarrow L$.

Remark 2.2.11 Let $(T,\tilde{T}),(S,\tilde{S}) : \mathcal{S} \longrightarrow \mathcal{S}'$ be scenic functors, $\lambda : T \longrightarrow S$ a pseudo-natural transformation. $(\lambda_X)_*$ defines an oplax transformation $\tilde{T} \longrightarrow \tilde{S}$ if and only if λ^*_X defines a lax transformation $\tilde{S} \longrightarrow \tilde{T}$. ■

Definition 2.2.12 Let $\mathcal{S} = (\mathbf{S},\mathbf{C},\mathbf{D})$ and $\mathcal{S}' = (\mathbf{S}',\mathbf{C}',\mathbf{D}')$ be scenes. A scenic functor $(F_1,F_2) : \mathcal{S} \longrightarrow \mathcal{S}'$ is *based* provided we have a (necessary unique) factorization

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{E} & \mathbf{C} \\ {\scriptstyle F_0}\,\vdots & & \downarrow{\scriptstyle F_1} \\ \mathbf{S}' & \xrightarrow[E']{} & \mathbf{C}' \end{array}$$

Based scenic functors form a subcategory **bScene** $\hookrightarrow$ **Scene**.

Actor monads In what follows we will in general not distinguish between the $\mathbf{C}$ and the $\mathbf{D}$ part of a scene-morphism. So we will simply write U instead of (T,U), sometimes writing $U_{\mathbf{C}}$ for T.

Definition 2.2.13 An *actor-monad* or *monadic actor* is a monad structure in the 2-category **Scene**.

To spell the above definition out, an actor-monad $\mathsf{T} = (T,e,m)$ on $\mathcal{S}$ consists of an actor $T : \mathcal{S} \longrightarrow \mathcal{S}$ such that $(T_{\mathbf{C}},e,m)$ is a pseudo-monad on $\mathbf{C}$.

2.2.1 Scenes vs. pre-scenes

For any pre-scene $\mathfrak{O} = \left(\mathbf{S} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathbf{O} \right)$ we obtain a discrete scene $S\mathfrak{O}$, defined by

$$\mathbf{S} = \mathbf{S} \underset{(\,)^\circ}{\overset{(\,)_\circ}{\rightrightarrows}} \mathbf{O}.$$

We define a *morphism of pre-scenes* $\mathfrak{O} \longrightarrow \mathfrak{O}'$ to be any scenic functor $S\mathfrak{O} \longrightarrow S\mathfrak{O}'$, and a transformation between such morphisms $T, T' : \mathfrak{O} \longrightarrow \mathfrak{O}'$ to be any slim transformation between the corresponding scenic functors $S\mathfrak{O} \longrightarrow S\mathfrak{O}'$. The (illegitimate) 2-category **PreScene** of pre-scenes defined by this data is thus the 1-full and 2-full subcategory of **Scene** spanned by the $S\mathfrak{O}$.

Given any scene $\mathcal{S}$ as in (2.1), we can form its *underlying pre-scene* $U\mathcal{S}$ by forgetting the "discrete part" of $\mathcal{S}$, that is, $U\mathcal{S}$ is defined to be

$$\mathbf{C} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathbf{D},$$

and U defines a functor $U : \mathbf{Scene} \longrightarrow \mathbf{PreScene}$.

Clearly every scenic functors between pre-scenes is based. Moreover, given a scene $\mathcal{S}$ as in (2.1), we can define its *discretezation* $D\mathcal{S}$ to be the pre-scene

$$\mathbf{S} \underset{(\,)^* \cdot E}{\overset{(\,)_* \cdot E}{\rightrightarrows}} \mathbf{D}',$$

where $\mathbf{D}'$ is the 1-full and 2-full subcategory of $\mathbf{D}$ spanned by the image of $(\,)_* \cdot E$. This clearly defines a functor $\mathbf{bScene} \xrightarrow{D} \mathbf{PreScene}$.

2.3 Constructing actors

We will now demonstrate various ways of constructing an actor $(T, \tilde{T})$ from a given functor T. In a way the most basic of these "extensions" is the Rel-construction, see Proposition 1.5.3, which produces an actor on $(\mathbf{C}, \operatorname{Rel} \mathbf{C})$ out of every functor $T : \mathbf{C} \longrightarrow \mathbf{C}$. A simple modification of the Rel-construction will allow us to extend functors from **Ord** to **Dist**. These two constructions will be the starting point for the construction of actors on $(\mathbf{Set}, \operatorname{Mat} \mathbf{V})$ and on $\mathcal{O}(\mathbf{V})$.

We will also present a different method using distributive laws. This has, despite its obvious categorical appeal, "been largely overlooked" [MRW02, p. 215]. One reason for this omission might be that this construction gives rise to strict functors, while the focus so far has been on lax functors. It should be worthwhile to develop — either in an order-enriched context or for full-fledged 2-categories — a notion of "lax distributive law" which corresponds to lax extensions .

Since, as we have seen in Examples 2.2.7(1), refined properties of $(T, \mathrm{Rel}\, T)$ depend on the behavior of T with respect to pullbacks we discuss now inverse image preserving or *taut*[1] functors; see [Möb81], [Man02].

2.3.1 Taut functors

Definition 2.3.1 ([Man02]) A functor $T : \mathbf{C} \longrightarrow \mathbf{C}'$ is *taut* if T sends every pullback

$$\begin{array}{ccc} f^{-1}[n] & \xrightarrow{i} & X \\ \downarrow & & \downarrow f \\ B & \xrightarrow[n]{} & Y \end{array} \tag{2.5}$$

in $\mathbf{C}$ with n a monomorphism to a pullback-diagram in $\mathbf{C}'$. A natural transformation $\lambda : T \longrightarrow S$ is *taut* provided the λ-naturality square

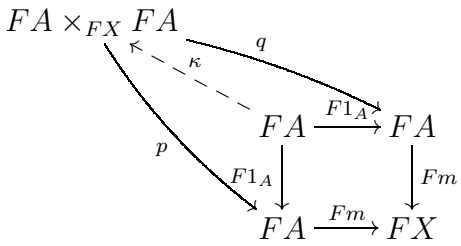
$$\begin{array}{ccc} A & TA & \xrightarrow{\lambda_A} SA \\ {\scriptstyle m}\downarrow & {\scriptstyle Tm}\downarrow & \downarrow{\scriptstyle Sm} \\ X & TX & \xrightarrow[\lambda_X]{} SX \end{array}$$

is a pullback for all *monomorphisms* m. A monad (T, e, m) is *taut* if T, e, and m are taut.

Every taut functor preserves monomorphisms. Weakening the concept of taut functor to "nearly taut" has no effect:

Lemma 2.3.2 *Let $\mathbf{C}$, $\mathbf{C}'$ be categories with pullbacks. Every functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ which nearly preserves inverse images (in the sense that for any $f : X \longrightarrow Y$ and any monomorphism $n : B \longrightarrow Y$ in $\mathbf{C}$ the canonical morphism $F(f^{-1}[n]) \xrightarrow{\kappa} FX \times_{FY} FB$ is a regular epimorphism) is already taut.*

Proof Let $F : \mathbf{C} \longrightarrow \mathbf{C}'$ be nearly inverse image preserving. We first show that F preserves monomorphisms. For any monomorphism $m : A \longrightarrow X$ in $\mathbf{C}$, we obtain a commutative diagram



[1]This usage of the term "taut" differs from the one in, e.g., Wyler's taut lift theorem; see [AHS04].

in which κ is a regular epimorphism. Thus $p = q$, and Fm is a monomorphism.

For an arbitrary inverse image diagram as (2.5), we have a factorization

$$F(f^{-1}[n]) \xrightarrow{Fi} FX = F(f^{-1}[n]) \xrightarrow{\kappa} FX \times_{FY} FB \xrightarrow{p} FX.$$

Since i is a monomorphism, also Fi is a monomorphism. This implies that κ is both a monomorphism and a regular epimorphism, hence an isomorphism. ∎

Corollary 2.3.3 *Every nearly pullback preserving functor between categories with pullbacks is taut.* ∎

Proposition 2.3.4 ([Möb81]) *Let $F, G, H : \mathbf{C} \longrightarrow \mathbf{C}'$ and $F', G', H' : \mathbf{C}' \longrightarrow \mathbf{C}''$ be functors, $F \xrightarrow{\kappa} G \xrightarrow{\lambda} H$ and $F' \xrightarrow{\kappa'} G' \xrightarrow{\lambda'} H'$ be natural transformations. Then the following statements hold:*

(1) *If F and F' are taut, so is $F'F$.*

(2) *If λ is taut, then κ is taut if and only if $\lambda \cdot \kappa$ is taut.*

(3) *If κ and G are taut, then F is taut.*

(4) *If H preserves monomorphisms and κ' is taut, then $\kappa' H$ is taut.*

(5) *If κ is taut and H' is taut, then $H'\kappa$ is taut.*

(6) *If κ, κ', and G' are taut and F preserves monomorphisms, then $\kappa' * \kappa$ is taut.*

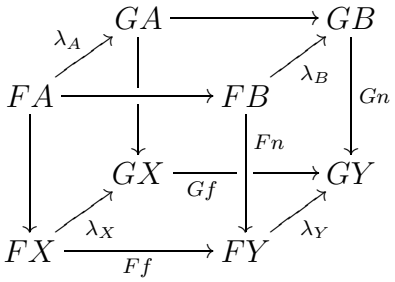
Proof (1), (4), and (5) are obvious.

To show (2), consider a monomorphism $A \xrightarrow{m} X$. We obtain a commutative diagram

$$\begin{array}{ccccc} FA & \xrightarrow{\kappa_A} & GA & \xrightarrow{\lambda_A} & HA \\ {\scriptstyle Fm}\downarrow & & \downarrow{\scriptstyle Gm} & & \downarrow{\scriptstyle Hm} \\ FX & \xrightarrow{\kappa_X} & GX & \xrightarrow{\lambda_X} & HX \end{array}$$

Tautness of λ implies that the right square is a pullback. Thus the left square is a pullback if and only if the outer frame is a pullback, and the claim follows.

For the proof of (3), let $X \xrightarrow{f} Y$ be a $\mathbf{C}$-morphism and $B \xrightarrow{n} Y$ be a monomorphism in $\mathbf{C}$ such that the inverse image $A = f^{-1}B$ exists. We obtain a commutative diagram



In the above cube the back- and both side-faces are pullbacks. Thus also the front-face is a pullback.

(6) follows immediately from (4), (5), and (2). ∎

Taut functors on Set

The two *trivial* monads on **Set** are (T_1, e_1, m_1) and (T_2, e_2, m_2) with $T_1X = \{*\}$ for all sets X and $T_2\emptyset = \emptyset$, $T_2X = \{*\}$ for $X \neq \emptyset$. For justification in calling these monads trivial see [MS04]. As is easily seen, T_2 is not a taut functor, and e_1 is not a taut natural transformation: consider the naturality square of the inclusion $\emptyset \hookrightarrow X$ for X non-empty. Hence neither of the trivial monads on **Set** is taut, and our definition of a taut monad is coextensive with the one of [Man02].

The filter functor We write F for the *filter-functor* **Set** $\longrightarrow$ **Set**; cf. (1.8). The powerset PX is called the *nullfilter* on X, every other filter is *proper*. We recall that Ff is defined by

$$Ff(\mathfrak{x}) = \{\, B \subset Y \mid f^{-1}B \in \mathfrak{x} \,\} \quad \text{for } X \xrightarrow{f} Y.$$

Lemma 2.3.5 *F is a taut functor.*

Proof We first remark that given a subset-inclusion $i : A \hookrightarrow X$ and $\mathfrak{x} \in FX$, we have

$$\mathfrak{x} \in \operatorname{Im} Fi \iff A \in \mathfrak{x}.$$

Hence F preserves monomorphisms, and we can consider FA as a subset of FX. To show tautness of F, suppose we are given $W \xrightarrow{f} X$. Assuming that all monomorphisms are subset-inclusions, we need to show that $(Ff)^{-1}[FA] = F(f^{-1}[A])$ holds. For $\mathfrak{w} \in FW$ we have $\mathfrak{w} \in (Ff)^{-1}[FA]$ iff $Ff(\mathfrak{w}) \in FA$ iff $A \in Ff(\mathfrak{w})$ iff $f^{-1}[A] \in \mathfrak{w}$ iff $\mathfrak{w} \in F(f^{-1}[A])$. ∎

Lemma 2.3.6 *Every functor T : **Set** $\longrightarrow$ **Set** which preserves binary coproducts is taut. Every natural transformation between binary coproduct preserving **Set**-functors is taut.*

Proof The proof shows that the lemma is valid in any extensive Boolean category; see [CLW93]:

Let $X \xrightarrow{f} Y$ be a function and $B \xrightarrow{m} Y$ be a monomorphisms. By Booleanness, we find $B' \xrightarrow{n} Y$ such that $B \xrightarrow{m} Y \xleftarrow{n} B'$ is a coproduct diagram. Forming pullbacks, we obtain

$$\begin{array}{ccccc} f^{-1}[B] & \longrightarrow & X & \longleftarrow & f^{-1}[B'] \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ B & \xrightarrow{m} & Y & \xleftarrow{n} & B' \end{array} \qquad (\dagger)$$

By extensitivity also the top row is a coproduct. Applying T to (†) we get a commutative diagram with top and bottom rows coproduct diagrams. By extensitivity, both squares in the T-image of (†) are pullbacks, and the first claim is proved.

Now given binary coproduct preserving functors T, S and a natural transformation $T \xrightarrow{\lambda} S$ we obtain for $B \xrightarrow{m} Y$ as above a commutative diagram

$$\begin{array}{ccccc} TB & \xrightarrow{Tm} & TY & \xleftarrow{Tn} & TB' \\ {\scriptstyle\lambda_B}\downarrow & & \downarrow{\scriptstyle\lambda_Y} & & \downarrow{\scriptstyle\lambda_{B'}} \\ SB & \xrightarrow{Sm} & SY & \xleftarrow{Sn} & SB' \end{array}$$

with top and bottom rows coproducts. Thus both squares are pullbacks, and λ is taut. ∎

Subfunctors of F We write F_0X for the subset of FX consisting of the proper filters, equivalently, the $\mathfrak{x} \in FX$ with $\emptyset \notin \mathfrak{x}$; and UX for the subset consisting of the ultrafilters, that is, maximal (with respect to $\subset$) proper filters. It is well-known that a filter $\mathfrak{x}$ on X is an ultrafilter if and only if for all $A \subset X$, either $A \in \mathfrak{x}$ or $X \setminus A \in \mathfrak{x}$.

Lemma 2.3.7 *The assignments $X \longmapsto F_0X$ and $X \longmapsto UX$ determine subfunctors F_0 and U of F. The inclusion $F_0 \hookrightarrow F$ is cartesian and the inclusion $U \hookrightarrow F$ is taut. In particular, both F_0 and U are taut. Moreover, $F_0\emptyset = \emptyset$ and U preserves finite coproducts.* ∎

Proposition 2.3.8 *The following statements are equivalent for a functor* $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$*:*

(1) *T is taut;*

(2) *there exists a (unique) taut transformation $\alpha : T \longrightarrow F$.*

Moreover, α factors through the inclusion $F_0 \hookrightarrow F$ if and only if T preserves the initial object, and α factors through the inclusion $U \hookrightarrow F$ if and only if T preserves finite coproducts.

The equivalence of (1) and (2) in Proposition 2.3.8 was first observed in [Möb81]; see also [Man02].

Proof [(2) ⇒ (1)]: Tautness of α and F implies tautness of T by Proposition 2.3.4(3).

[(1) ⇒ (2)]: Let $A \xrightarrow{m} X$ be a monomorphism. We wish to define α so that

$$\begin{array}{ccc} TA & \xrightarrow{\alpha_A} & FA \\ {\scriptstyle Tm}\downarrow & & \downarrow{\scriptstyle Fm} \\ TX & \xrightarrow{\alpha_X} & FX \end{array}$$

is a pullback-diagram. This is the case precisely when $\alpha_X(\mathfrak{X}) \in \operatorname{Im} Fm \iff \mathfrak{X} \in \operatorname{Im} Tm$ holds for all $\mathfrak{X} \in TX$. Thus the required tautness of α *forces us to define*

$$\alpha_X(\mathfrak{X}) = \{\, A \subset X \mid \mathfrak{X} \in \operatorname{Im} T(A \hookrightarrow X) \,\}. \tag{2.6}$$

Clearly $X \in \alpha_X(\mathfrak{X})$; and $A \in \alpha_X(\mathfrak{X})$ and $A \subset B$ imply $B \in \alpha_X(\mathfrak{X})$. If $A, A' \in \alpha_X(\mathfrak{X})$, then $A \cap A' \in \alpha_X(\mathfrak{X})$ since T preserves intersections by tautness. Thus $\alpha_X(\mathfrak{X})$ is a filter on X. We need to show that α is a natural transformation: let $f : X \longrightarrow Y$ be a function. We have for $B \subset Y$:

$$\begin{aligned}
B \in Ff \cdot \alpha_X(\mathfrak{X}) &\iff f^{-1}[B] \in \alpha_X(\mathfrak{X}) \\
&\iff \mathfrak{X} \in \operatorname{Im} T(f^{-1}[B] \hookrightarrow X) \\
&\iff \mathfrak{X} \in (Tf)^{-1} T(B \hookrightarrow Y) && \text{by tautness of } T \\
&\iff Tf(\mathfrak{X}) \in \operatorname{Im} T(B \hookrightarrow Y) && \text{by definition of } (Tf)^{-1} \\
&\iff B \in \alpha_Y(Tf(\mathfrak{X})).
\end{aligned}$$

Thus α is a taut natural transformation. Uniqueness follows by construction.

Now clearly $T\emptyset = \emptyset$ iff $\emptyset \notin \alpha_X(\mathfrak{X})$ for all sets X and $\mathfrak{X} \in TX$, thus α takes values in F_0 iff T preserves $\emptyset$.

Suppose T preserves finite coproducts and take $A \subset X$. Then $TX = \operatorname{Im} T(A \hookrightarrow X) + \operatorname{Im} T(X \setminus A \hookrightarrow X)$ holds. Thus $A \notin \alpha_X(\mathfrak{X})$ iff $\mathfrak{X} \notin \operatorname{Im} T(A \hookrightarrow X)$ iff $\mathfrak{X} \in \operatorname{Im} T(X \setminus A \hookrightarrow X)$ iff $(X \setminus A) \in \alpha_X(\mathfrak{X})$.

Assume now that α factors as $T \xrightarrow{\alpha'} U \hookrightarrow F$. We need to show that T preserves binary coproducts. Take any coproduct diagram $A \xrightarrow{m} X \xleftarrow{n} B$. We obtain a commutative diagram

$$\begin{array}{ccccc}
TA & \xrightarrow{Tm} & TX & \xleftarrow{Tn} & TB \\
{\scriptstyle \alpha'_A}\downarrow & & \downarrow{\scriptstyle \alpha'_X} & & \downarrow{\scriptstyle \alpha'_B} \\
UA & \xrightarrow{Um} & UX & \xleftarrow{Un} & UB
\end{array}$$

where the bottom row is a coproduct and both squares are pullbacks since α' is taut by Proposition 2.3.4. Thus, by extensivity of **Set**, also the top row is a coproduct. ■

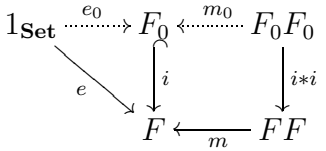
Corollary 2.3.9 *There are uniquely determined taut natural transformations $e : 1_{\mathbf{Set}} \longrightarrow F$ and $m : FF \longrightarrow F$. These make $\mathbf{F} = (F, e, m)$ into a taut monad. F_0 and U determine submonads $\mathbf{F}_0$ and $\mathbf{U}$ of $\mathbf{F}$. Moreover, the following statements are equivalent for a monad $\mathbf{T}$ on* **Set***:*

(1) $\mathbf{T}$ *is taut;*

(2) *there exists a (uniquely determined) taut monad-morphism $\mathbf{T} \longrightarrow \mathbf{F}$.*

Proof Clearly $1_{\mathbf{Set}}$ and FF are taut. Thus unique taut transformations e and m exist. Moreover, also $m \cdot Fe$, $m \cdot eF$, $m \cdot mF$, and $m \cdot Fm$ are taut by Proposition 2.3.4, thus uniqueness forces the required equations to hold.

Let $i : F_0 \longrightarrow F$ denote the inclusion. Since $1_{\mathbf{Set}}$ and F_0F_0 preserve $\emptyset$ and e as well as $m \cdot (i * i)$ are taut, we have factorizations



By Proposition 2.3.4, e_0 and m_0 are taut.

Analogously one shows that U determines a taut submonad of F.

Now if T is taut we obtain a unique transformation $T \xrightarrow{\alpha} F$, which is a monad-morphism by uniqueness of taut transformations $1_{\mathbf{Set}} \longrightarrow F$ and $TT \longrightarrow F$. On the other hand, if there is a taut monad morphism $\mathsf{T} \longrightarrow \mathsf{F}$, then T is taut by the cancellation properties exhibited in Proposition 2.3.4. ∎

The natural transformations $e : 1_{\mathbf{Set}} \longrightarrow F$ and $m : FF \longrightarrow F$ can be described easily without any recourse to Section 1.3.6. Using (2.6), we obtain $e_X(x) = \{\, A \subset X \mid x \in A \,\}$, thus $e_X(x)$ is just the *principal filter* of x. Now for $\mathfrak{X} \in FFX$ we have, using (2.6) and the fact that $A \in \mathfrak{x}$ if and only if $\mathfrak{x} \in FA$,

$$A \in m_X(\mathfrak{X}) \iff \{\, \mathfrak{x} \in FX \mid A \in \mathfrak{x} \,\} \in \mathfrak{X}.$$

Börger's Lemma Proposition 2.3.8 immediately implies Börger's Lemma [Bör87]:

Corollary 2.3.10 (Börger's Lemma) *If $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ preserves finite coproducts then there is a unique natural transformation $\alpha : T \longrightarrow U$.*

Proof Existence follows from Proposition 2.3.8 and Lemma 2.3.6. Uniqueness follows from the fact that any natural transformation $T \longrightarrow U$ is taut by Lemma 2.3.6. ∎

Börger's Lemma implies in particular that the monad structure on U exhibited above is the unique monad structure on U.

2.3.2 Constructing actors on $\mathcal{D}$

Recall the distributor scene $\mathcal{D}$, given by the diagram

$$\mathbf{Set} \xrightarrow{\text{discrete}} \mathbf{Ord} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathbf{Dist}.$$

We will show that every pseudo-functor $\mathbf{Ord} \longrightarrow \mathbf{Ord}$ determines an actor on $\mathcal{D}$.

Factoring distributors For a distributor $X \xrightarrow{r} Y$, we write $R = \{\,(x,y) \mid x\,r\,y\,\}$, ordered as a subset of $X \times Y$. The distributor r is determined by the diagram

$$X \xleftarrow{r_0} R \xrightarrow{r_1} Y$$

in **Ord**, here r_0 and r_1 are the evident projections.

Lemma 2.3.11 *Every distributor $X \xrightarrow{r} Y$ factors as $X \xrightarrow{(r_0)^*} R \xrightarrow{(r_1)_*} Y$.*

Proof Let us denote the order relation of an ordered set X by o_X. Recall that r factors in **Rel** as $r = r_1 \cdot r_0^\circ$. Thus, in **Rel** we have $(r_1)_* \cdot (r_0)^* = o_Y \cdot r_1 \cdot r_0^\circ \cdot o_X = o_Y \cdot r \cdot o_X = r$ since r is a distributor. Hence, $(r_1)_* \cdot (r_0)^* = r$ holds in **Dist**. ∎

Extending pseudo-functors Given a pseudo-functor $T : \mathbf{Ord} \longrightarrow \mathbf{Ord}$, we can, in analogy to the case of functors between regular categories, define an extension $\tilde{T} : \mathbf{Dist} \longrightarrow \mathbf{Dist}$ by

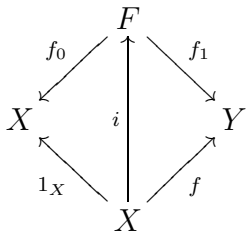
$$\tilde{T}X = TX, \qquad \tilde{T}r = (Tr_1)_* \cdot (Tr_0)^*. \tag{2.7}$$

Proposition 2.3.12 *$(T, \tilde{T})$ is an actor on $\mathcal{D}$.*

Proof We will first show that

$$\begin{array}{ccc} \mathbf{Ord} & \xrightarrow{(\,)_*} & \mathbf{Dist} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \tilde{T}} \\ \mathbf{Ord} & \xrightarrow[(\,)_*]{} & \mathbf{Dist} \end{array}$$

commutes. Take any $X \xrightarrow{f} Y$ in **Ord**; f_* factors as $f_* = (f_1)_* \cdot (f_0)^*$ where $X \xleftarrow{f_0} F \xrightarrow{f_1} Y$ are the projections from $F = \{\,(x,y) \mid f(x) \leq y\,\}$. Writing $i : X \longrightarrow F$ for the function $x \longmapsto (x, f(x))$, we obtain a diagram

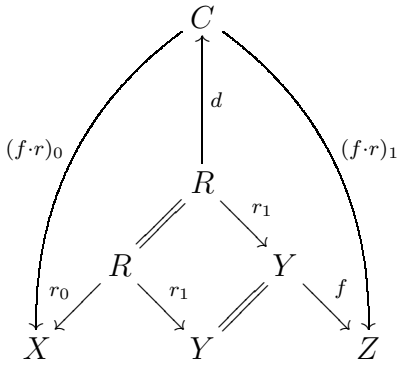


in **Ord** in which both triangles commute. Moreover, $f \cdot f_0 \leq f_1$ holds. Thus $Tf \cdot Tf_0 \leq Tf_1$, which implies $(Tf_1)_* \subset (Tf)_* \cdot (Tf_0)_*$. This gives $\tilde{T}(f_*) = (Tf_1)_* \cdot (Tf_0)^* \subset (Tf)_* \cdot (Tf_0)_* \cdot (Tf_0)^* \subset (Tf)_*$. On the other hand, we have $(Tf)_* = (Tf_1)_* \cdot (Ti)_* \subset (Tf_1)_* \cdot (Tf_0)^* \cdot (Tf_0)_* \cdot (Ti)_* = (Tf_1)_* \cdot (Tf_0)^* \cdot (T1_X)_* = \tilde{T}(f_*)$, and thus $(Tf)_* = \tilde{T}(f_*)$.

Now take any distributor $X \xrightarrow{r} Y$ and monotone functions $Y \xrightarrow{f} Z$, $W \xrightarrow{g} X$. First we will show that $\tilde{T}(f_*) \cdot \tilde{T}r \subset \tilde{T}(f_* \cdot r)$ holds.

Observe that $\tilde{T}s$ is actually defined for *any* relation s and is still locally monotone on this larger domain. Since $f \subset f_*$ holds in **Rel**, it suffices to show $(Tf)_* \cdot \tilde{T}r = \tilde{T}f_* \cdot \tilde{T}r \subset \tilde{T}(f \cdot r)$.

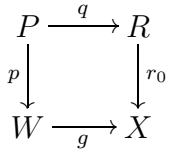
The two "legs" of $f \cdot r$ are formed as follows:



Hence, we have:

$$\begin{aligned}(Tf)_* \cdot \tilde{T}r &= (Tf)_* \cdot (Tr_1)_* \cdot (Tr_0)^* \\ &= (T(f \cdot r_1))_* \cdot (Tr_0)^* \\ &= (T(f \cdot r)_1)_* \cdot (Td)_* \cdot (Td)^* \cdot (T(f \cdot r)_0)^* \\ &\subset (T(f \cdot r)_1)_* \cdot (T(f \cdot r)_0)^* \\ &= \tilde{T}(f \cdot r).\end{aligned}$$

To show $\tilde{T}(r \cdot g_*) \subset \tilde{T}r \cdot \tilde{T}(g_*)$, first observe that $r \cdot g_* = r \cdot g$ holds. Moreover, for every pullback diagram



(r_0, r_1) being a monosource implies that also $(p, r_1 \cdot q)$ is a monosource. We may assume $P \subset W \times Y$, and obtain $(r \cdot g)_0 = p$, $(r \cdot g)_1 = r_1 \cdot q$. Thus

$$\begin{aligned}\tilde{T}(r \cdot g_*) &= (T(r \cdot g)_1)_* \cdot (T(r \cdot g)_0)^* \\ &= (T(r_1 \cdot q))_* \cdot (Tp)^* \\ &= (Tr_1)_* \cdot (Tq)_* \cdot (Tp)^* \\ &\subset (Tr_1)_* \cdot (Tr_0)^* \cdot (Tg)_* \\ &= \tilde{T}r \cdot \tilde{T}(g_*).\end{aligned}$$

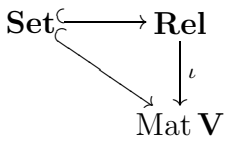
■

2.3.3 Constructing actors on $\mathrm{Mat}\,\mathbf{V}$

Let $\mathbf{V} = (V, \otimes, k)$ be a (non-trivial) quantale. Given a functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$, we want to construct an actor $(T, \tilde{T})$ on $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$. We need to construct a 2-graph morphism $\tilde{T} : \mathrm{Mat}\,\mathbf{V} \longrightarrow \mathrm{Mat}\,\mathbf{V}$. Since $\mathbf{V}$ is non-trivial, we have an embedding

$$\mathbf{Rel} \cong \mathrm{Mat}\,\mathbf{2} \xrightarrow{\iota} \mathrm{Mat}\,\mathbf{V}$$

which is induced by $\mathbf{2} \xrightarrow{\iota_{\mathbf{V}}} \mathbf{V}$, $\bot \longmapsto \bot_V$, $\top \longmapsto k_{\mathbf{V}}$, and makes



commute.

Since we already have a quite satisfying theory for extending **Set**-functors to actors on **Rel**, we will concentrate on extending actors on **Rel** further along ι.

This problem was first addressed by Clementino and Hofmann [CH04b] for a *lax monad* $\mathbf{T}$ on **Rel**. They performed a 2-step construction, first extending $\mathbf{T}$ to $\mathrm{Mat}\,\mathbb{D}\mathbf{V}$ (here $\mathbb{D}V$ is equipped with a suitable quantale structure), and then "transferring" this extension to $\mathrm{Mat}\,\mathbf{V}$. For this transfer to work out properly one needs to assume $\mathbf{V}$ to be *constructive completely distributive* (*ccd*).

We present a one-step approach, based on their "approximation from below" formula (2.9). This will allow us to extend actors, lax *functors*, and transformations from **Rel** to $\mathrm{Mat}\,\mathbf{V}$, for *any* $\mathbf{V}$. Alas, we are not able to extend lax monad structures, basically because the extension process in this general setting does not respect composition. We show that it is much better behaved if $\mathbf{V}$ is ccd. This allows us to recover the results of Clementino and Hofmann.

Although our extension formula (2.9) is taken from [CH04b], and this article has been a great inspiration for the results in this section, we have developed independent proofs. Also be warned that we use some notation in a slightly different way.

Levels For a $\mathbf{V}$-matrix $X \xrightarrow{a} Y$ and $v \in V$, we define a relation $a_v \subset X \times Y$ by

$$x\, a_v\, y \iff v \leq a(x, y). \tag{2.8}$$

We call a_v the *level of a at v*. Observe that $(\ )_v = \mathrm{Mat}(\uparrow_v) : \mathrm{Mat}\,\mathbf{V} \longrightarrow \mathbf{Rel}$, with $\uparrow_v (u) = \top$ if $v \leq u$ and $\bot$ else. These level-functors are crucial to the construction of extensions. We collect some of their basic properties in the following lemma:

Lemma 2.3.13 *Let $X \xrightarrow{a} Y$, $Y \xrightarrow{b} Z$ be $\mathbf{V}$-matrices and $u, v \in V$. Then the following statements hold:*

(1) *$a \leq a'$ for some $\mathbf{V}$-matrix a' implies $a_v \subset a'_v$;*

(2) $a_{\bigvee W} = \bigcap_{w\in W} a_w$ *for all* $W \subset V$*; in particular,* $u \le v$ *implies* $a_v \subset a_u$*;*

(3) $a_v \cdot r \subset (a \cdot \iota(r))_v$ *and* $t \cdot a_v \subset (\iota(t) \cdot a)_v$ *for all relations* $r : W \mathrel{-\!\!\!\rightarrow} X$, $t : Y \mathrel{-\!\!\!\rightarrow} Z$*;*

(4) $a_v \cdot f = (a \cdot f)_v$ *for any function* $f : X \longrightarrow Y$*;*

(5) $m \cdot a_v = (m \cdot a)_v$ *for any injective function* $m : Y \longrightarrow W$*;*

(6) $b_u \cdot a_v \subset (b \cdot a)_{u\otimes v}$.

The following notation will be useful: Given a relation $X \overset{r}{\mathrel{-\!\!\!\rightarrow}} Y$ and a subset $A \subset X$ we write $r[A] = \{\, y \in Y \mid \exists a \in A : a\, r\, y \,\}$.

Proof Statements (1) and (2) are immediate; (4) and (5) follow from [3] below.

[3] We will concentrate on the first inclusion, the other is proved analogously. Take any $w \in W$, $y \in Y$; $w\,(a_v \cdot r)\,y$ implies that there exists $x \in r[w]$ with $v \le a(x, y)$, thus $v \le \bigvee_{x\in r[w]} a(x, y) = (a \cdot \iota(r))(w, y)$, that is, $w\,(a \cdot \iota(r))_v\, y$.

[6] Suppose we have $x\,(b_u \cdot a_v)\,z$, that is, there exists $y \in Y$ with $x\, a_v\, y$ and $y\, b_u\, z$. Then $u \otimes v \le b(y, z) \otimes a(x, y) \le (b \cdot a)(x, z)$, and hence $x\,(b \cdot a)_{u\otimes v}\, z$. ∎

Remark 2.3.14 For a relation $r \subset X \times Y$ and $v \in V$ we have

$$(\iota(r))_v = \begin{cases} X \times Y & v = \bot, \\ r & v \ne \bot \text{ and } v \le k_{\mathbf{V}}, \\ \emptyset & \text{else.} \end{cases}$$

∎

The extension For any 2-graph morphism T : **Rel** $\longrightarrow$ **Rel** we define a 2-graph morphism $T_{\mathbf{V}} : \operatorname{Mat}\mathbf{V} \longrightarrow \operatorname{Mat}\mathbf{V}$ by "approximation from below":

$$T_{\mathbf{V}}(a)(\mathfrak{x}, \mathfrak{y}) = \bigvee \{\, v \in V \mid \mathfrak{x}\,(Ta_v)\,\mathfrak{y} \,\}. \tag{2.9}$$

Clearly $T_{\mathbf{V}}$ is locally monotone; $T_{\mathbf{V}}$ preserves the bottom $\mathbf{V}$-matrices provided T preserves the empty relation. Moreover, if T commutes with the involution of **Rel**, then $T_{\mathbf{V}}$ commutes with the one of $\operatorname{Mat}\mathbf{V}$.

Proposition 2.3.15 *For any 2-graph morphism* T : **Rel** $\longrightarrow$ **Rel***, we have*

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\ \iota\ } & \operatorname{Mat}\mathbf{V} \\ {\scriptstyle T}\downarrow & \le & \downarrow{\scriptstyle T_{\mathbf{V}}} \\ \mathbf{Rel} & \xrightarrow[\ \iota\ ]{} & \operatorname{Mat}\mathbf{V}. \end{array} \tag{2.10}$$

If $\mathbf{V}$ *is integral or* T *preserves the* $\bot$*-matrix, then* (2.10) *commutes. In this case we say that* $T_{\mathbf{V}}$ extends T.

The extension $\mathrm{Rel}\,T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ of a **Set**-endofunctor T preserves the bottom matrix if and only if $T(\emptyset) = \emptyset$ holds. We remark that the restrictions on T and $\mathbf{V}$ are necessary, as the following simple counterexample shows:

Let T be (the extension to **Rel** of) the **Set**-functor given by $X \longmapsto X+1$, and $\mathbf{V}$ the chain $\{0,1,2\}$ with quantale-structure as in Example 1.5.8(5). Since $T1_\emptyset = 1_1$ holds, we obtain $\iota \cdot T(1_\emptyset) = 1_1 = (1)$, but $T_\mathbf{V} \cdot \iota(1_\emptyset) = (2)$, that is, $T_\mathbf{V}$ does not extend T.

Proof The first assertion is obvious. Assume now that T preserves the $\bot$-matrix or that $\mathbf{V}$ is integral. By our hypothesis on T and $\mathbf{V}$ we obtain from Remark 2.3.14

$$T(\iota(r)_v) = \begin{cases} Tr & v \leq k_\mathbf{V}, \\ \emptyset & \text{else}, \end{cases}$$

for all $v \in V \setminus \{\bot\}$. This in turn implies

$$T_\mathbf{V}(\iota(r))(\mathfrak{x},\mathfrak{y}) = \bigvee \{\, v \in V \setminus \{\bot\} \mid \mathfrak{x}\,\big(T(\iota(r))_v\big)\,\mathfrak{y} \,\} = \begin{cases} k_\mathbf{V} & \text{if } \mathfrak{x}\,(Tr)\,\mathfrak{y}, \\ \bot & \text{else}, \end{cases}$$

and hence $T_\mathbf{V} \cdot \iota = \iota \cdot T$. ∎

Formula (2.9) is not the only possibility to define an extension of T to $\mathrm{Mat}\,\mathbf{V}$; see [CHT04] for an example.

Lemma 2.3.16 *Suppose $T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ is a 2-graph morphism. For a $\mathbf{V}$-matrix $Y \overset{a}{\nrightarrow} Y$ and* relations *$X \overset{r}{\nrightarrow} TY$, $TZ \overset{t}{\nrightarrow} W$, the following "whisker" formulas hold:*

$$T_\mathbf{V}a \cdot \iota(r)(x,\mathfrak{z}) = \bigvee\{\, v \in V \mid x\,(Ta_v \cdot r)\,\mathfrak{z}\,\}, \quad \iota(t) \cdot T_\mathbf{V}a(\mathfrak{y},w) = \bigvee\{\, v \in V \mid \mathfrak{y}\,(t \cdot Ta_v)\,w\,\},$$

for all $x \in X$, $\mathfrak{y} \in TY$, $\mathfrak{z} \in TZ$, $w \in W$.

Proof Consider the first equation. The left hand side simplifies to $T_\mathbf{V}a \cdot \iota(r)(x,\mathfrak{z}) = \bigvee_{\mathfrak{y} \in r[x]} T_\mathbf{V}a(\mathfrak{y},\mathfrak{z}) = \alpha$. Write $\beta = \bigvee\{\, v \in V \mid x\,(Ta_v \cdot r)\,\mathfrak{z}\,\}$ for the right hand side.

We first show $\alpha \leq \beta$. For this it suffices to show $T_\mathbf{V}a(\mathfrak{y},\mathfrak{z}) \leq \beta$ for all $\mathfrak{y} \in r[x]$. Since $\mathfrak{y}\,Ta_v\,\mathfrak{z}$ and $x\,r\,\mathfrak{y}$ imply $x\,(Ta_v \cdot r)\,\mathfrak{z}$, we obtain, for any $\mathfrak{y} \in r[x]$:

$$T_\mathbf{V}a(\mathfrak{y},\mathfrak{z}) = \bigvee\{\, v \in V \mid \mathfrak{y}\,Ta_v\,\mathfrak{z}\,\} \leq \bigvee\{\, v \in V \mid x\,(Ta_v \cdot r)\,\mathfrak{z}\,\} = \beta.$$

To show $\beta \leq \alpha$, observe that for any $u \in V$ such that $x\,(Ta_u \cdot r)\,\mathfrak{z}$ there exists $\mathfrak{y} \in r[x]$ such that $\mathfrak{y}\,Ta_u\,\mathfrak{z}$, hence $u \leq \bigvee\{\, v \in V \mid \mathfrak{y}\,Ta_v\,\mathfrak{z}\,\} = T_\mathbf{V}a(\mathfrak{y},\mathfrak{z}) \leq T_\mathbf{V}a \cdot \iota(r)(x,\mathfrak{z})$. Since this holds for all u with $x\,(Ta_u \cdot r)\,\mathfrak{z}$, we obtain $\beta \leq \alpha$.

The second equation is proved analogously. ∎

Proposition 2.3.17 *If $\mathbf{Rel} \overset{T}{\longrightarrow} \mathbf{Rel}$ is a lax functor, then also $\mathrm{Mat}\,\mathbf{V} \overset{T_\mathbf{V}}{\longrightarrow} \mathrm{Mat}\,\mathbf{V}$ is a lax functor.*

Observe that it does not matter whether $T_{\mathbf{V}}$ extends T.

Proof We have $1_{T_{\mathbf{V}}X} = \iota(1_{TX}) \leq \iota(T(1_X)) \leq T_{\mathbf{V}}\iota 1_X = \tilde{T}1_X$ for each set X.

Let $X \xrightarrow{a} Y \xrightarrow{b} Z$ be $\mathbf{V}$-matrices, and $\mathfrak{x} \in TX$, $\mathfrak{z} \in TZ$. We write $\alpha = T_{\mathbf{V}}(b \cdot a)(\mathfrak{x}, \mathfrak{z})$. Since $T_{\mathbf{V}}b \cdot T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{z}) = \bigvee_{\mathfrak{y} \in TY} T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) \otimes T_{\mathbf{V}}b(\mathfrak{y}, \mathfrak{z})$, it suffices to show that

$$T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) \otimes T_{\mathbf{V}}b(\mathfrak{y}, \mathfrak{z}) \leq \alpha$$

holds for any $\mathfrak{y} \in TY$. For each $\mathfrak{y} \in TY$ we have

$$T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) \otimes T_{\mathbf{V}}b(\mathfrak{y}, \mathfrak{z}) = \Bigl(\bigvee_{v \in V\colon \mathfrak{x} T a_v\, \mathfrak{y}} v \Bigr) \otimes \Bigl(\bigvee_{u \in V\colon \mathfrak{y} T b_u\, \mathfrak{z}} u \Bigr) = \bigvee_{\substack{v \in V \\ \mathfrak{x} T a_v\, \mathfrak{y}}} \bigvee_{\substack{u \in V \\ \mathfrak{y} T b_u\, \mathfrak{z}}} v \otimes u.$$

Thus it suffices to show $u \otimes v \leq \alpha$ whenever $\mathfrak{x}\, T a_v\, \mathfrak{y}$, $\mathfrak{y}\, T b_u\, \mathfrak{z}$. For each such v, u, we obtain

$$\begin{aligned} \mathfrak{x}\, (T b_u \cdot T a_v)\, \mathfrak{z} &\implies \mathfrak{x}\, \bigl(T(b_u \cdot a_v)\bigr)\, \mathfrak{z} && \text{since } T \text{ is a lax functor} \\ &\implies \mathfrak{x}\, T(b \cdot a)_{u \otimes v}\, \mathfrak{z} && \text{by Lemma 2.3.13(6).} \end{aligned}$$

Thus $u \otimes v \leq \bigvee \{\, w \in V \mid \mathfrak{x}\, T(b \cdot a)_w\, \mathfrak{z} \,\} = \alpha$. ∎

Extending actors Let us write $\mathcal{M}$ for the class of $\mathbf{V}$-graphs of injective functions.

Proposition 2.3.18 *Let* $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$ *be a functor. If* $T_{\mathbf{V}}$ *extends* $\operatorname{Rel} T$*, then* $(T, T_{\mathbf{V}})$ *is an actor on* $(\mathbf{Set}, \operatorname{Mat} \mathbf{V})$*. Moreover, we have the following results:*

(1) $(T, T_{\mathbf{V}})$ *is left* $\mathcal{M}$*-strict and right* $\mathcal{M}^*$*-strict;*

(2) $(T, T_{\mathbf{V}})$ *is right* $\mathcal{M}$*-strict and left* $\mathcal{M}^*$*-strict provided* T *is taut;*

(3) $T_{\mathbf{V}}$ *is a lax functor provided* T *nearly preserves pullbacks.*

Proof Let $X \xrightarrow{f} Y$, $Z \xrightarrow{g} W$ be functions, and $Y \xrightarrow{a} Z$ be a $\mathbf{V}$-matrix. (M_1) holds by assumption. To show (M_{2a}), take any $\mathfrak{y} \in TY$, $\mathfrak{w} \in TW$; we obtain:

$$\begin{aligned} T_{\mathbf{V}}g \cdot T_{\mathbf{V}}a(\mathfrak{y}, \mathfrak{w}) &= Tg \cdot T_{\mathbf{V}}a(\mathfrak{y}, \mathfrak{w}) \\ &= \bigvee \{\, v \in V \mid \mathfrak{y}\, (Tg \cdot T a_v)\, \mathfrak{w} \,\} && \text{by Lemma 2.3.16} \\ &\leq \bigvee \{\, v \in V \mid \mathfrak{y}\, T(g \cdot a_v)\, \mathfrak{w} \,\} && \text{since } T \text{ is an actor} \qquad (2.11) \\ &\leq \bigvee \{\, v \in V \mid \mathfrak{y}\, T(g \cdot a)_v\, \mathfrak{w} \,\} && \text{by Lemma 2.3.13(3)} \\ &= T_{\mathbf{V}}(g \cdot a)(\mathfrak{y}, \mathfrak{w}). \end{aligned}$$

To verify (M_{2b}), take $\mathfrak{x} \in TX$ and $\mathfrak{z} \in TZ$, then

$$\begin{aligned} T_{\mathbf{V}}(a \cdot f)(\mathfrak{x}, \mathfrak{z}) &= \bigvee \{\, v \in V \mid \mathfrak{x}\, T(a \cdot f)_v\, \mathfrak{z} \,\} \\ &= \bigvee \{\, v \in V \mid \mathfrak{x}\, T(a_v \cdot f)\, \mathfrak{z} \,\} && \text{by Lemma 2.3.13(4)} \\ &\leq \bigvee \{\, v \in V \mid \mathfrak{x}\, (Ta_v \cdot Tf)\, \mathfrak{z} \,\} && \text{since } T \text{ is an actor} \\ &= T_{\mathbf{V}} a \cdot Tf(\mathfrak{x}, \mathfrak{z}) && \text{by Lemma 2.3.16} \\ &= T_{\mathbf{V}} a \cdot T_{\mathbf{V}} f(\mathfrak{x}, \mathfrak{z}). \end{aligned} \tag{2.12}$$

Hence $\tilde{T}$ is an actor.

To show (1), assume that g is a monomorphism. Then the first inequality in (2.11) is in fact an equality by Proposition 1.5.4 and the second inequality in (2.11) is an equality by Lemma 2.3.13(5). The other assertion follows using the involution of $\operatorname{Mat} \mathbf{V}$.

To show (2), observe that the inequality in (2.12) is, by Proposition 1.5.4, an equality provided $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ preserves pullbacks along f.

Finally, if T nearly preserves pullbacks, then $\operatorname{Rel} T$ is a lax functor by Proposition 1.5.3, and thus $T_{\mathbf{V}}$ is a lax functor by Proposition 2.3.17. ∎

Extending natural transformations

Proposition 2.3.19 *Let $S, T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ be 2-graph morphisms. Every oplax (lax, natural, respectively) transformation $\lambda : S \longrightarrow T$ extends to an oplax (lax, natural respectively) transformation $S_{\mathbf{V}} \longrightarrow T_{\mathbf{V}}$.*

Proof We will proof the claim for oplax transformations. Suppose $\lambda : S \longrightarrow T$ is oplax and let $X \xrightarrow{a} Y$ be a $\mathbf{V}$-matrix. Thus we have for $\mathfrak{X} \in SX$ and $\mathfrak{y} \in TY$:

$$\begin{aligned} \lambda_Y \cdot S_{\mathbf{V}}(a)(\mathfrak{X}, \mathfrak{y}) &= \bigvee \{\, v \in V \mid \mathfrak{X}\, (\lambda_Y \cdot Sa_v)\, \mathfrak{y} \,\} && \text{by Lemma 2.3.16} \\ &\leq \bigvee \{\, v \in V \mid \mathfrak{X}\, (Ta_v \cdot \lambda_X)\, \mathfrak{y} \,\} && \text{since } S \xrightarrow{\lambda} T \text{ is oplax} \\ &= T_{\mathbf{V}} a \cdot \lambda_X(\mathfrak{X}, \mathfrak{y}) && \text{by Lemma 2.3.16.} \end{aligned}$$

The proof for the case of a lax or natural transformation is now obvious. ∎

Corollary 2.3.20 *Let $S, T : \mathbf{Set} \longrightarrow \mathbf{Set}$ be functors and $S \xrightarrow{\lambda} T$ a natural transformation. Then the $\mathbf{V}$-graphs of the λ_X form an oplax transformation $S_{\mathbf{V}} \longrightarrow T_{\mathbf{V}}$, which is a natural transformation if λ is nearly cartesian.*

Proof Propositions 1.5.6 and 2.3.19. ∎

Proposition 2.3.21 *We have that $(1_{\mathbf{Rel}})_{\mathbf{V}} = 1_{\operatorname{Mat} \mathbf{V}}$ and $(TS)_{\mathbf{V}} \leq T_{\mathbf{V}} S_{\mathbf{V}}$ hold for all 2-graph morphisms $S, T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$; that is, the $(\ )_{\mathbf{V}}$-construction is normally oplax-functorial.*

Proof Clearly $(1_{\mathbf{Rel}})_{\mathbf{V}} = 1_{\mathrm{Mat}\,\mathbf{V}}$ holds. To prove the second claim, observe that $S(a_v) \subset (S_{\mathbf{V}}a)_v$ holds for all $\mathbf{V}$-matrices a and all $v \in V$. Thus $(TS)a_v \subset T(S(a_v)) \subset T(S_{\mathbf{V}}a)_v$ and hence $(TS)_{\mathbf{V}}(a) \leq T_{\mathbf{V}}S_{\mathbf{V}}(a)$. ∎

Corollary 2.3.22 *We have* $(1_{\mathbf{Set}})_{\mathbf{V}} = 1_{\mathrm{Mat}\,\mathbf{V}}$ *and* $(GF)_{\mathbf{V}} \leq G_{\mathbf{V}}F_{\mathbf{V}}$ *for all* **Set***-functors* F, G.

Proof Remark 1.5.5 and Proposition 2.3.21. ∎

Even though for a **Set**-monad (T, e, m) with T nearly pullback-preserving we obtain by the above construction a lax functor $T_{\mathbf{V}} : \mathrm{Mat}\,\mathbf{V} \longrightarrow \mathrm{Mat}\,\mathbf{V}$ together with oplax transformations $e : 1_{\mathrm{Mat}\,\mathbf{V}} \longrightarrow T_{\mathbf{V}}$ and $m : (TT)_{\mathbf{V}} \longrightarrow T_{\mathbf{V}}$, in general these data does *not* form a lax monad in the sense of [CH04b], simply because m does *not* need to be an oplax transformation $T_{\mathbf{V}}T_{\mathbf{V}} \longrightarrow T_{\mathbf{V}}$ (since $(TT)_{\mathbf{V}} \neq T_{\mathbf{V}}T_{\mathbf{V}}$).

Involving complete distributivity The preceding results can be improved if V is constructive completely distributive (ccd); see Section 1.3.6. We do not know whether V being ccd is necessary for the following.

Proposition 2.3.23 *If* V *is ccd, then the* $(\ )_{\mathbf{V}}$*-construction is preserves composition.*

Proof Assume V to be ccd and let $B : V \longrightarrow \mathbb{D}V$ be the left adjoint of $\bigvee$. In view of Proposition 2.3.21, it suffices to show that $T_{\mathbf{V}}S_{\mathbf{V}} \leq (TS)_{\mathbf{V}}$ holds for all 2-graph morphisms $S, T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$.

Let $X \overset{a}{\nrightarrow} Y$ be a $\mathbf{V}$-matrix and $\mathfrak{x} \in TSX$, $\mathfrak{y} \in TSY$. Since $T_{\mathbf{V}}S_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\, v \in V \mid \mathfrak{x}\, T(S_{\mathbf{V}}a)_v\, \mathfrak{y} \,\}$, it suffices to show

$$v \leq (TS)_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\, w \in W \mid \mathfrak{x}\, TS(a_w)\, \mathfrak{y} \,\} \tag{$\dagger$}$$

for all $v \in V$ such that $\mathfrak{x}\, T(S_{\mathbf{V}}a)_v\, \mathfrak{y}$. Fix any such v. Since the set on the right side of ($\dagger$) is a downset, ($\dagger$) is equivalent to

$$B(v) \subset \{\, w \in V \mid \mathfrak{x}\, TS(a_w)\, \mathfrak{y} \,\}. \tag{$\ddagger$}$$

Take any $u \in B(v)$. We claim that $(S_{\mathbf{V}}a)_v \subset S(a_u)$ holds. Indeed, for $\mathfrak{a} \in SX$, $\mathfrak{b} \in SY$, we have

$$\mathfrak{a}\, (S_{\mathbf{V}}a)_v\, \mathfrak{b} \iff v \leq S_{\mathbf{V}}a(\mathfrak{a}, \mathfrak{b}) = \bigvee\{\, w \in V \mid \mathfrak{a}\, Sa_w\, \mathfrak{b} \,\}$$

where, again, the set involved is a downset. Thus, by the description (1.6) of $B(v)$ given above, $\mathfrak{a}\, Sa_u\, \mathfrak{b}$. Since T is locally monotone, we arrive at $T(S_{\mathbf{V}}a)_v \subset TS(a_u)$ for all $u \in B(v)$. In particular, we obtain $\mathfrak{x}\, TS(a_u)\, \mathfrak{y}$ for all $u \in B(v)$, and ($\ddagger$) is proved. ∎

Proposition 2.3.24 *Assume* $T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ *to be a 2-functor and* $T_{\mathbf{V}}$ *to extend* T. *If* V *is ccd, then*

$$T_{\mathbf{V}}a \cdot T_{\mathbf{V}}\iota(r) = T_{\mathbf{V}}(a \cdot \iota(r)) \qquad \textit{and} \qquad T_{\mathbf{V}}\iota(s) \cdot T_{\mathbf{V}}a = T_{\mathbf{V}}(\iota(s) \cdot a)$$

hold for all relations $X \xrightarrow{r} Y$, $Z \xrightarrow{s} W$ *and every* $\mathbf{V}$*-matrix* $Y \xrightarrow{a} Z$.

Proof We will only prove the first equation. Let $B : V \longrightarrow \mathbb{D}V$ be the left adjoint of $\bigvee$. First we shall show that for any $v \in V$, $(a \cdot \iota(r))_v \subset a_u \cdot r$ holds for all $u \in B(v)$. Indeed, $v \leq a \cdot \iota(r)(x,z) = \bigvee_{y \in r[x]} a(y,z)$ implies (after forming down-closures) that there exists $y \in r[x]$ with $u \leq a(y,z)$, that is, $x\,(a_u \cdot r)\,z$. From this we obtain $T(a \cdot \iota(r))_v \subset T(a_u \cdot r) = Ta_u \cdot Tr$ for all $u \in B(v)$. Let us write $W = \{\, w \in V \mid \mathfrak{x}\,(Ta_w \cdot Tr)\mathfrak{z} \,\}$.

Thus $\mathfrak{x}\,T(a \cdot \iota(r))_v\,\mathfrak{z}$ implies $B(v) \subset W^{\downarrow}$, equivalently, $v \leq \bigvee W^{\downarrow} = \bigvee W = \tilde{T}a \cdot \iota(Tr)(\mathfrak{x},\mathfrak{z})$, where the last equations holds by Lemma 2.3.16. Therefore,

$$T_{\mathbf{V}}(a \cdot \iota(r))(\mathfrak{x},\mathfrak{z}) = \bigvee\{\, v \in V \mid \mathfrak{x}\,T(a \cdot \iota(r))_v\,\mathfrak{z} \,\} \leq T_{\mathbf{V}}a \cdot \iota(Tr)(\mathfrak{x},\mathfrak{z}) = T_{\mathbf{V}}a \cdot T_{\mathbf{V}}\iota(r)(\mathfrak{x},\mathfrak{z}).$$

Since $T_{\mathbf{V}}$ is a lax functor by Proposition 2.3.17, we are done. ∎

Proposition 2.3.25 *If* V *is ccd and* $\mathbf{V}$ *is a frame, then* $T_{\mathbf{V}} : \operatorname{Mat}\mathbf{V} \longrightarrow \operatorname{Mat}\mathbf{V}$ *is a 2-functor provided* $T : \mathbf{Rel} \longrightarrow \mathbf{Rel}$ *is one.*

Proof First observe that our hypotheses imply that $T_{\mathbf{V}}$ extends T, thus preserves identities. Let us denote the left adjoint of $\bigvee$ by B. We only need to show that $T_{\mathbf{V}}(b \cdot a)(\mathfrak{x},\mathfrak{z}) \leq T_{\mathbf{V}}b \cdot T_{\mathbf{V}}a(\mathfrak{x},\mathfrak{z})$ holds for all $X \xrightarrow{a} Y \xrightarrow{b} Z$ and $\mathfrak{x} \in TX$, $\mathfrak{z} \in TZ$.

We write $W = \{\, w \in V \mid \exists \mathfrak{y} \in TY : w = T_{\mathbf{V}}b(\mathfrak{y},\mathfrak{z}) \wedge T_{\mathbf{V}}a(\mathfrak{x},\mathfrak{y}) \,\}$ and observe that it suffices to show $v \leq \bigvee W = \bigvee W^{\downarrow}$ for all $v \in V$ such that $\mathfrak{x}\,T(b{\cdot}a)_v\,\mathfrak{z}$ holds. For any such $v \in V$ and any $u \in B(v)$, we claim that $(b \cdot a)_v \subset b_u \cdot a_u$ holds. Indeed, $v \leq b \cdot a(x,z) = \bigvee_y b(y,z) \wedge a(x,y)$ implies that there exists $y_0 \in Y$ with $u \leq b(y_0,z) \wedge a(x,y_0)$. This in turn implies

$$T(b \cdot a)_v \subset T(b_u \cdot a_u) = Tb_u \cdot Ta_u.$$

Hence there exists $\mathfrak{y} \in TY$ with $\mathfrak{x}\,Ta_u\,\mathfrak{y}$ and $\mathfrak{y}\,Tb_u\,\mathfrak{z}$, thus $u \leq T_{\mathbf{V}}a(\mathfrak{x},\mathfrak{y}) \wedge T_{\mathbf{V}}b(\mathfrak{y},\mathfrak{z})$, which implies $u \in W^{\downarrow}$. That is, $B(v) \subset W^{\downarrow}$, and we are done. ∎

Changing V

Proposition 2.3.26 *Let* $\mathbf{V}$, $\mathbf{W}$ *be quantales,* $V \xrightarrow{\phi} W$ *a functor, and* $\mathbf{Rel} \xrightarrow{T} \mathbf{Rel}$ *a 2-graph morphism. The following statements hold:*

(1) *If* ϕ *preserves suprema, then* $\operatorname{Mat}\phi \cdot T_{\mathbf{V}} \leq T_{\mathbf{W}} \cdot \operatorname{Mat}\phi$ *holds.*

(2) *If ϕ preserves infima, then we have $T_{\mathbf{W}} \cdot \mathrm{Mat}\,\phi \leq \mathrm{Mat}\,\phi \cdot T_{\mathbf{V}}$.*

The counterexample to Proposition 2.3.15 shows that (2) need not hold if ϕ does not preserve infima, which for $\iota : \mathbf{2} \longrightarrow \mathbf{V}$ is equivalent to $\mathbf{V}$ being non-integral. In this sense the above proposition extends Proposition 2.3.15 (observe that $\iota : \mathbf{2} \longrightarrow \mathbf{V}$ always preserves suprema).

Proof Let $X \xrightarrow{a} Y$ be a $\mathbf{V}$-matrix.

[1] Monotonicity of ϕ implies $a_v \subset (\mathrm{Mat}\,\phi(a))_{\phi(v)}$, and thus

$$\begin{aligned}
\mathrm{Mat}\,\phi(T_{\mathbf{V}}(a))(\mathfrak{x},\mathfrak{y}) &= \phi\big(\bigvee\{\, v \in V \mid \mathfrak{x}\, T a_v\, \mathfrak{y}\,\}\big) \\
&= \bigvee\{\, w \in W \mid \exists v \in V : \phi(v) = w,\ \mathfrak{x}\, T a_v\, \mathfrak{y}\,\} \\
&\leq \bigvee\{\, w \in W \mid \exists v \in V : \phi(v) = w,\ \mathfrak{x}\, T(\mathrm{Mat}\,\phi(a))_w\, \mathfrak{y}\,\} \\
&\leq \bigvee\{\, w \in W \mid \mathfrak{x}\, T(\mathrm{Mat}\,\phi(a))_w\, \mathfrak{y}\,\} \\
&= T_{\mathbf{W}}(\mathrm{Mat}\,\phi(a))(\mathfrak{x},\mathfrak{y})
\end{aligned}$$

for all $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$.

[2] Preservation of infima of ϕ is equivalent to the existence of $\chi \dashv \phi$. For each $x \in X$, $y \in Y$, we have $x\,(\mathrm{Mat}\,\phi(a))_w\,y \iff w \leq \phi(a(x,y)) \iff \chi(w) \leq a(x,y) \iff x\,a_{\chi(w)}\,y$, and thus, for any $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$:

$$\begin{aligned}
\chi\Big(\bigvee\{\, w \in W \mid \mathfrak{x}\, T(a_{\chi(w)})\, \mathfrak{y}\,\}\Big) &= \bigvee\{\, \chi(w) \mid w \in W,\ \mathfrak{x}\, T(a_{\chi(w)})\, \mathfrak{y}\,\} \\
&\leq \bigvee\{\, v \in V \mid \mathfrak{x}\, T a_v\, \mathfrak{y}\,\}
\end{aligned}$$

which implies $T_{\mathbf{W}}(\mathrm{Mat}\,\phi(a))(\mathfrak{x},\mathfrak{y}) \leq \phi\big(\bigvee\{v \in V \mid \mathfrak{x}\, T a_v\, \mathfrak{y}\,\} = \mathrm{Mat}\,\phi(T_{\mathbf{V}}(a))(\mathfrak{x},\mathfrak{y})$. ■

2.3.4 From Dist to OrdMat$\mathbf{V}$

We are going to use the techniques from the preceding section to extend a 2-graph morphism on $\mathbf{Dist} = \mathrm{Dist}(\mathbf{Set}, \mathbf{Rel}) = \mathrm{OrdMat}(\mathbf{2})$ to $\mathrm{OrdMat}(\mathbf{V})$. First observe that Lemma 2.3.13(3) implies:

Corollary 2.3.27 *If $(X,a) \xrightarrow{r} (Y,b)$ is a morphism in $\mathrm{OrdMat}(\mathbf{V})$ and $v \in V$, then the level r_v is a distributor $(X,a) \nrightarrow (Y,b)$.* ■

For a 2-graph morphism $\mathbf{Dist} \xrightarrow{T} \mathbf{Dist}$ we define $\mathrm{OrdMat}(\mathbf{V}) \xrightarrow{T_{\mathbf{V}}} \mathrm{OrdMat}(\mathbf{V})$ as before (2.9) by approximation from below:

$$T_{\mathbf{V}} r(\mathfrak{x},\mathfrak{y}) = \bigvee\{\, v \in V \mid \mathfrak{x}\, T r_v\, \mathfrak{y}\,\}. \tag{2.13}$$

We need to show that $T_{\mathbf{V}}r$ is an monotone matrix whenever r is one: suppose $\mathfrak{x} \leq \mathfrak{x}'$ in TX and $\mathfrak{y}' \leq \mathfrak{y}$ in TY, we obtain, using that Tr_v is a distributor for all $v \in V$:

$$T_{\mathbf{V}}r(\mathfrak{x}', \mathfrak{y}') = \bigvee \{\, v \in V \mid \mathfrak{x}'\, Tr_v\, \mathfrak{y}' \,\} \leq \bigvee \{\, v \in V \mid \mathfrak{x}\, Tr_v\, \mathfrak{y} \,\} = T_{\mathbf{V}}r(\mathfrak{x}, \mathfrak{y}).$$

Proposition 2.3.28 *Let $T : \mathbf{Dist} \longrightarrow \mathbf{Dist}$ be a 2-graph morphism and $\mathbf{V}$ be a quantale. Define $T_{\mathbf{V}} : \mathrm{OrdMat}(\mathbf{V}) \longrightarrow \mathrm{OrdMat}(\mathbf{V})$ by (2.13). Then*

$$\begin{array}{ccc} \mathbf{Dist} & \xrightarrow{\;T\;} & \mathbf{Dist} \\ \downarrow & & \downarrow \\ \mathrm{OrdMat}(\mathbf{V}) & \xrightarrow[T_{\mathbf{V}}]{} & \mathrm{OrdMat}(\mathbf{V}) \end{array} \tag{2.14}$$

commutes provided $\mathbf{V}$ is integral or T preserves the bottom-distributor $\emptyset$. In this case the following statements hold:

(1) *If T is an actor on $\mathcal{D}$, then $T_{\mathbf{V}}$ is an actor on $\mathcal{O}(\mathbf{V})$;*

(2) *if T is a lax functor, so is $T_{\mathbf{V}}$;*

(3) *if T is a 2-functor and $\mathbf{V}$ is a frame with V ccd, then also $T_{\mathbf{V}}$ is a 2-functor.*

Proof Commutativity of (2.14) is proved as in Proposition 2.3.15.

[1] (M_1) follows from the commutativity of (2.14) and the fact that T satisfies (M_1). (M_{2a}) is shown exactly as in Proposition 2.3.18. Also (M_{2b}) is shown as in Proposition 2.3.18, but we have to take some care: since $a \cdot f_* = a \cdot f$ holds in **Dist** and in $\mathrm{OrdMat}(\mathbf{V})$ for all monotone functions f and all distributors a, we obtain $(a \cdot f_*)_v = (a \cdot f)_v = a_v \cdot f$ by Lemma 2.3.13(4). Thus $T(a \cdot f_*)_v = T(a_v \cdot f_*)$ and we can argue like in the proof of Proposition 2.3.18.

(2) and (3) follow from Proposition 2.3.17 and Proposition 2.3.25, respectively, since ordered $\mathbf{V}$-matrices are composed like (ordinary) $\mathbf{V}$-matrices. ∎

2.3.5 Extending functors to Kleisli categories

In this section we will show how to extend an endofunctor G on a category $\mathbf{C}$ to a functor on a Kleisli category $\mathbf{C_T}$. As we will see, this process is governed by a *distributive law of G over $\mathbf{T}$*. In fact, it is dual (in an appropriate 2-categorical sense) to the well-known lifting of G to the Eilenberg–Moore category $\mathbf{C^T}$, which is governed by a distributive law of $\mathbf{T}$ over G; see [BW00]. For a proof "by duality" see, for instance, [MRW02].

Nonetheless, I decided to include an elementary proof (which I could not find in the literature) of the correspondence between distributive laws and extensions since I feel that it leads to better understanding of the situation. Also, it might point the

way to an extension of the correspondence to one between "lax" extensions and "lax" distributive laws in an order-enriched context.

We shall use this technique to construct functors on **Dist**. Hopefully, it will also shed some light on the extension-process from **Set** to **Rel**.

Let $\mathbf{T} = (T, e, m)$ be a monad on a category $\mathbf{C}$ and $G : \mathbf{C} \longrightarrow \mathbf{C}$ be a functor. Recall from Section 1.3.6 that a distributive law of G over $\mathbf{T}$ is a natural transformation $GT \xrightarrow{\lambda} TG$ which satisfies $\lambda[e]$ and $\lambda[m]$, that is, $\lambda \cdot Ge = eG$ and $\lambda \cdot Gm = mG \cdot T\lambda \cdot \lambda T$ hold.

Proposition 2.3.29 *There is a one-to-one correspondence between distributive laws of G over $\mathbf{T}$ and* extensions *of G to $\mathbf{C}_\mathbf{T}$, that is, functors $\tilde{G} : \mathbf{C}_\mathbf{T} \longrightarrow \mathbf{C}_\mathbf{T}$ such that*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F_\mathbf{T}} & \mathbf{C}_\mathbf{T} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \tilde{G}} \\ \mathbf{C} & \xrightarrow{F_\mathbf{T}} & \mathbf{C}_\mathbf{T} \end{array} \tag{2.15}$$

commutes.

Proof Given a distributive law $GT \xrightarrow{\lambda} TG$ of G over $\mathbf{T}$, we define $G_\lambda : \mathbf{C}_\mathbf{T} \longrightarrow \mathbf{C}_\mathbf{T}$ by $G_\lambda X = GX$ and

$$G_\lambda(f^\flat) = (GX \xrightarrow{Gf} GTY \xrightarrow{\lambda_Y} TGY)^\flat.$$

First we show that $G_\lambda \cdot F_\mathbf{T} = F_\mathbf{T} \cdot G$ holds: given any $f \in \mathbf{C}$, we have

$$\begin{aligned} G_\lambda(F_\mathbf{T} f) &= \lambda_Y \cdot G(Tf \cdot e_X) \\ &= TGf \cdot \lambda_X \cdot Ge_X \\ &= TGf \cdot e_{GX} && \text{by } \lambda[e] \\ &= F_\mathbf{T}(Gf) \end{aligned}$$

Here we omit the superscript $\flat$ to increase readability. Preservation of identities is immediate. For $f : X \longrightarrow TY$, $g : Y \longrightarrow TZ$ in $\mathbf{C}$, we have

$$\begin{aligned} \lambda_Z \cdot G(m_Z \cdot Tg \cdot f) &= \lambda_Z \cdot Gm_Z \cdot GTg \cdot Gf \\ &= m_{GZ} \cdot T\lambda_Z \cdot \lambda_{TZ} \cdot GTg \cdot Gf && \text{by } \lambda[m] \\ &= m_{GZ} \cdot T\lambda_Z \cdot TGg \cdot \lambda_Y \cdot Gf \\ &= m_{GZ} \cdot T(\lambda_Z \cdot Gg) \cdot (\lambda_Y \cdot Gf), \end{aligned}$$

that is, $G_\lambda(g^\flat * f^\flat) = G_\lambda(g^\flat) * G_\lambda(f^\flat)$.

Now suppose we are given a functor $\tilde{G} : \mathbf{C}_\mathbf{T} \longrightarrow \mathbf{C}_\mathbf{T}$ such that (2.15) commutes, that is, $\tilde{G}(e_Y \cdot f) = e_{GY} \cdot Gf$ for any $f : X \longrightarrow Y$ in $\mathbf{C}$. We define $\lambda : GT \longrightarrow TG$ by

$\lambda_X = \tilde{G}(1^\flat_{TX})$. We show naturality of λ: observe

$$\begin{aligned} TGf \cdot \lambda_X &= m_{GY} \cdot Te_{GY} \cdot TGf \cdot \lambda_X \\ &= m_{GY} \cdot T(F_{\mathbf{T}} Gf) \cdot \tilde{G}(1_{TX}) \\ &= \tilde{G}(F_{\mathbf{T}} f) * \tilde{G}(1_{TX}) \\ &= \tilde{G}(F_{\mathbf{T}} f * 1_{TX}) \\ &= \tilde{G}(Tf) \end{aligned}$$

and that, since $b \cdot f = b^\flat * F_{\mathbf{T}} f$ holds for any $X \xrightarrow{f} Y \xrightarrow{b} TZ$, we have

$$\lambda_Y \cdot Gh = \tilde{G}(1_{TY}) * F_{\mathbf{T}}(Gh) = \tilde{G}(1_{TY} * F_{\mathbf{T}} h) = \tilde{G}(1_{TY} \cdot h) = \tilde{G}(h). \qquad (\dagger)$$

for any $X \xrightarrow{h} TY$ in $\mathbf{C}$. Specializing to $h = Tf$, we see that λ is natural. Also $\lambda[e]$ follows from (†) (with $h = e_X$) and the fact that $\tilde{G}$ preserves identities: $\lambda_X \cdot G(e_X) = \tilde{G}(e_X) = e_{\tilde{G}X} = e_{GX}$. To establish $\lambda[m]$, observe that

$$m_{GX} \cdot T\lambda_X \cdot \lambda_{TX} = \lambda_X * \lambda_{TX} = \tilde{G}(1_{TX}) * \tilde{G}(1_{TTX}) = \tilde{G}(1_{TX} * 1_{TTX}) = \tilde{G}(m_X)$$

holds. Thus, by (†), $m_{GX} \cdot T\lambda_X \cdot \lambda_{TX} = \lambda_X \cdot Gm_X$.

The proof that the constructions are mutually inverse is straightforward. ∎

In case $\mathbf{C}$ is an ordered category and T, G are 2-functors, also $\mathbf{C}_{\mathbf{T}}$ is ordered, with 2-cells inherited from $\mathbf{C}$, and each G_λ will be a 2-functor.

Observe that we used the same $(\ )_\lambda$-notation as for the composite of monads formed using λ. This should not cause any confusion, cf. Lemma 2.3.34.

Recall that, ignoring 2-categorical aspects for a moment, $\mathrm{Mat}\,\mathbf{V} \cong \mathbf{Set}_{P_{\mathbf{V}}}$, where $P_{\mathbf{V}}$ is the $\mathbf{V}$-valued fuzzy powerset monad. In particular, $\mathbf{Rel} \cong \mathbf{Set}_P$. Thus, whenever we can extend a functor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ along the embedding to a functor $\mathrm{Mat}\,\mathbf{V} \longrightarrow \mathrm{Mat}\,\mathbf{V}$, we have in fact a distributive law $FP_{\mathbf{V}} \longrightarrow P_{\mathbf{V}}F$.

In particular, we obtain a distributive law $FP \longrightarrow PF$ over the powerset monad for any nearly-pullback (and regular epimorphism) preserving $\mathbf{Set}$-endofunctor F. For any such F and any non-trivial ccd lattice V, Proposition 2.3.25 shows that the extension $\tilde{F}$ of F to $\mathrm{Mat}\,\mathbf{V}$ is a 2-functor (here $\mathbf{V} = (V, \wedge, \top)$; observe that any ccd lattice is a frame, and thus $\mathbf{V}$ is indeed an integral quantale). Thus, Proposition 2.3.29 provides us with a distributive law $FP_{\mathbf{V}} \longrightarrow P_{\mathbf{V}}F$.

Functorial properties of the extension Let $\mathbf{C}$ be a category, $\mathbf{T} = (T, e, m)$ be a monad on $\mathbf{C}$, and $G, H : \mathbf{C} \longrightarrow \mathbf{C}$ be functors with distributive laws $GT \xrightarrow{\lambda} TG$, $HT \xrightarrow{\mu} TH$ over $\mathbf{T}$.

Lemma 2.3.30 *The identity transformation $1_T : T \longrightarrow T$ gives a distributive law of $1_{\mathbf{C}}$ over $\mathbf{T}$, and $\nu = \lambda H \cdot G\mu$ is a distributive law of GH over $\mathbf{T}$.*

Proof The first claim is obvious. Clearly ν is a natural transformation $GHT \longrightarrow TGH$. Moreover, we have $\nu \cdot GHe = \lambda H \cdot G\mu \cdot GHe = \lambda H \cdot GeH = eGH$, thus $\nu[e]$ holds. To establish $\nu[m]$, we proceed as follows:

$$\begin{aligned} mGH \cdot T\nu \cdot \nu T &= mGH \cdot T\lambda H \cdot TG\mu \cdot \lambda HT \cdot G\mu T \\ &= mGH \cdot T\lambda H \cdot \lambda TH \cdot GT\mu \cdot G\mu T \\ &= \lambda H \cdot GmH \cdot GT\mu \cdot G\mu T && \text{by } \lambda[m] \\ &= \lambda H \cdot G(mH \cdot T\mu \cdot \mu T) \\ &= \lambda H \cdot G(\mu \cdot Hm) && \text{by } \mu[m] \\ &= \lambda H \cdot G\mu \cdot GHm \\ &= \nu \cdot GHm. \end{aligned}$$

■

Proposition 2.3.31 *Given functors G, H as above, we have $G_\lambda H_\mu = (GH)_\nu$, where $\nu = \lambda H \cdot G\mu$, and $(1_{\mathbf{C}})_{1_T} = 1_{\mathbf{C}_\mathbf{T}}$.*

Proof We have $(GH)_\nu(f) = \nu_Y \cdot GHf = \lambda_{HY} \cdot G\mu_Y \cdot GHf = \lambda_{HY} \cdot G(\mu_Y \cdot Hf) = G_\lambda(H_\nu(f))$ for any $f : X \longrightarrow TY$. ■

Proposition 2.3.32 *Let $G \xrightarrow{\alpha} H$ be a natural transformation. Then $(F_\mathbf{T}\alpha_X)_X$ is a natural transformation $G_\lambda \longrightarrow H_\mu$ if and only if*

$$\begin{array}{ccc} GT & \xrightarrow{\lambda} & TG \\ {\scriptstyle \alpha T}\downarrow & & \downarrow{\scriptstyle T\alpha} \\ HT & \xrightarrow{\mu} & TH \end{array} \tag{2.16}$$

commutes. In this case we say that α extends along *$F_\mathbf{T}$.*

Proof $(F_\mathbf{T}\alpha_X)_X$ is a natural transformation precisely when $F_\mathbf{T}(\alpha_Y) * G_\lambda(f) = H_\mu(f) * F_\mathbf{T}(\alpha_X)$ holds for any $f : X \longrightarrow Y$ in $\mathbf{C}_\mathbf{T}$. In any case we have

$$F_\mathbf{T}(\alpha_Y) * G_\lambda f = T\alpha_Y \cdot G_\lambda f = T\alpha_Y \cdot \lambda_Y \cdot Gf,$$

and

$$H_\mu(f) * F_\mathbf{T}(\alpha_X) = \mu_Y \cdot Hf \cdot \alpha_X = \mu_Y \cdot \alpha_{TY} \cdot Gf.$$

Thus in case (2.16) commutes, $(F_\mathbf{T}\alpha_X)_X$ is a natural transformation. Conversely, commutativity of (2.16) follows from the above equations for $f = 1^\flat_{TY}$. ■

Extending monads Let $\mathsf{S} = (S, d, n)$ be a monad on $\mathbf{C}$ and $\mu : ST \longrightarrow TS$ a distributive law of S over T. Then d and n extend to natural transformations $1_{\mathbf{C}_\mathsf{T}} \longrightarrow S_\mu$ and $S_\mu S_\mu \longrightarrow S_\mu$ precisely when d and n satisfy the extension condition (2.16) for $\lambda = 1_T$ and $\lambda = \mu S \cdot S\mu$, respectively. But this is the case precisely when μ satisfies $\mu[n]$ and $\mu[d]$, that is, if and only if μ is a distributive law of S over T in the sense of Section 1.3.6. Thus we have shown:

Proposition 2.3.33 *There is a one-to-one correspondence between distributive laws μ of S over T and extensions of S to $\mathbf{C}_\mathsf{T}$.* ∎

We recall that μ as above allows us to construct a monad structure on the composite functor TS, and that this monad is denoted by $(\mathsf{TS})_\mu$.

Lemma 2.3.34 *There is an isomorphism $H : \mathbf{C}_{(\mathsf{TS})_\mu} \longrightarrow (\mathbf{C}_\mathsf{T})_{\mathsf{S}_\mu}$ which makes*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F_\mathsf{T}} & \mathbf{C}_\mathsf{T} \\ {\scriptstyle F_{(\mathsf{TS})_\mu}}\downarrow & & \downarrow{\scriptstyle F_{\mathsf{S}_\mu}} \\ \mathbf{C}_{(\mathsf{TS})_\mu} & \xrightarrow{H} & (\mathbf{C}_\mathsf{T})_{\mathsf{S}_\mu} \end{array} \tag{2.17}$$

commute. ∎

An example We will now try to illustrate the results of this section with the example of the upper-set monad $\mathbb{U}$ on **Ord**. Using the distributive law $\mathbb{U}\mathbb{D} \xrightarrow{r} \mathbb{D}\mathbb{U}$ we obtain a 2-monad $(\mathbb{U}_r, (\ )^\uparrow, \bigcup)$ on $\mathbf{Ord}_\mathbb{D} \cong \mathbf{Dist}^{\mathrm{op}}$. We can consider $\mathbb{U}_r$ as a 2-functor $\mathbf{Dist} \longrightarrow \mathbf{Dist}$ which makes

$$\begin{array}{ccc} \mathbf{Ord} & \xrightarrow{(\ )^*} & \mathbf{Dist} \\ {\scriptstyle \mathbb{U}}\downarrow & & \downarrow{\scriptstyle \mathbb{U}_r} \\ \mathbf{Ord} & \xrightarrow{(\ )^*} & \mathbf{Dist} \end{array}$$

commute. $(\mathbb{U}, \mathbb{U}_r)$ forms an actor on $(\mathbf{Ord}, \mathbf{Dist})$, which we simply denote by $\mathbb{U}$.

Since $((\ )^\uparrow_X)^*_X$ and $(\bigcup_X)^*_X$ form natural transformations $\mathbb{U}_r \longrightarrow 1$ and $\mathbb{U}_r \longrightarrow \mathbb{U}_r \cdot \mathbb{U}_r$, respectively, on **Dist**, thus $(\ )^\uparrow$ and $\bigcup$ form oplax transformation $1_\mathbf{Dist} \longrightarrow \mathbb{U}_r$ and $\mathbb{U}_r\mathbb{U}_r \longrightarrow \mathbb{U}_r$. We give an elementary description of $\mathbb{U}_r$: for a distributor $X \xrightarrow{a} Y$ and $A \in \mathbb{U}X$, $B \in \mathbb{U}Y$, we have

$$\begin{aligned} A\,(\mathbb{U}_r a)\,B &\iff A \in (r_X \cdot \mathbb{U}\check{a})(B) \\ &\iff \forall A' \in \mathbb{U}\check{a}(B)\,\exists x \in A : x \in A'. \end{aligned} \tag{$*$}$$

From $\mathbb{U}\check{a}(B) = (\check{a}[B])^\uparrow$ we obtain that $A' \in \mathbb{U}\check{a}(B)$ holds if and only if there exists $y \in B$ such that $\check{a}(y) \subset A'$. Clearly, in $(*)$ we only have to consider those A' of the

form $\check{a}(y)$ for some $y \in B$. Thus $A\,(\mathbb{U}_r a)\,B$ holds iff for every $y \in B$ there exists $x \in A$ with $x \in \check{a}(y)$. That is,

$$A\,(\mathbb{U}_r a)\,B \iff \forall y \in B\, \exists x \in A : x\,a\,y.$$

The same reasoning applies to any subKZ-monad $\mathbb{U}'$ of $\mathbb{U}$. Using the restricted distributive law $\mathbb{U}'\mathbb{D} \xrightarrow{r'} \mathbb{D}\mathbb{U}'$ we obtain an extension $\tilde{\mathbb{U}}'$ of $\mathbb{U}'$ to **Dist**. For example, consider the finitely generated upperset monad $\mathbb{U}_\omega$ and the inhabited upperset monad $\mathbb{U}_0$ with their inclusions $\mathbb{U}_\omega \xrightarrow{i} \mathbb{U}$ and $\mathbb{U}_0 \xrightarrow{j} \mathbb{U}$. Lemma 1.3.9 shows that the premises of Proposition 2.3.32 are satisfied. Thus we have shown:

Remark 2.3.35 $(i_X^*)_X$ and $(j_X^*)_X$ form natural transformations $\tilde{\mathbb{U}} \longrightarrow \tilde{\mathbb{U}}_\omega$ and $\tilde{\mathbb{U}} \longrightarrow \tilde{\mathbb{U}}_0$. Hence, we obtain *oplax* transformations $(i_X)_* : \tilde{\mathbb{U}}_\omega \longrightarrow \tilde{\mathbb{U}}$ and $(j_X)_* : \tilde{\mathbb{U}}_0 \longrightarrow \tilde{\mathbb{U}}$. ■

Application to distributive laws We have seen that we can use the basic distributive law r to construct a *2-comonad* $(\mathbb{U}_r, (\ )^{\uparrow *}, \bigcup^*)$ on **Dist**. For any 2-comonad $\mathsf{T} = (T, e, m)$ on **Dist** and any ccd-lattice V, Proposition 2.3.28 and (a modification of) Proposition 2.3.19 give a 2-comonad $(T_{\mathbf{V}}, e, m)$ on $\mathrm{OrdMat}(\mathbf{V})$, where $\mathbf{V} = (V, \wedge, \top)$, which extends T in the obvious sense. Using $\mathrm{OrdMat}(\mathbf{V}) \cong \mathbf{Ord}_{D_{\mathbf{V}}}^{\mathrm{op}}$ we obtain a 2-monad on $\mathbf{Ord}_{D_{\mathbf{V}}}$.

Proposition 2.3.36 *Let T be a 2-monad on* **Ord** *and $\mathbf{V} = (V, \wedge, \top)$ a frame with V ccd. If there exists a (necessary unique) distributive law $\lambda : \mathsf{T}\mathbb{D} \longrightarrow \mathbb{D}\mathsf{T}$, then there exists also a distributive law $\mathsf{T}D_{\mathbf{V}} \longrightarrow D_{\mathbf{V}}\mathsf{T}$.*

Proof λ gives rise to an extension of T to a 2-comonad T_λ on **Dist**, which extends further to a 2-comonad $(\mathsf{T}_\lambda)_{\mathbf{V}}$ on $\mathrm{OrdMat}(\mathbf{V})$, which "is" a 2-monad on $\mathbf{Ord}_{D_{\mathbf{V}}}$. This 2-monad is an extension of T along $F_{D_{\mathbf{V}}} : \mathbf{Ord} \longrightarrow \mathbf{Ord}_{D_{\mathbf{V}}}$. This extension corresponds to the sought distributive law. ■

Uniqueness of extensions We have already remarked that formula (2.9) is not the only possibility for extending a **Rel**-functor to $\mathrm{Mat}\,\mathbf{V}$. This raises the question whether there are other possible extensions. At least for the case of extensions from **Ord** to $\mathrm{OrdMat}(\mathbf{V})$ this can be answered negatively for a wide class of monads.

Lemma 2.3.37 *Fix a quantale $\mathbf{V}$. If $\mathsf{T} = (T, e, m)$ is either a KZ- or a coKZ-monad on* **Ord**, *then any extension $(T, T_{\mathbf{V}})$ of T to an actor on $\mathcal{O}(\mathbf{V})$ such that $T_{\mathbf{V}}$ is a 2-functor and e^* and m^* are natural transformations $T_{\mathbf{V}} \longrightarrow 1$ and $T_{\mathbf{V}} \longrightarrow T_{\mathbf{V}}T_{\mathbf{V}}$, respectively, is unique.*

Proof Any such extension corresponds to a distributive law $\mathsf{T}D_{\mathbf{V}} \longrightarrow D_{\mathbf{V}}\mathsf{T}$, which is unique by virtue of T being KZ or coKZ, see [MRW02, Proposition 4.1]. ■

In summary, the use of distributive laws enables us to give a detailed analysis on the possibility of extending functors. Unfortunately, lax functors and monads do not seem to be captured easily using distributive laws.

2.4 Some conditions on scenes

Beck–Chevalley conditions Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ be a scene and $\mathcal{F} \subset \operatorname{mor}\mathbf{S}$ a class of morphisms.

Definition 2.4.1 We say that $\mathcal{S}$ has the *Beck–Chevalley property (*or *(BC)-property) with respect to* $\mathcal{F}$, if every pullback-square

$$\begin{array}{ccc} P & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array} \tag{2.18}$$

in $\mathbf{S}$ with $f \in \mathcal{F}$ is a Beck–Chevalley square in $\mathbf{D}$. Given a scenic functor $\mathcal{S} \xrightarrow{T} \mathcal{S}'$, we say that T has the Beck–Chevalley property (or (BC)-property, for short) with respect to $\mathcal{F}$ if T sends any pullback square (2.18) in $\mathbf{S}$ with $f \in \mathcal{F}$ to a Beck–Chevalley square in $\mathbf{D}'$.

Given a slim transformation $\lambda : T \longrightarrow T'$ between scenic functors $T, T' : \mathcal{S} \longrightarrow \mathcal{S}'$, we say that λ has the Beck–Chevalley property (or (BC)-property) with respect to $\mathcal{F}$ if the λ-naturality square

$$\begin{array}{ccc} TA & \xrightarrow{\lambda_A} & T'A \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle T'f} \\ TB & \xrightarrow{\lambda_B} & T'B \end{array} \tag{2.19}$$

is a Beck–Chevalley square in $\mathbf{D}$ for any $f \in \mathcal{F}$.

In case if $\mathcal{F} = \operatorname{mor}\mathbf{S}$ we drop the suffix "with respect to $\mathcal{F}$" and simply speak about the Beck–Chevalley property (or (BC)-property).

For instance, the actor on $(\mathbf{C}, \operatorname{Rel}\mathbf{C})$ with $\mathbf{C}$ regular defined by a $\mathbf{C}$-endofunctor T has the Beck–Chevalley property if and only if T nearly preserves pullbacks, and has the Beck–Chevalley property with respect to monomorphisms if and only if T is taut; analogously for natural transformations.

Remark 2.4.2 If a scene $\mathcal{S}$ has the (BC)-property with respect to $\mathcal{F}$, then also $\operatorname{Dist}\mathcal{S}$ has the (BC)-property with respect to $\mathcal{F}$. ■

Splitable factorization systems To effectively use factorization systems in the context of scenes we need to single out a special kind of well-behaved factorization systems, which may be called splitable.

Definition 2.4.3 A factorization system (for morphisms) (E, M) on $\mathbf{S}$ is called *splitable* if every $e \in E$ is a cover in $\mathbf{D}$ and every $m \in M$ is an injection in $\mathbf{D}$.

This is extended in the obvious way to factorization systems for sinks and sources: if (E, M) is a factorization system for sinks in $\mathbf{S}$, then (E, M) is called *splitable* provided

every sink in E is jointly covering in $\mathbf{D}$ and every morphism in M is an injection in $\mathbf{D}$. Splitable factorization system for sources are define dually.

Remark 2.4.4 Every splitable factorization system (E, M) is proper, that is, $M \subset \operatorname{Mono}\mathbf{S}$ and $E \subset \operatorname{Epi}\mathbf{S}$.

Proof Suppose $m \in M$ and $m \cdot f = m \cdot g$ holds in $\mathbf{S}$. Then $f_* \approx m^* \cdot m_* \cdot f_* \approx m^* \cdot m_* \cdot g_* \approx g_*$ holds in $\mathbf{D}$, and hence $f = g$ holds in $\mathbf{S}$. ■

Examples

(1) The only splitable factorization system on $(\mathbf{C}, \operatorname{Rel}\mathbf{C})$ is $(\mathrm{RegEpi}, \mathrm{Mono})$.

(2) For any quantale $\mathbf{V}$, $(\mathrm{Epi}, \mathrm{Mono})$, $(\mathrm{Epi}, \mathrm{Monosource})$, and $(\mathrm{Episink}, \mathrm{Mono})$ are splitable factorization systems on $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$.

(3) $(\mathrm{Epi}, \mathrm{InitMono})$ is a splitable factorization system on $(\mathbf{Ord}, \mathbf{Dist})$.

Complete and topological scenes

Definition 2.4.5 We say that a scene $(\mathbf{S}, \mathbf{C}, \mathbf{D})$ is *(finitely) complete* if $\mathbf{S}$ is (finitely) complete, and $\mathbf{D}$ is locally (finitely) complete.
(Finitely) cocomplete scenes are defined dually.

Definition 2.4.6 A scene $(\mathbf{S}, \mathbf{C}, \mathbf{D})$ is called *topological* provided $\mathbf{D}$ is locally (possibly large) complete. That is, each $\mathbf{D}(A, B)$ has all (large) infima and suprema.

For example, $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ as well as $(\mathbf{Set}, \mathbf{Ord}, \operatorname{OrdMat}(\mathbf{V}))$ are complete as well as topological.

3 Lax algebras

3.1 Basics

Definition 3.1.1 Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ be a scene and $T : \mathcal{S} \longrightarrow \mathcal{S}$ an actor. A *lax algebra* for T is a pair (X, a), where X is an $\mathbf{S}$-object and $a : TX \nrightarrow X$ is a $\mathbf{D}$-morphism. A *morphism* of lax algebras $(X, a) \xrightarrow{f} (Y, b)$ is given by an $\mathbf{S}$-morphism $X \xrightarrow{f} Y$ such that $f \cdot a \leq b \cdot Tf$ holds.

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \leq & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3.1}$$

This leads to the category $\mathbf{Alg}(T, \mathcal{S})$ with composition and identities induced by the ones of $\mathbf{S}$. We have the obvious forgetful functor

$$\mathbf{Alg}(T, \mathcal{S}) \xrightarrow{|\,|} \mathbf{S}, \qquad (X, a) \longmapsto X, \qquad f \longmapsto f.$$

Often we will write $\mathbf{Alg}\, T$ instead of $\mathbf{Alg}(T, \mathcal{S})$.

The following lemma will be used subsequently without further reference:

Lemma 3.1.2 *The following statements are equivalent for a* $\mathbf{S}$*-morphism* $X \xrightarrow{f} Y$*:*

(1) $f : (X, a) \longrightarrow (Y, b)$ *is a lax morphism, that is,* $f \cdot a \leq b \cdot Tf$*;*

(2) $a \leq f^* \cdot b \cdot Tf$*;*

(3) $a \cdot Tf^* \leq f^* \cdot b$*;*

(4) $f \cdot a \cdot Tf^* \leq b$*.*

Proof Just apply the adjunctions $f \cdot (\,) \dashv f^* \cdot (\,)$ and $(\,) \cdot Tf^* \dashv (\,) \cdot Tf$. ∎

Proposition 3.1.3 *Let* (Y, b) *be a lax* T*-algebra. Every* $\mathbf{S}$*-morphism* $X \xrightarrow{f} Y$ *has an initial (or cartesian) lift, given by*

$$(X, f^* \cdot b \cdot Tf) \xrightarrow{f} (Y, b),$$

and every $\mathbf{S}$*-morphism* $Y \xrightarrow{g} Z$ *has a final (or opcartesian) lift, given by*

$$(Y, b) \xrightarrow{g} (Z, g \cdot b \cdot Tg^*).$$

Hence $|\ | : \mathbf{Alg}\, T \longrightarrow \mathbf{S}$ *is a fibration as well as an opfibration.*

Proof Let a lax algebra (W, d) and a $\mathbf{S}$-morphism $w : W \longrightarrow X$ be given such that $(W, d) \xrightarrow{f \cdot w} (Y, b)$ is a morphism. We obtain

$$w \cdot d \leq f^* \cdot f \cdot w \cdot d \leq f^* \cdot b \cdot T(f \cdot w) \approx f^* \cdot b \cdot Tf \cdot Tw,$$

so $(W, d) \xrightarrow{w} (X, f^* \cdot b \cdot Tf)$ is a morphism.

The claim concerning final morphisms is proved analogously. ∎

3.1.1 First examples

We will introduce some basic examples of categories of lax algebras. Later on we will develop two methods to construct more interesting examples: the use of more evolved scenes and the use of extensions of monads instead of mere endofunctors.

Graphs Consider the (discrete) scene $(\mathbf{Set}, \mathbf{Rel})$ together with the identity-actor 1. An algebra for 1 is given by a set X together with a relation r on X. A morphism $(X, r) \xrightarrow{f} (Y, s)$ is given by a function $f : X \longrightarrow Y$ which preserves the relation:

$$x\, r\, x' \implies f(x)\, s\, f(x')$$

for all $x, x' \in X$. Hence $\mathbf{Alg}1$ is just the category $\mathbf{Bin}$ of *binary endorelations* or *thin graphs*. If we consider instead an elementary topos $\mathbf{E}$, and the scene $(\mathbf{E}, \mathrm{Rel}\, \mathbf{E})$, we obtain the category of internal binary relations in $\mathbf{E}$ instead.

T-graphs Let $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ be a functor. By Proposition 1.5.3, T extends to an actor on $(\mathbf{Set}, \mathbf{Rel})$. $\mathbf{Alg}\, T$ has as objects sets X equipped with a relation $TX \xrightarrow{a} X$, and as morphisms $(X, a) \xrightarrow{f} (X, b)$ functions $f : X \longrightarrow Y$ such that

$$\mathfrak{x}\, a\, x \implies Tf(\mathfrak{x})\, b\, f(x)$$

for all $\mathfrak{x} \in TX$, $x \in X$.

Consider for example a non-empty set L (of *labels*) and the functor $T = L \times (\ )$: $\mathbf{Set} \longrightarrow \mathbf{Set}$. Lax algebras for T are *labelled transition systems* with labels from L. Note however that the morphisms of $\mathbf{Alg}\, T$ are not the ones usually chosen for the category of labelled transition systems.

Fuzzy sets Let $\mathbf{V}$ be a quantale. Consider the scene $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ and the actor T which is given by the constant functor with value $\{*\}$. Algebras for T are pairs $(X, X \xrightarrow{\phi} V)$ where X is a set and ϕ is an arbitrary function. A morphism $(X,a) \xrightarrow{f} (Y,b)$ must satisfy

$$a(x) \leq b(f(x))$$

for all $x \in X$. Hence $\mathbf{Alg}\,T$ is the category $\mathbf{Fuz}(V)$ of *V-valued fuzzy sets*; see [Wyl91].

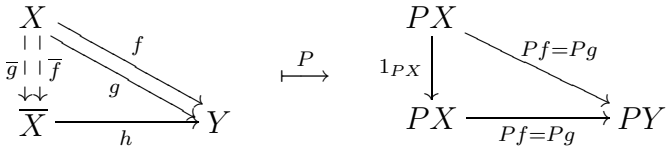
3.2 Faithful fibrations

Although fibrations play a prominent role in present-day category theory, there seems to be no general account on *faithful fibrations*. The results in this section are mostly adaptions of the ones found in [Gro71], [Gra67], [Man67], or [Str03].

Let $P : \mathbf{X} \longrightarrow \mathbf{B}$ be a functor and f a $\mathbf{B}$-morphism. We say $\alpha \in \operatorname{mor}\mathbf{X}$ is *over* f if $P\alpha = f$. For a $\mathbf{B}$-object B, let P_B denote the *fiber* of B, that is, the subcategory of $\mathbf{X}$ consisting of all morphisms over 1_B.

Proposition 3.2.1 *A fibration $P : \mathbf{X} \longrightarrow \mathbf{B}$ is faithful if and only if P_B is thin for all $B \in \mathbf{B}$.*

Proof Clearly P_B is thin provided P is faithful. To show the converse, suppose $Pf = Pg$ holds for $f, g : X \longrightarrow Y$ in $\mathbf{X}$. Let $\overline{X} \xrightarrow{h} Y$ be an initial lift of $(Pf, Y) = (Pg, Y)$. By initiality of h, we get morphisms $\overline{f}, \overline{g} : X \longrightarrow \overline{X}$ over 1_{PX} with $h \cdot \overline{f} = f$ and $h \cdot \overline{g} = g$.



But $\overline{f} = \overline{g}$ since P_{PX} is thin, and hence $f = g$. ∎

Given a faithful functor P, the order on P_B is given by

$$X \leq X' \iff X \xrightarrow{1_B} X' \text{ is a } \mathbf{X}\text{-morphism.}$$

Definition 3.2.2 ([Man67]) A functor $F : \mathbf{X} \longrightarrow \mathbf{B}$ is called a *fibering* if it is faithful and both a fibration and an opfibration.

By Proposition 3.1.3, the forgetful functor $\mathbf{Alg}\,T \longrightarrow \mathbf{S}$ is always a fibering. We will note some basic properties of fiberings. Proofs can be found, for instance, in [Str03].

Theorem 3.2.3 (1) *$P : \mathbf{X} \longrightarrow \mathbf{B}$ is a fibering if and only if $P^{\mathrm{op}} : \mathbf{X}^{\mathrm{op}} \longrightarrow \mathbf{B}^{\mathrm{op}}$ is a fibering, thus the concept of fibering is self-dual.*

(2) *Every isomorphism is a fibering.*

(3) *Fiberings are closed under composition.*

(4) *Fiberings are stable under pullback.*

(5) *If* $P : \mathbf{X} \longrightarrow \mathbf{B}$ *is a fibering and* $X \in \operatorname{obj}\mathbf{X}$*, then the obvious functor* $P' : \mathbf{X}/X \longrightarrow \mathbf{B}/PX$ *which makes*

$$\begin{array}{ccc} \mathbf{X}/X & \xrightarrow{\partial_0} & \mathbf{X} \\ {\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbf{B}/PX & \xrightarrow[\partial_0]{} & \mathbf{B} \end{array}$$

commute is a fibering. ■

Suppose $P : \mathbf{X} \longrightarrow \mathbf{B}$ is a fibering, and let $(A \xrightarrow{f} B) \in \mathbf{B}$. We obtain functions $f_\diamond : P_A \longrightarrow P_B$ and $f^\diamond : P_B \longrightarrow P_A$ as follows: choose a final of (X, f) for every $X \in P_A$; this defines $f_\diamond$. Analogously, define $f^\diamond$ by choosing, for each $Y \in P_B$, an initial lift of (f, Y).

The use of the axiom of choice in the definition of $f_\diamond$ and $f^\diamond$ can be avoided by using fiberings equipped with a *cleavage* and an *opcleavage*. For instance, if P is the forgetful functor of a category of lax algebras, we can define $f_\diamond$ and $f^\diamond$ choice-free using the construction of Proposition 3.1.3. Choice is not needed in case P is amnestic.

Lemma 3.2.4 *The following statements hold:*

(1) $f_\diamond \dashv f^\diamond$*; in particular,* $f_\diamond$ *and* $f^\diamond$ *are monotone;*

(2) $g_\diamond \cdot f_\diamond \approx (g \cdot f)_\diamond$ *and* $f^\diamond \cdot g^\diamond \approx (g \cdot f)^\diamond$*;*

(3) $(1_A)_\diamond \approx 1_{P_A} \approx (1_A)^\diamond$.

Proof To show (1), observe that for all $(A \xrightarrow{f} B) \in \mathbf{B}$, $X \in P_A$, $Y \in P_B$ we have:

$$\begin{aligned} f_\diamond X \le Y &\iff f_\diamond X \xrightarrow{1_B} Y \text{ is a } \mathbf{X}\text{-morphism} \\ &\iff X \xrightarrow{f} Y \text{ is a } \mathbf{X}\text{-morphism (by finality)} \\ &\iff X \xrightarrow{1_A} f^\diamond Y \text{ is a } \mathbf{X}\text{-morphism (by initiality)} \\ &\iff X \le f^\diamond Y. \end{aligned}$$

(2) and (3) follow from the fact that initial (final) morphisms compose and that initial (final) lifts are essentially unique. ■

Proposition 3.2.5 *Suppose* $P : \mathbf{X} \longrightarrow \mathbf{B}$ *is a fibering,* $(A \xrightarrow{f_i} PX_i)_I$ *is a source in* $\mathbf{B}$*. The following statements are equivalent:*

(1) $D = \bigwedge_I f_i^\diamond X_i$ *exists in* P_A*;*

(2) *there exists an initial lift* L *of the source* $(f_i)_I$*, that is,* $L \in P_A$*,* $L \xrightarrow{f_i} X_i$ *in* $\mathbf{X}$ *for all* $i \in I$*, and given* $PW \xrightarrow{g} PL$ *in* $\mathbf{B}$ *such that* $W \xrightarrow{f_i \cdot g} X_i$ *for all* $i \in I$*, then* $W \xrightarrow{g} L$ *in* $\mathbf{X}$*.*

In this case, $L \approx D$ *holds in* P_A*.*

Proof (1) implies (2): From $W \xrightarrow{f_i \cdot g} X_i$ in $\mathbf{X}$, we obtain $W \xrightarrow{g} f_i^\diamond X_i$ by initiality, which is equivalent to $W \xrightarrow{g} g_\diamond W \xrightarrow{1_A} f_i^\diamond X$, that is, $g_\diamond W \leq f_i^\diamond X$. Since this holds for all $i \in I$, we obtain $g_\diamond W \leq \bigwedge_I f_i^\diamond X_i = D$. This in turn implies that $W \xrightarrow{g} g_\diamond W \xrightarrow{1_A} D$ are $\mathbf{X}$-morphisms, and hence $W \xrightarrow{g} D$ in $\mathbf{X}$.

(2) implies (1): We need to show that L is an infimum of $(f_i^\diamond X_i)_I$. Clearly, we have $L \leq f_i^\diamond X_i$ for all $i \in I$. Given any other lower bound K, we obtain $K \leq L$ by initiality. ■

Corollary 3.2.6 *The following statements are equivalent for* $P : \mathbf{X} \longrightarrow \mathbf{B}$*:*

(1) P *is topological;*

(2) P *is a fibering and* P_B *is a (large-)complete lattice for all* $B \in \mathbf{B}$*.* ■

Corollary 3.2.7 *If* $\mathcal{S}$ *is a topological scene, then* $\mathbf{Alg}\, T$ *is topological for every actor* T *on* $\mathcal{S}$*.* ■

Proposition 3.2.8 *The following statements are equivalent for a faithful fibration* $\mathbf{X} \xrightarrow{P} \mathbf{B}$*:*

(1) *each* P_B *has a least element* $\bot_B$*;*

(2) P *has a left adjoint* D*.*

In case any of the above holds, we have $D(B) = \bot_B$*.* ■

Limits and colimits in fiberings

Proposition 3.2.9 *Let* $\mathbf{X} \xrightarrow{P} \mathbf{B}$ *be a fibering and* $\mathbf{I} \xrightarrow{D} \mathbf{X}$ *be a diagram.* D *has a limit which is preserved by* P *if and only if the diagram* $\mathbf{I} \xrightarrow{PD} \mathbf{B}$ *has a limit* (L, l_i) *and* $X = \bigwedge_{\mathrm{obj}\,\mathbf{I}} l_i^\diamond D(i)$ *exists in* P_L*.*

Proof Follows immediately from [AHS04, Proposition 13.15] and Proposition 3.2.5. ■

The above proposition implies immediately:

Theorem 3.2.10 *If* $\mathcal{S}$ *is a (finitely) complete scene, then* $\mathbf{Alg}\, T$ *is (finitely) complete for any actor* T *on* $\mathcal{S}$*. In this case the forgetful functor* $\mathbf{Alg}\, T \xrightarrow{|\,|} \mathbf{S}$ *lifts (finite) limits and has a right adjoint. If* $\mathcal{S}$ *is complete, then* $\mathbf{Alg}\, T \xrightarrow{|\,|} \mathbf{S}$ *also has a left adjoint.* ■

3.3 Lax algebras are universal

For a fibering $P : \mathbf{X} \longrightarrow \mathbf{B}$ we will construct a scene $\mathcal{S}$ together with an actor T such that $\mathbf{Alg}\, T$ is concretely isomorphic to $\mathbf{X}$. Hence we can reduce the study of fiberings to the study of lax algebras.

3.3.1 Change of base

The category of fiberings Let **Fiber** denote the following category:

- objects are fiberings;
- given fiberings $\mathbf{X} \xrightarrow{P} \mathbf{B}$ and $\mathbf{X}' \xrightarrow{P'} \mathbf{B}'$, a morphism $P \longrightarrow P'$ is given by a pair of functors (F, G) such that

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{G} & \mathbf{X}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathbf{B} & \xrightarrow[F]{} & \mathbf{B}' \end{array}$$

  commutes.

We write $\mathbf{Fiber}(\mathbf{B})$ for the subcategory of **Fiber** given by the morphism of the form $(1_{\mathbf{B}}, G)$.

The category of actors The *category of actors* **Actor** is defined as follows: objects are pairs $(T, \mathcal{S})$ with $\mathcal{S}$ a scene and T an actor on $\mathcal{S}$; morphism $(T, \mathcal{S}) \xrightarrow{F} (T', \mathcal{S}')$ are based scenic functors $\mathcal{S} \xrightarrow{F} \mathcal{S}'$ such that

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F_1} & \mathbf{C}' \\ {\scriptstyle T}\downarrow & \approx & \downarrow{\scriptstyle T'} \\ \mathbf{C} & \xrightarrow[F_1]{} & \mathbf{C}' \end{array} \tag{3.2}$$

holds pointwise. Observe that we do *not* require the functors to be comparable at the **D**-level.

Examples Suppose we have quantales $\mathbf{V}$ and $\mathbf{W}$ together with a functor $V \xrightarrow{\phi} W$ which satisfies $\phi(k_{\mathbf{V}}) = k_{\mathbf{W}}$ and $\phi(\bot_V) = \bot_W$, and moreover a functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ such that the extensions $T_{\mathbf{V}} : \operatorname{Mat}\mathbf{V} \longrightarrow \operatorname{Mat}\mathbf{V}$ and $T_{\mathbf{W}} : \operatorname{Mat}\mathbf{W} \longrightarrow \operatorname{Mat}\mathbf{W}$ commute with the embeddings of **Set**. By Proposition 1.5.18, $\operatorname{Mat}\phi$ is a based scenic functor $(\mathbf{Set}, \operatorname{Mat}\mathbf{V}) \longrightarrow (\mathbf{Set}, \operatorname{Mat}\mathbf{W})$, and $\operatorname{Mat}\phi$ defines an **Actor**-morphism

$$\big(T_{\mathbf{V}}, (\mathbf{Set}, \operatorname{Mat}\mathbf{V})\big) \xrightarrow{(1, \operatorname{Mat}\phi)} \big(T_{\mathbf{W}}, (\mathbf{Set}, \operatorname{Mat}\mathbf{W})\big).$$

Suppose we have regular categories $\mathbf{C}$ and $\mathbf{C}'$ with functors $\mathbf{C} \xrightarrow{T} \mathbf{C}$ and $\mathbf{C}' \xrightarrow{T'} \mathbf{C}'$. A functor $\mathbf{C} \xrightarrow{F} \mathbf{C}'$ defines a morphism $(\mathrm{Rel}\, T, (\mathbf{C}, \mathrm{Rel}\, \mathbf{C})) \longrightarrow (\mathrm{Rel}\, T', (\mathbf{C}', \mathrm{Rel}\, \mathbf{C}'))$ provided F preserves regular epimorphisms and $F \cdot T = T' \cdot F$ holds.

Turning Alg into a functor Suppose we are given $(T, \mathcal{S}) \xrightarrow{F} (T', \mathcal{S}')$ in **Actor**, $F = (F_0, F_1, F_2)$. We define

$$\mathbf{Alg}\, F : \mathbf{Alg}(T, \mathcal{S}) \longrightarrow \mathbf{Alg}(T', \mathcal{S}'), \qquad (X, a) \longmapsto (F_0 X, F_2 a), \qquad f \longmapsto F_0 f.$$

It is easy to check that $(\mathbf{Alg}\, F, F_0)$ is a morphism in **Fiber**, which we also denote by $\mathbf{Alg}\, F$, and we obtain:

Proposition 3.3.1 $\mathbf{Alg} : \mathbf{Actor} \longrightarrow \mathbf{Fiber}$ *is a functor.* ∎

3.3.2 The construction

Let $P : \mathbf{X} \longrightarrow \mathbf{B}$ be a faithful fibering. We will construct a scene

$$\mathbf{B} \xrightarrow{E} \mathbf{C} \underset{(\,)^*}{\overset{(\,)_*}{\rightrightarrows}} \mathbf{D}$$

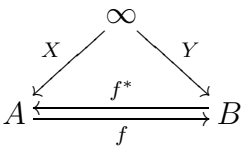
Constructing C Let $\mathbf{T}$ denote a terminal category and let $\mathbf{B} \xrightarrow{E} \mathbf{C} \xleftarrow{E'} \mathbf{T}$ be a coproduct. We identify $\mathbf{B}$-objects and morphisms with their images under E and write ∞ for the $\mathbf{C}$-object in the image of E'. Let $T : \mathbf{C} \longrightarrow \mathbf{C}$ be the constant functor with value ∞.

Constructing D: The construction of $\mathbf{D}$ proceeds in the following 3 steps:

Step 1: Construct a graph $\mathbf{G}$ as follows:

$$\begin{aligned} \mathrm{obj}\, \mathbf{G} &= \mathrm{obj}\, \mathbf{C} \\ \mathbf{G}(A, B) &= \mathbf{C}(A, B) + \{\, f^* \mid f \in \mathbf{C}(B, A) \,\} \\ \mathbf{G}(\infty, A) &= P_A \end{aligned}$$

for all $A, B \in \mathrm{obj}\, \mathbf{B}$; all other hom-sets are empty. So $\mathbf{G}$ is formed out of $\mathbf{C}$ by first adding a "formal right adjoint" f^* for each $\mathbf{C}$-morphism f, and then "suspending" the elements of the fibers P_A. Hence we may picture a part of $\mathbf{G}$ as



Step 2: Let $\mathbf{D}_0$ be the free category on $\mathbf{G}$, subject to the following commutativity conditions:

(C1) $A \xrightarrow{f} B \xrightarrow{g} C \equiv A \xrightarrow{g \cdot f} C$ for all $f, g \in \mathbf{C}$;

(C2) $\infty \xrightarrow{X} A \xrightarrow{f} B \equiv \infty \xrightarrow{f \diamond X} B$ for each $f \in \mathbf{C}$ and $X \in P_A$;

(C3) $\infty \xrightarrow{Y} B \xrightarrow{f^*} A \equiv \infty \xrightarrow{f^\diamond Y} A$ for each $f \in \mathbf{C}$ and $Y \in P_B$.

The purpose of the conditions of type (C2) and (C3) is to force $\mathbf{D}_0(\infty, A)$ and P_A to be isomorphic as sets.

Step 3: We define a locally related category $\mathbf{D}_1$ by equipping $\mathbf{D}_0$ with the following relations

(O1) For each $A \in \operatorname{obj} \mathbf{B}$, $\mathbf{D}_0(\infty, A)$ is ordered like P_A.

(O2) The following hold for each $A \xrightarrow{f} B \in \mathbf{B}$: $(1_A, f^* \cdot f)$, $(f \cdot f^*, 1_B)$.

Let $\mathbf{D}$ be the free ordered category on $\mathbf{D}_1$. By (O2), the formal right adjoints f^* become right adjoints in $\mathbf{D}$.

Clearly $\mathbf{D}(\infty, A)$ is order-isomorphic to P_A for all $\mathbf{B}$-objects A. We have the obvious assignments $\mathbf{C} \underset{(\ )^*}{\overset{(\ )_*}{\rightrightarrows}} \mathbf{D}$, which satisfy $u_* \dashv u^*$ for all $u \in \mathbf{C}$ by virtue of (O2). Moreover, by (C1), $(\ )_*$ is a functor, which, by essential uniqueness of adjoints, forces $(\ )^*$ to be a pseudo-functor. Hence

$$\mathbf{B} \xrightarrow{E} \mathbf{C} \underset{(\ )^*}{\overset{(\ )_*}{\rightrightarrows}} \mathbf{D}.$$

is a scene. We extent T along $(\ )_*$ to a 2-graph morphism $\mathbf{D} \longrightarrow \mathbf{D}$ in the obvious way, and in this way T is an actor on $(\mathbf{B}, \mathbf{C}, \mathbf{D})$.

Definition 3.3.2 The scene $(\mathbf{B}, \mathbf{C}, \mathbf{D})$ together with T constructed above from the fibering $\mathbf{X} \xrightarrow{P} \mathbf{B}$ forms an object in **Actor** which we denote by $\operatorname{Act}(\mathbf{X}, P)$.

Theorem 3.3.3 *For each fibering $(\mathbf{X}, P)$ the concrete functor*

$$(\mathbf{X}, P) \longrightarrow \mathbf{Alg}\operatorname{Act}(\mathbf{X}, P)$$

defined by $X \longmapsto (P(X), X)$ is a concrete isomorphism.

Proof Clearly $(\mathbf{X}, P)$ and $\mathbf{Alg}\mathrm{Act}(\mathbf{X}, P)$ have isomorphic fibers. So we only need to show that a $\mathbf{B}$-morphism $f : A \longrightarrow B$ is a morphism $X \longrightarrow Y$ in $\mathbf{X}$ (where $X \in P_A$ and $Y \in P_B$) if and only if it is a morphism $(A, X) \longrightarrow (B, Y)$ in $\mathbf{Alg}\mathrm{Act}(\mathbf{X}, P)$.

$$\begin{aligned} X \xrightarrow{f} Y \text{ in } \mathbf{X} &\iff f_\diamond X \xrightarrow{1_B} Y \text{ in } \mathbf{X} \\ &\iff f_\diamond X \leq Y \text{ in } P_B \\ &\iff f \cdot X \leq Y \cdot Tf \text{ in } \mathbf{D}(TB, B), \text{ since } f_\diamond X = f \cdot X \text{ by (C2)} \\ &\iff (A, X) \xrightarrow{f} (B, Y) \text{ in } \mathbf{Alg}\mathrm{Act}(\mathbf{X}, P). \end{aligned}$$

■

3.3.3 Conclusions

Size issues Of course, the category $\mathbf{D}$ constructed above will in general not be locally small. We do not run into problems here. One way of avoiding them is to use a foundation like Grothendieck-universes. But even in a more restricted metatheory this fact does not hinder us in any practical way: as the reader will notice, we never need to consider hom-"sets" of the form $\mathbf{D}(A, B)$ with $A, B \in \mathbf{B}$, but only those of the form $\mathbf{D}(TA, B)$ or $\mathbf{C}(A, B)$. In the scene constructed above, hom-sets of the form $\mathbf{D}(TA, B)$ are small as long as $(\mathbf{X}, P)$ is fiber-small, and $\mathbf{C}$ is locally small as long as $\mathbf{B}$ is so.

The fact that we are not hindered in any practical way becomes more transparent if one takes a look at a typical argument involving fibrations: only the fibers and morphisms in the base-category (or the induced change-of-base functors) play a role.

So our construction does not make things worse, for practical purposes.

On the role of the representation theorem Although the possibility to represent every fibering as a category of lax algebras is surely interesting, it does not really help us in the study of fiberings. What the representation theorem really tells us is that we cannot prove anything about lax algebras for a scene which we cannot prove for fiberings.

One of the reasons why it is hard to prove any facts about, say, topological spaces, using the construction outlined above is that the constructed scene is quite hard to work with.

One possible solution is to study a relaxed version of Eilenberg–Moore algebras for a monad instead of "naked" lax algebras. This is the context of most of the literature on lax algebras so far. It allows us to choose a scene with "nice" properties and a monad on it to study fiberings.

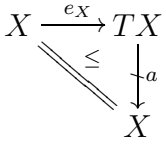
3.4 Lax algebras for a monad

We introduce reflexive, unitary, and transitive lax algebras for a *monadic actor*.

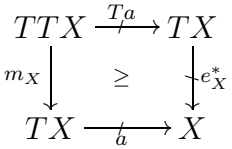
Let $\mathbf{T} = (T, e, m)$ be an actor monad on a scene $\mathcal{S}$.

Definition 3.4.1 We call a lax T-algebra (X, a)

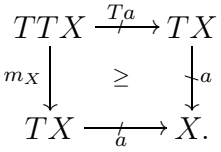
- *reflexive* provided $1_X \leq a \cdot e_X$ holds,



- *unitary* provided $e_X^* \cdot Ta \leq a \cdot m_X$ holds,



- *transitive* provided $a \cdot Ta \leq a \cdot m_X$ holds,



We write $\mathbf{Alg}_r(\mathsf{T}, \mathcal{S})$ or $\mathbf{Alg}_r\mathsf{T}$ for the full subcategory of $\mathbf{Alg}\,T$ spanned by the reflexive algebras, $\mathbf{Alg}_u(\mathsf{T}, \mathcal{S})$ or $\mathbf{Alg}_u\mathsf{T}$ for the full subcategory of $\mathbf{Alg}\,T$ spanned by the reflexive *and* unitary lax algebras, and $\mathbf{Alg}_t(\mathsf{T}, \mathcal{S})$ or $\mathbf{Alg}_t\mathsf{T}$ for the full subcategory of $\mathbf{Alg}\,T$ spanned by the reflexive *and* transitive lax algebras.

Obviously, we should call a lax algebra T-reflexive instead of reflexive since the notion of reflexivity depends on the transformation e, and similarly for unitary and transitive algebras. We will continue with this abuse of notation, assuring the reader that no danger lies ahead.

Remark 3.4.2 A more natural setting for reflexive lax algebras would be that of *pointed actors*, that is, actors T on $\mathcal{S}$ equipped with a slim transformation $1_{\mathcal{S}} \longrightarrow T$; see [CHT03b] . ■

Lemma 3.4.3 ([CH03]) (1) *(X, a) is reflexive if and only if $e_X^* \leq a$ holds, in this case $a \leq a \cdot Ta \cdot m_X^*$;*

(2) *(X, a) is transitive if and only if $a \cdot Ta \cdot m_X^* \leq a$ holds, and unitary if and only if $e_X^* \cdot Ta \cdot m_X^* \leq a$ holds;*

(3) *every reflexive and transitive lax algebra is unitary.* ■

As a corollary we obtain that each (X, e_X^*) is reflexive, unitary, and transitive. Thus we have full embeddings

$$\mathbf{Alg}_t\mathsf{T} \hookrightarrow \mathbf{Alg}_u\mathsf{T} \hookrightarrow \mathbf{Alg}_r\mathsf{T} \hookrightarrow \mathbf{Alg}\,T \tag{3.3}$$

We do not have any representation theorem for the categories of reflexive or reflexive and transitive lax algebras for a monad. Experience for the "algebraic side" suggests that such results might be feasible, at least if we restrict ourselves to base categories with special properties.

Indeed, Eilenberg–Moore algebras for a monad are more general than algebras for a mere endofunctor T, at least over a sufficiently well-behaved base category: we can form the free monad on T [BW00], Eilenberg-Moore algebras for this monad correspond precisely to algebras for T.

We will first present some well-known examples of this string of embeddings, and then study it in more detail.

3.4.1 Topological spaces and approach spaces: the motivating example

Barr's original presentation [Bar70] of topological spaces as lax algebras may be viewed as the starting point for the study of lax algebras.

Barr's Theorem
Let $\mathsf{U} = (U, e, m)$ be the ultrafilter-monad on **Set**; see Section 2.3.1. Manes showed in [Man69] that the Eilenberg–Moore algebras for U are precisely the compact Hausdorff spaces. We can extent U to an actor on the discrete scene $(\mathbf{Set}, \mathbf{Rel})$, which we will also denote by U. In fact, U extends to a 2-functor: to see this, we only need (assuming the Axiom of Choice to hold) to show that U nearly preserves pullbacks. This is shown in [CHT03a, 2.9]. For a relation r, Ur can be described by

$$\mathfrak{x}\,(Ur)\,\mathfrak{y} \iff \forall A \in \mathfrak{x}\,\forall B \in \mathfrak{y}\,\exists x \in A\,\exists y \in B : x\,r\,y. \tag{3.4}$$

The unit e extends oplax and m extends even natural transformation.

Barr's Theorem (1970) $\mathbf{Alg}_t\mathsf{U}$ *is concretely isomorphic to the category* **Top** *of topological spaces and continuous maps.* ∎

For a proof see Barr's original paper [Bar70], [Pis99], the very readable [Wyl95], or [HT06]. Moreover, we have:

Theorem 3.4.4

(1) $\mathbf{Alg}_r\mathsf{U}$ *is concretely isomorphic to the category* **PsTop** *of pseudotopological spaces.*

(2) $\mathbf{Alg}_u\mathsf{U}$ *is concretely isomorphic to the category* **PrTop** *of pretopological spaces.* ∎

Statement (1) of the above theorem was first observed by Wyler [Wyl95], while statement (2) is due to Hofmann [Hof05]. For information on **PsTop** and **PrTop** see [HLCS91].

Trivially, $\mathbf{Alg}\,\mathsf{U}$ is just the category **URS** of *ultrarelational* or *grizzly spaces*, as used in [CHT03a]. Thus we have exhibited diagram (3.3) for $\mathsf{T} = \mathsf{U}$ as

$$\mathbf{Top} \hookrightarrow \mathbf{PrTop} \hookrightarrow \mathbf{PsTop} \hookrightarrow \mathbf{URS}.$$

Observe that the categories on the right side have better exactness properties, while the categories on the left side are what a general topologist might consider more interesting. The fact that **Top** embeds fully into **PsTop** (a topological quasitopos; see [AHS04], [Wyl91]) and **URS** (even a topological universe; see [AHS04]), was exploited by Clementino, Hofmann, Tholen, and others to study descent morphisms and exponentiability in **Top**. Some of these results can be generalized to lax algebras; see [CH04a], [CHT03b], and Section 3.6.2.

One of the advantages of representing categories as categories of reflexive and transitive lax algebras (or reflexive and unitary or plainly reflexive lax algebras) is that these come with a canonical embedding into an "ultra-convenient" category, namely the corresponding category of ("naked") lax algebras.

Hence we have the following strategy for dealing with a problem in one of the categories "at the left side" of diagram (3.3):

(α) "Go the right" until the solution of the problem becomes easy by "convenience" of the super-category,

(β) study the corresponding embedding to pull the solution back to the category we have been starting with.

In the following sections we will study the embeddings in (3.3) in detail, showing that under mild completeness conditions on the scene they are reflective. In Chapter 6 we will use this approach to study exactness properties of categories of lax algebras.

First let us return to the ultrafilter monad.

Of course Barr's Theorem requires some form of the axiom of choice. Later on we will present topological spaces as lax algebras using the convergence of filters instead of ultrafilter convergence. This allows us to avoid the axiom of choice.

Approach spaces

Using the results of Section 2.3.3, we cannot only extend the ultrafilter monad to an actor on $(\mathbf{Set}, \mathbf{Rel})$, but to *any* discrete scene of the form $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ for a quantale $\mathbf{V}$. We will write $\mathbf{U}_{\mathbf{V}}$ for this extension.

In case $\mathbf{V} = \overline{\mathbb{R}}_+$ (see Example 1.5.8(3)) we obtain:

Theorem 3.4.5 *For the actor* $\mathbf{U}_{\overline{\mathbb{R}}_+}$ *on the scene* $(\mathbf{Set}, \mathrm{Mat}\,\overline{\mathbb{R}}_+)$ *we obtain:*

(1) $\mathbf{Alg}_r\mathbf{U}_{\overline{\mathbb{R}}_+}$ *is concretely isomorphic to the category* **PsAp** *of pseudo-approach spaces.*

(2) $\mathbf{Alg}_u\mathsf{U}_{\overline{\mathbb{R}}_+}$ *is concretely isomorphic to the category* $\mathbf{PrAp}$ *of pre-approach spaces.*

(3) $\mathbf{Alg}_t\mathsf{U}_{\overline{\mathbb{R}}_+}$ *is concretely isomorphic to the category* $\mathbf{Ap}$ *of approach spaces.* ■

Hence diagram (3.3) specializes to

$$\mathbf{Ap} \hookrightarrow \mathbf{PrAp} \hookrightarrow \mathbf{PsAp} \hookrightarrow \mathbf{Alg}\mathsf{U}_{\overline{\mathbb{R}}_+}.$$

For information on approach spaces see [Low97]. We wish to point out that the description of approach spaces as reflexive and transitive lax algebras leads to a very concise definition of $\mathbf{Ap}$. Statements (1) and (3) have been proved in [CH03]. Observe that the extension of the ultrafilter-monad used by Clementino and Hofmann coincides with the one constructed in this work.

3.4.2 Further examples

Categories

Let $\mathcal{S}$ be any scene and $\mathbf{1}$ be the identity-actor on $\mathcal{S}$. Then $\mathbf{Alg}_t\mathbf{1}$ is just $\operatorname{Cat} D(\mathcal{S})$, where $D(\mathcal{S})$ is the discrete scene associated with $\mathcal{S}$. This is the precise sense in which the study of lax algebras encompasses the study of $\mathfrak{O}$-categories.

More generally we have:

Remark 3.4.6 If T is a based scenic actor on a scene $\mathcal{S}$, then

$$\mathbf{Alg}(T, \mathcal{S}) \cong \mathbf{Alg}(D(T), D(\mathcal{S}))$$

where $D(\mathcal{S})$ is the discrete scene associated with $\mathcal{S}$. ■

Observe that every lax $\mathbf{1}$-algebra is unitary.

For example, our string of full embeddings (3.3) can be described for $\mathsf{T} = \mathbf{1}_{(\mathbf{Set},\mathbf{Rel})}$ as

$$\mathbf{Ord} \hookrightarrow \mathbf{RBin} = \mathbf{RBin} \hookrightarrow \mathbf{Bin}.$$

Here $\mathbf{RBin}$ is the category of reflexive thin graphs. This example was the motivation behind the names "reflexive" and "transitive".

Recall the filter-on-downsets monad F on $\mathbf{Ord}$ and the corresponding Kleisli-prescene $\operatorname{Klei}\mathsf{F}$. Here, for $\mathsf{T} = \mathbf{1}$ (3.3) can be described as:

$$\mathbf{Top} \hookrightarrow \mathbf{PrTop} = \mathbf{PrTop} \hookrightarrow \mathbf{Alg}(\mathbf{1}, \operatorname{Klei}\mathsf{F}).$$

Indeed, reflexivity of a lax algebra (X, a) just means that the function $X \xrightarrow{a} FX$ satisfies $e_X(x) \leq a(x)$ pointwise, where $e_X(x)$ is the principal ultrafilter of x.

Closure spaces

Let $\mathbb{U}$ be the upperset monad on **Ord**. In Section 2.3.5, we have constructed an extension $\tilde{\mathbb{U}}$ of $\mathbb{U}$ to an actor on $(\mathbf{Set}, \mathbf{Ord}, \mathbf{Dist})$. A lax $\tilde{\mathbb{U}}$-algebra is just a pair (X, α), X a set, $\alpha : PX \nrightarrow X$ a distributor; associating to each distributor $a : PX \nrightarrow X$ the monotone function $\hat{a} : PX \longrightarrow PX$ (see Remark 1.3.6), we define a concrete isomorphism between $\mathbf{Alg}\,\tilde{\mathbb{U}}$ and the category **Mop** of *sets with monotone operators*, defined as follows:

- objects are pairs (X, c), X a set and $c : PX \longrightarrow PX$ a monotone function;
- morphisms $(X, c) \longrightarrow (Y, c')$ are functions $f : X \longrightarrow Y$ such that $f[cA] \subset c'f[A]$.

A monotone operator c on a set X is called expansive if $A \subset cA$ for all $A \in PX$ and idempotent if $ccA \subset cA$ holds for all $A \in PX$. A *closure space* is a set equipped with an expansive and idempotent monotone operator. This defines the category **Clos** of closure spaces as a full, concrete subcategory of **Mop**. We will now characterize closure spaces terms of lax $\mathbb{U}$-algebras:

Proposition 3.4.7 *A lax $\mathbb{U}$-algebra (X, a) is*

(1) *reflexive if and only if $\hat{a}$ is expansive;*

(2) *transitive if and only if $\hat{a}$ is idempotent.*

Proof (1) is immediate. (2) follows easily from the second statement of the following lemma. ∎

Lemma 3.4.8 *Given distributors $a : \mathbb{U}X \nrightarrow Y$, $b : \mathbb{U}Y \nrightarrow Z$, we have $\tilde{\mathbb{U}}a \cdot m_X^* = (\hat{a})_*$ and $\widehat{b * a} = \hat{b} \cdot \hat{a}$.*

Proof First observe that $A\,(\tilde{\mathbb{U}}a \cdot m_X^*)\,B$ holds iff there exists $\mathcal{A} \in \mathbb{U}\mathbb{U}X$ with $\bigcup \mathcal{A} \subset A$ and $\mathcal{A}\,\tilde{\mathbb{U}}a\,B$. This in turn implies that for all $y \in B$, there exists $A' \in \mathcal{A}$ with $A'\,a\,y$, thus $A\,a\,y$ since a is a distributor and hence $B \subset \hat{a}(A)$.

On the other hand, if $B \subset \hat{a}(A)$ holds, we can set $\mathcal{A} = \{\, A' \in \mathbb{U}X \mid A \supset A' \,\} = (A)^{\uparrow}_{\mathbb{U}X}$, and obtain $\mathcal{A}\,\tilde{\mathbb{U}}a\,B$ by construction.

To prove the second claim, observe that, by the first statement, $z \in \widehat{b * a}(A)$ holds iff there exists $B \in \mathbb{U}Y$ with $B\,b\,z$ and $B \subset \hat{a}(A)$, iff $z \in \hat{b} \cdot \hat{a}(A)$ since b is a distributor. ∎

Corollary 3.4.9 $\mathbf{Alg}_t\mathbb{U}$ *is concretely isomorphic to* **Clos**. ∎

Remark 3.4.10 We wish to point out that the above proof as well as the construction of the extension of $\mathbb{U}$ to **Dist** is constructive. This implies that the results hold in any topos, and that we can describe *internal closure spaces* using lax algebras. ∎

Every lax $\mathbb{U}$-algebra is unitary. This fact follows from the following remark and the observation made in Section 2.3.5.

Remark 3.4.11 If $(e_X^*)_X$ forms an oplax transformation $T \longrightarrow 1_{\mathbf{D}}$, then every lax T-algebra is unitary.

Proof Oplaxness of (e_X^*) implies $e_X^* \cdot Ta \leq a \cdot e_{TX}^*$ and thus $e_X^* \cdot Ta \cdot m_X^* \leq a \cdot e_{TX}^* \cdot m_X^* \approx a \cdot (m_X \cdot e_{TX})^* \approx a$ for each $TX \overset{a}{\nrightarrow} X$. Thus any (X, a) is unitary. ■

3.4.3 The embeddings $\mathbf{Alg}_*\mathsf{T} \longrightarrow \mathbf{Alg}\, T$

As before, we will write

$$f^\diamond(b) = f^* \cdot b \cdot Tf, \qquad f_\diamond(a) = f \cdot a \cdot Tf^*$$

for $(X \xrightarrow{f} Y) \in \mathbf{S}$, $a \in \mathbf{D}(TX, X)$, $b \in \mathbf{D}(TY, Y)$.

Lemma 3.4.12 *If (Y, b) is reflexive (or unitary, transitive), then also $(X, f^\diamond(b))$ is reflexive (or unitary, transitive).*

Proof We will prove the claim concerning transitivity. The other claims are proved analogously:

$$\begin{aligned} f^* \cdot b \cdot Tf \cdot T(f^* \cdot b \cdot Tf) &\leq f^* \cdot b \cdot T(f \cdot f^* \cdot b \cdot Tf) \\ &\leq f^* \cdot b \cdot T(b \cdot Tf) \\ &\leq f^* \cdot b \cdot Tb \cdot TTf \\ &\leq f^* \cdot b \cdot m_Y \cdot TTf \\ &\approx f^* \cdot b \cdot Tf \cdot m_X \end{aligned}$$

■

Given an ordered set O and a monotone function $f : O \longrightarrow O$, we call $x \in O$ a *post-fixpoint* of f if $f(x) \leq x$ holds. Clearly, the set of post-fixpoints is closed under any infima which exist in O.

Define

$$\begin{aligned} \mathbf{D}(TX, X) &\xrightarrow{\kappa} \mathbf{D}(TX, X) & \qquad \mathbf{D}(TX, X) &\xrightarrow{\lambda} \mathbf{D}(TX, X) \\ a &\longmapsto a \cdot Ta \cdot m_X^* & \qquad a &\longmapsto e_X^* \cdot Ta \cdot m_X^* \end{aligned}$$

Clearly κ and λ are monotone. We may rephrase Lemma 3.4.3 by saying that (X, a) is transitive if and only if a is a post-fixpoint of κ, and a reflexive lax algebra (X, a) is transitive if and only if a is a fixpoint of κ. (X, a) is unitary if and only if it is a post-fixpoint of λ. Hence we have:

Lemma 3.4.13 *The sets of reflexive, of unitary, of reflexive and unitary, of transitive, and of reflexive and transitive structures are closed under any infima which exist in $\mathbf{D}(TX, X)$.* ■

Combining Lemma 3.4.13, Lemma 3.4.12, and Proposition 3.2.5, we obtain:

Proposition 3.4.14 *Each of the full embeddings below is closed under intial sources:*

$$\mathbf{Alg}_t\mathsf{T} \longhookrightarrow \mathbf{Alg}_u\mathsf{T} \longhookrightarrow \mathbf{Alg}_r\mathsf{T} \longhookrightarrow \mathbf{Alg}\,T. \qquad \blacksquare$$

Thus under suitable completeness conditions (for instance if $\mathcal{S}$ is topological) these embeddings will be reflective. This was described by Clementino and Hofmann [CH03]. We will present weaker conditions for the results to hold.

The embedding $\mathbf{Alg}_r\mathsf{T} \longhookrightarrow \mathbf{Alg}\,T$

Proposition 3.4.15 *The reflexive reflection of a lax algebra (X, a) is given by*

$$(X, a) \xrightarrow{1_X} (X, a \vee e_X^*),$$

provided the supremum exists. Hence $\mathbf{Alg}_r\mathsf{T} \longhookrightarrow \mathbf{Alg}\,T$ is reflective if and only if $(\ ) \vee e_X^* : \mathbf{D}(TX, X) \longrightarrow \mathbf{D}(TX, X)$ exists for all $X \in \mathbf{S}$.

Proof If $(X, a) \xrightarrow{f} (Y, b)$ is a morphism with (Y, b) reflexive, then also $e_X^* \leq f^\diamond b$ holds, hence $a \vee e_X^* \leq f^\diamond b$, that is, $(X, a \vee e_X^*) \xrightarrow{f} (Y, b)$ is a morphism. $\blacksquare$

The embedding $\mathbf{Alg}_t\mathsf{T} \longhookrightarrow \mathbf{Alg}_r\mathsf{T}$ Lemma 3.4.3(1) gives $a \leq \kappa(a)$ for all a such that (X, a) is reflective. Hence, for any such a, we have a restriction

$$\kappa_a : a^\uparrow \longrightarrow a^\uparrow$$

Proposition 3.4.16 *The following statements are equivalent for $\overline{a} \in a^\uparrow$.*

(1) *$\overline{a}$ is a least fixpoint of κ_a;*

(2) *$(X, a) \xrightarrow{1_X} (X, \overline{a})$ is a transitive and reflexive reflection of (X, a).*

Hence we may, depending on the properties of the ordered set $\mathbf{D}(TX, X)$, use various ways to contruct least fixpoints.

Proof [(2) $\Rightarrow$ (1)]: Clearly $\overline{a}$ is a fixpoint. Any other fixpoint x defines a reflexive and transitive lax algebra (X, x). Since $a \leq x$, $(X, a) \xrightarrow{1_X} (X, x)$ is a morphism, hence there is a (unique) $(X, \overline{a}) \xrightarrow{h} (X, x)$ such that $h \cdot 1_X = 1_X$. Clearly $h = 1_X$ and $\overline{a} \leq x$.

[(1) $\Rightarrow$ (2)]: Clearly $(X, \overline{a})$ is reflexive and transitive, and $(X, a) \xrightarrow{1_X} (X, \overline{a})$ is a morphism. Now let $(X, a) \xrightarrow{f} (Y, b)$ be a morphism such that (Y, b) is reflexive and transitive. We have to show: $\overline{a} \leq f^\diamond(b)$.

$a \leq f^\diamond(b)$ holds and $(X, f^\diamond(b))$ is reflexive and transitive by Lemma 3.4.12, thus $f^\diamond(b)$ is a fixpoint of κ_a. This implies $\overline{a} \leq f^\diamond(b)$ since $\overline{a}$ is a least fixpoint of κ_a. $\blacksquare$

The embedding $\mathbf{Alg}_u\mathbf{T} \hookrightarrow \mathbf{Alg}_r\mathbf{T}$ If e extends oplax, then $a \leq \lambda(a)$ holds for all a. Indeed, $e_X \cdot a \leq Ta \cdot e_{TX}$ is equivalent to $a \cdot e^*_{TX} \leq e^*_X \cdot Ta$, hence $a \approx a \cdot e^*_{TX} \cdot m^*_X \leq e^*_X \cdot Ta \cdot m^*_X = \lambda(a)$. Thus λ restricts to $\lambda_a : a^\uparrow \longrightarrow a^\uparrow$ for any a.

Proposition 3.4.17 *If e extends oplax, the following statements are equivalent for $\overline{a} \in a^\uparrow$:*

(1) *$\overline{a}$ is a least fixpoint of λ_a;*

(2) *$(X, a) \xrightarrow{1_X} (X, \overline{a})$ is a unitary (and reflexive) reflection of (X, a).*

Proof Analogous to the one of Proposition 3.4.16. ∎

Under additional conditions on $\mathbf{T}$, least fixpoints of λ_a are particularly easy to construct.

Proposition 3.4.18 *If e and m extend oplax and T is right $\{ m^*_X \mid X \in \mathbf{C} \}$-strict, then $\lambda(a)$ is unitary for all a.*

Proof For any $a \in \mathbf{D}(TX, X)$, we have

$$\begin{aligned} e^*_X \cdot T(e^*_X \cdot Ta \cdot m^*_X) \cdot m^*_X &\leq e^*_X \cdot Te^*_X \cdot TTa \cdot Tm^*_X \cdot m^*_X && \text{strictness of } T \\ &\approx e^*_X \cdot Te^*_X \cdot TTa \cdot m^*_{TX} \cdot m^*_X \\ &\leq e^*_X \cdot Te^*_X \cdot m^*_X \cdot Ta \cdot m^*_X && \text{since } m \text{ extends oplax} \\ &\approx e^*_X \cdot Ta \cdot m^*_X && . \end{aligned}$$ ∎

Constructing least fixpoints

Proposition 3.4.19 *If $\mathbf{D}(TX, X)$ is a set and has suprema of (arbitrary) chains for all $X \in \mathbf{S}$, then $\mathbf{Alg}_t\mathbf{T} \hookrightarrow \mathbf{Alg}_r\mathbf{T}$ is reflective.*

Proof Let $(X, a) \in \mathbf{Alg}_r\mathbf{T}$. If suffices to construct a least fixpoint of κ_a. We define the following ascending transfinite sequence in $\mathbf{D}(TX, X)$:

$$a_0 = a, \qquad a_{\beta+1} = \kappa(a_\beta), \qquad a_\gamma = \bigvee_{\beta<\gamma} a_\beta$$

for all ordinals β and limit-ordinals γ. Since $\mathbf{D}(TX, X)$ is a set, this chain has to become stationary at some a_{β_0}, which is the least fixpoint of κ_a. ∎

The coKleisli-composition As a generalization of κ and λ above Hofmann [Hof05] defined the *coKleisli-composition*: for $a : TX \nrightarrow Y$, $b : TY \nrightarrow Z$, set

$$b * a := b \cdot Ta \cdot m^*_X. \tag{3.5}$$

This coKleisli-composition is monotone in each variable, and we have $a * e^*_X \approx a$ for $a : TX \nrightarrow X$. A lax algebra (X, a) is unitary if and only if $e^*_X * a \approx a \approx a * e^*_X$ holds — which explains the term "unitary" — and transitive if and only if $a * a \leq a$ holds.

Final structures

Proposition 3.4.20 *Let $(X_i \xrightarrow{f_i} Y)_I$ be a sink in* $\mathbf{S}$*. Consider:*

(1) *(f_i) is jointly covering;*

(2) *for every family $(X_i, a_i)_I$ of reflexive lax algebras, if $(|(X_i, a_i)| \xrightarrow{f_i} Y)_I$ has a final lift (Y, b), then (Y, b) is reflexive.*

(1) *implies* (2)*, and the converse holds if $\bigvee f_i \cdot f_i^*$ exists and e_Y is an injection.*

Proof [(1) $\Rightarrow$ (2)]: By the dual of Proposition 3.2.5, b is isomorphic to $\bigvee f_i \cdot a_i \cdot Tf_i^*$. If (f_i) is jointly covering, $\bigvee f_i \cdot f_i^*$ exists and $(\bigvee f_i \cdot f_i^*) \cdot e_Y^* \approx \bigvee f_i \cdot e_{X_i}^* \cdot Tf_i^*$ holds, thus we obtain $e_Y^* \approx (\bigvee f_i \cdot f_i^*) \cdot e_Y^* \approx \bigvee f_i \cdot e_{X_i}^* \cdot Tf_i^* \leq \bigvee f_i \cdot a_i \cdot Tf_i^*$.

[(2) $\Rightarrow$ (1)]: If $\bigvee f_i \cdot f_i^*$ exists also $\bigvee f_i \cdot f_i^* \cdot e_Y^* \approx \bigvee f_i \cdot e_{X_i}^* \cdot Tf_i^*$ exists, and is reflexive by (1). Thus $1_Y \leq (\bigvee f_i \cdot e_{X_i}^* \cdot Tf_i^*) \cdot e_Y \approx (\bigvee f_i \cdot f_i^*) \cdot e_Y^* \cdot e_Y \approx \bigvee f_i \cdot f_i^*$ holds, that is, (f_i) is jointly covering. ∎

3.4.4 Ordering morphisms

Fix a scene $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ and an actor monad $\mathbf{T} = (T, e, m)$ on $\mathcal{S}$. We will assume that e extends oplax.

Definition 3.4.21 ([CHT04]) Let (X, a) and (Y, b) be reflexive and transitive lax algebras. Given morphisms $f, g : (X, a) \longrightarrow (Y, b)$, we say $f \Rightarrow g$ if $g \leq b \cdot e_Y \cdot f$.

Proposition 3.4.22 *The above defined notion of 2-cell turns $\mathbf{Alg}_t\mathbf{T}$ into an ordered category.*

Proof Let $f, g, h : (X, a) \longrightarrow (Y, b)$ be morphisms. Reflexivity of (Y, b) implies $f \Rightarrow f$. Assume $f \Rightarrow g$ and $g \Rightarrow h$, then

$$\begin{aligned} h &\leq b \cdot e_Y \cdot g \leq b \cdot e_Y \cdot b \cdot e_Y \cdot f \leq b \cdot Tb \cdot e_{TY} \cdot e_Y \cdot f \\ &\leq b \cdot m_Y \cdot e_{TY} \cdot e_Y \cdot f \approx b \cdot e_Y \cdot f, \end{aligned}$$

thus $\Rightarrow$ is transitive. That $\Rightarrow$ is compatible is composition is straightforward. ∎

Examples 3.4.23 (1) In case $\mathbf{T} = \mathbf{1}_{\mathfrak{O}}$, we obtain 2-cells of $\mathfrak{O}$-categories as in Definition 1.6.1.

(2) For $\mathbf{Alg}_t\mathbf{U} \cong \mathbf{Top}$, and continuous functions $f, g : X \longrightarrow Y$, we get $f \Rightarrow g$ if and only if $e_Y(f(x)) \to g(x)$ for all $x \in X$, equivalently, $g(x) \in \mathrm{cl}_Y\{f(x)\}$. Hence, $\Rightarrow$ is the pointwise generalization order.

(3) For $\mathbf{Alg}_t\mathbb{U} \cong \mathbf{Clos}$ we obtain $f \Rightarrow g$ if and only if $g(x) \in \mathrm{cl}_Y\{f(x)\}$ for all $x \in X$.

3.5 Strict and costrict morphisms

Definition 3.5.1 A morphism $(X,a) \xrightarrow{f} (Y,b)$ of lax algebras is called *strict* provided $f \cdot a \approx b \cdot Tf$ and *costrict* provided $a \cdot Tf^* \approx f^* \cdot b$.

The notions of strict and costrict morphisms as presented here are due to Möbus [Möb81].

3.5.1 Examples

Topological spaces We recall that a *proper* map between topological spaces is a closed continuous map with compact fibers.

Proposition 3.5.2 ([Möb81], [Sch02], [CH04a]) *The strict morphisms in* $\mathbf{Alg}_t\mathbf{U}$ *are precisely the proper maps and the costrict morphisms are precisely the open maps.*

The proof rests on the following well-known characterizations of proper and open maps; see [Her86], [Bou98]:

Proposition 3.5.3 *The following statements are equivalent for* $X \xrightarrow{f} Y$ *in* **Top***:*

(1) *f is proper;*

(2) *every pullback of f is closed;*

(3) *for every proper filter $\mathfrak{x}$ on X and every cluster point y of $Ff(\mathfrak{x})$ there exists a cluster point x of $\mathfrak{x}$ in X such that $f(x) = y$;*

(4) *for every ultrafilter $\mathfrak{a}$ on X and every convergence point y of $Uf(\mathfrak{a})$ there exists a convergence point x of $\mathfrak{a}$ in X such that $f(x) = y$.* ∎

Now condition (4) is equivalent to the inclusion $b \cdot Uf \subset f \cdot a$, and thus the first claim in Proposition 3.5.2 is proved. To show the statement concerning open maps, let us introduce some notation. If X is any topological space and $x \in X$, we write $\mathfrak{u}_x$ for the neighborhood filter of x in X. Clearly a map $X \xrightarrow{f} Y$ between topological spaces is continuous if and only if $\mathfrak{u}_{f(x)} \subset Ff(\mathfrak{u}_x)$ holds for all $x \in X$.

Proposition 3.5.4 *The following statements are equivalent for* $X \xrightarrow{f} Y$ *in* **Top***:*

(1) *f is open;*

(2) *$Ff(\mathfrak{u}_x) = \mathfrak{u}_{f(x)}$ for all $x \in X$;*

(3) *for any $\mathfrak{y} \in FY$ with $\mathfrak{y} \longrightarrow f(x)$ for some $x \in X$, there exists $\mathfrak{x} \in FX$ with $\mathfrak{x} \longrightarrow x$ and $Ff(\mathfrak{x}) \subset \mathfrak{y}$;*

(4) *for any* $\mathfrak{b} \in UY$ *with* $\mathfrak{b} \longrightarrow f(x)$ *for some* $x \in X$*, there exists* $\mathfrak{a} \in UX$ *with* $\mathfrak{a} \longrightarrow x$ *and* $Uf(\mathfrak{a}) = \mathfrak{b}$*.*

Proof [(1) $\Rightarrow$ (2)]: We only need to show $Ff(\mathfrak{u}_x) \subset \mathfrak{u}_{f(x)}$: $B \in Ff(\mathfrak{u}_x)$ implies the existence of an open set $U \subset X$ with $x \in U$ and $U \subset f^{-1}[B]$, which in turn implies $f(x) \in f[U] \subset f[f^{-1}[B]] \subset B$, with $f[U]$ open by hypothesis. Thus $B \in \mathfrak{u}_{f(x)}$.

[(2) $\Rightarrow$ (3)]: Suppose $\mathfrak{y} \longrightarrow f(x)$. Then $Ff(\mathfrak{u}_x) = \mathfrak{u}_{f(x)} \subset \mathfrak{y}$ and clearly $\mathfrak{u}_x \longrightarrow x$ in X.

[(3) $\Rightarrow$ (1)]: Let $A \subset X$ be open. If suffices to show that $f[A] \in \mathfrak{u}_y$ for all $y \in f[A]$. Given any $y \in f[A]$, we obtain $x \in A$ with $\mathfrak{u}_y \longrightarrow y = f(x)$. Hence there exists $\mathfrak{x} \in FX$ with $Ff(\mathfrak{x}) \subset \mathfrak{u}_y$ and $\mathfrak{x} \longrightarrow x$. Thus $A \in \mathfrak{x}$ and hence $f[A] \in Ff(\mathfrak{x}) \subset \mathfrak{u}_y$.

[(3) $\Rightarrow$ (4)]: Straightforward.

[(4) $\Rightarrow$ (2)]: It suffice to show $Ff(\mathfrak{u}_x) \subset \mathfrak{u}_{f(x)}$. Since $\mathfrak{u}_{f(x)}$ is the intersection of all $\mathfrak{b} \in UX$ with $\mathfrak{b} \longrightarrow f(x)$, it suffices to show that $Ff(\mathfrak{u}_x) \subset \mathfrak{b}$ whenever $\mathfrak{b} \in UY$, $\mathfrak{b} \longrightarrow f(x)$. For any such $\mathfrak{b}$ there exists $\mathfrak{a} \in UX$ with $Ff(\mathfrak{a}) = Uf(\mathfrak{a}) = \mathfrak{b}$ and $\mathfrak{a} \longrightarrow x$, thus $\mathfrak{u}_x \subset \mathfrak{a}$, which implies $Ff(\mathfrak{u}_x) \subset Ff(\mathfrak{a}) = \mathfrak{b}$. ■

To finish the proof of Proposition 3.5.2, observe that statement (4) of the above proposition is equivalent to $f^* \cdot b \subset a \cdot Uf^*$.

Corollary 3.5.5 *The strict monomorphisms in* $\mathbf{Alg}_t\mathbf{U}$ *are precisely the closed embeddings and the costrict monomorphisms in* $\mathbf{Alg}_t\mathbf{U}$ *are precisely the open embeddings.* ■

Having characterizations of proper and of open morphisms in **Top** in terms of (reflexive and transitive) lax algebras allows us to generalize the concepts of proper and of open morphism to pretopological, pseudotopological, and ultrarelational spaces. Precisely this has been done in [CHT03a].

Approach spaces Analogously to topological spaces, we have that the strict morphisms in $\mathbf{Alg}_t\mathbf{U}_{\overline{\mathbb{R}}_+} \cong \mathbf{Ap}$ are precisely the *proper contractions* (cf. [CLW03]), while the costrict morphisms give a good notion of *openness* for contractions.

Graphs and categories Let us first consider the identity actor on scenes of the form $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$. Here we have that a morphism $(X, a) \xrightarrow{f} (Y, b)$

$$\text{is strict} \iff \forall x \in X, y \in Y : b(f(x), y) \leq \bigvee_{x' \in f^{-1}\{y\}} a(x, x'), \tag{3.6}$$

$$\text{is costrict} \iff \forall x \in X, y \in Y : b(y, f(x)) \leq \bigvee_{x' \in f^{-1}\{y\}} a(x', x). \tag{3.7}$$

Using these equations, it is easy to verify the following results:

- To describe strictness in **Ord**, observe that any monotone functions $f : X \longrightarrow Y$ has restrictions $f_x : x^\uparrow \longrightarrow (f(x))^\uparrow$ for each $x \in X$. Now, f is strict if and only if each of the f_x is surjective. In particular, the strict embeddings in **Ord** are precisely the up-closed embeddings. Dually, f is costrict in **Ord** precisely when each of the down-set restrictions $x^\downarrow \longrightarrow (f(x))^\downarrow$ is surjective. Thus, the costrict embeddings in **Ord** are precisely the down-closed embeddings. The same reasoning applies to $\mathbf{Ord}(\mathbf{E})$ for a topos $\mathbf{E}$.
- For **Met**, we do not have a "conceptual" interpretation of strictness and costrictness which goes beyond the formulas (3.6) and (3.7). Note however that in case f is an embedding, we have that strictness if equivalent to $b(y, y') = \infty$ for all $y \in \operatorname{Im} f$, $y' \notin \operatorname{Im} f$, we may say that it is "impossible to exit" $\operatorname{Im} f$. Analogously, we have that the costrict embeddings are precisely those f for which it is "impossible to enter" $\operatorname{Im} f$, that is, those f as above with $b(y', y) = \infty$ for all $y' \notin \operatorname{Im} f$, $y \in \operatorname{Im} f$.

Topological spaces via neighborhoods For the filter-on-downsets monad F on **Ord**, we have that the strict Klei F-functors are those morphisms $(X, a) \xrightarrow{f} (Y, b)$ which satisfy $Ff \cdot a = b \cdot f$. Stated in terms of neighborhood-filters, this means precisely that $Ff(\mathfrak{u}_x) = \mathfrak{u}_{f(x)}$ holds for all $x \in X$, thus, by Proposition 3.5.4, to the fact that f is open.

T-graphs Let T be a **Set**-endofunctor. An $\mathbf{Alg}(T, (\mathbf{Set}, \mathbf{Rel}))$-morphism $(X, a) \xrightarrow{f} (Y, b)$ is strict if and only if

$$Tf(\mathfrak{x})\, b\, y \iff \exists x \in X : f(x) = y,\ \mathfrak{x}\, a\, x$$

holds for all $\mathfrak{x} \in TX$ and $y \in Y$.

Let P denote the powerset-functor on **Set** and for any relation $A \xrightarrow{r} B$ define $\check{r} : B \longrightarrow PA$ by $\check{r}(b) = \{\, a \in A \mid a\, r\, b \,\}$. Then strictness of f as above means precisely that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \check{a}}\downarrow & & \downarrow{\scriptstyle \check{b}} \\ PTX & \xrightarrow{PTf} & PTY \end{array}$$

commutes, that is, that f is a PT-coalgebra-morphism $(X, \check{a}) \longrightarrow (Y, \check{b})$. Thus we have that the category of coalgebras for endofunctors of the form $P \cdot T$ is the category of lax T-algebras with *strict* morphisms.

Closure spaces Strict and costrict morphisms in $\mathbf{Alg}\mathbb{U}$ can be described as follows:

Proposition 3.5.6 *Let $(X, a) \xrightarrow{f} (Y, b)$ be a morphism of lax $\mathbb{U}$-algebras.*

(1) *f is strict if and only if $f : (X, c_a) \longrightarrow (Y, c_b)$ is* closed, *that is, $c_b f[A] = f[c_a A]$ holds for all $A \subset X$;*

(2) *f is costrict if and only if $f : (X, c_a) \longrightarrow (Y, c_b)$ is* open, *that is, $f^{-1}[c_b B] = c_a\, f^{-1}[B]$ holds for all $B \subset Y$.*

Proof (1) is straightforward. To establish (2), first observe that $c_a f^{-1}[B] \subset f^{-1} c_b[B]$ holds for any $f \in \mathbf{Alg}\mathbb{U}$. To show the converse inclusion, note that $x \in f^{-1}[c_b B]$ iff $f(x) \in c_b B$ iff $B\,(f^\circ \cdot b)\,x$. On the other hand, we clearly have $f^{-1} B\, a\, x$ iff $B\,(a \cdot (\mathbb{U}f)^*)\,x$, and the result is proved. ∎

3.5.2 Basic properties

The following proposition summarizes the basic properties of strict and costrict morphisms:

Proposition 3.5.7 *Let $(X, a) \xrightarrow{f} (Y, b)$ be a morphism in $\mathbf{Alg}\, T$.*

(1) *The classes of strict and costrict morphisms are closed under composition and contain all isomorphisms. In fact, f is an isomorphism in $\mathbf{Alg}\, T$ if and only if it is an isomorphism in $\mathbf{S}$ and is costrict or strict or initial or final.*

(2) *Let $(Y, b) \xrightarrow{g} (Z, c)$ be a morphism in $\mathbf{Alg}\, T$. Then the following statements hold:*

   (a) *$g \cdot f$ strict, Tf covers $\implies$ g strict.*

   (b) *$g \cdot f$ strict, g injects $\implies$ f strict.*

   (c) *$g \cdot f$ costrict, f covers $\implies$ g costrict.*

   (d) *$g \cdot f$ costrict, Tg injects $\implies$ f costrict.*

(3) *If f is strict and injects, or if f is costrict and Tf injects, then f is initial.*

(4) *If f is strict and Tf covers, or if f is costrict and f covers, then f is final, i.e. $b \approx f \cdot a \cdot Tf^*$.* ∎

Let $\mathcal{F} \subset \operatorname{mor} \mathbf{S}$. We write

$$\mathcal{F}_{\mathrm{S}} = \{f \in \mathbf{Alg}\, T \mid f \text{ is strict}, \ |f| \in \mathcal{F}\}, \quad \mathcal{F}_{\mathrm{C}} = \{f \in \mathbf{Alg}\, T \mid f \text{ is costrict}, \ |f| \in \mathcal{F}\},$$

and $\mathcal{F}_{\mathrm{B}} = \mathcal{F}_{\mathrm{S}} \cap \mathcal{F}_{\mathrm{C}}$. Thus $\mathcal{F}_{\mathrm{S}}$ is the class of all strict $\mathbf{Alg}\, T$-morphisms over $\mathcal{F}$.

Proposition 3.5.8 *Suppose that $\mathcal{S}$ is finitely complete and has the (BC)-property. Let $\mathcal{F} \subset \operatorname{mor} \mathbf{S}$ be pullback-stable such that $(\mathbf{D}, \mathcal{F})$ satisfies the Frobenius law. Then $\mathcal{F}_{\mathrm{S}}$ is stable under pullback.*

Proof Suppose

$$\begin{array}{ccc} (P,d) & \xrightarrow{q} & (Y,b) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Z,c) \end{array} \tag{3.8}$$

is a pullback in $\mathbf{Alg}\,T$ such that $\big((X,a) \xrightarrow{f} (Z,c)\big) \in \mathcal{F}_{\mathrm{S}}$. By Theorem 3.2.10, the underlying diagram in $\mathbf{S}$ is a pullback as well, which implies $q\cdot p^*\cdot a\cdot Tp \approx g^*\cdot f\cdot a\cdot Tp \approx g^*\cdot c\cdot Tf\cdot Tp \approx g^*\cdot c\cdot Tg\cdot Tq$ by the (BC)-property. From $d \approx q^*\cdot b\cdot Tq \wedge p^*\cdot a\cdot Tp$ we obtain, using the Frobenius law,

$$\begin{aligned} q\cdot d &\approx q\cdot(q^*\cdot b\cdot Tq \wedge p^*\cdot a\cdot Tp) \\ &\approx b\cdot Tq \wedge q\cdot p^*\cdot a\cdot Tp \\ &\approx b\cdot Tq \wedge g^*\cdot c\cdot Tg\cdot Tq \\ &\approx (b \wedge g^*\cdot c\cdot Tg)\cdot Tq \\ &\approx b\cdot Tq \end{aligned}$$

since $b \leq g^*\cdot c\cdot Tg$. Hence $(P,d) \xrightarrow{q} (Y,b)$ is strict. ∎

Proposition 3.5.9 *Suppose that $\mathcal{S}$ is finitely complete. Let $\mathcal{F} \subset \operatorname{mor}\mathbf{S}$ be pullback-stable such that T has the (BC)-property with respect to $\mathcal{F}$ and $(\mathbf{D}, T[\mathcal{F}])$ satisfies the coFrobenius law. Then $\mathcal{F}_{\mathrm{C}}$ is stable under pullback.*

Proof Given a pullback (3.8) such that $\big((X,a) \xrightarrow{f} (Z,c)\big) \in \mathcal{F}_{\mathrm{C}}$, we obtain, since the T-image of the underlying $\mathbf{S}$-pullback of (3.8) is a (BC)-square, $p^*\cdot a\cdot Tp\cdot Tq^* \approx p^*\cdot a\cdot Tf^*\cdot Tg \approx p^*\cdot f^*\cdot c\cdot Tg \approx q^*\cdot g^*\cdot c\cdot Tg$, and thus

$$\begin{aligned} d\cdot Tq^* &\approx (p^*\cdot a\cdot Tp \wedge q^*\cdot b\cdot Tq)\cdot Tq^* \\ &\approx p^*\cdot a\cdot Tp\cdot Tq^* \wedge q^*\cdot b \\ &\approx q^*\cdot g^*\cdot c\cdot Tg \wedge q^*\cdot b \\ &\approx q^*\cdot(g^*\cdot c\cdot Tg \wedge b) \\ &\approx q^*\cdot b, \end{aligned}$$

hence also $(P,d) \xrightarrow{q} (Y,b)$ is costrict. ∎

Observe that Propositions 3.5.8 and 3.5.9 hold not only in $\mathbf{Alg}\,T$ but also in $\mathbf{Alg}_r\mathsf{T}$, $\mathbf{Alg}_u\mathsf{T}$, and $\mathbf{Alg}_t\mathsf{T}$ since these subcategories are closed under finite limits by assumption.

Examples Since the premises of Propositions 3.5.8 and 3.5.9 hold for $(\mathbf{Set}, \mathbf{Rel})$ (with $\mathcal{F} = \operatorname{mor}\mathbf{Set}$) and $\mathsf{T} = \mathsf{U}$, we see that open and proper continuous maps of topological spaces are stable under pullback.

Preservation of costrict morphisms The following result which will be crucial in Chapter 6 stems from [MST06].

Proposition 3.5.10 *Assume* $\mathbf{D}$ *to be a quantaloid and let* $(X,a) \xrightarrow{f} (Y,b)$ *be a costrict morphism in* $\mathbf{Alg}_r\mathbf{T}$ *such that* T *to preserve* $\{f^*\}$*-whiskers. The following statements hold:*

(1) *If* (X,a) *is unitary and* $(Y,\overline{b})$ *is the unitary reflection of* (Y,b)*, then also* $(X,a) \xrightarrow{f} (Y,\overline{b})$ *is costrict;*

(2) *if* (X,a) *is transitive and* $(Y,\overline{b})$ *is the transitive reflection of* (Y,b)*, then also* $(X,a) \xrightarrow{f} (Y,\overline{b})$ *is costrict.*

Proof We will only show (2), the proof of (1) is similar: we may assume $\overline{b}$ to be constructed as in Proposition 3.4.19. Thus is suffices to show $f^* \cdot b_\alpha \approx a \cdot Tf^*$ for all ordinals α. For $\alpha = 0$, the result holds trivially, and for any ordinal α, we obtain

$$\begin{aligned} f^* \cdot b_{\alpha+1} &\approx f^* \cdot b_\alpha \cdot Tb_\alpha \cdot m_Y^* \\ &\approx a \cdot Tf^* \cdot Tb_\alpha \cdot m_Y^* \\ &\approx a \cdot T(f^* \cdot b_\alpha) \cdot m_Y^* \\ &\approx a \cdot T(a \cdot Tf^*) \cdot m_Y^* \\ &\approx a \cdot Ta \cdot m_X^* \cdot Tf^* \\ &\approx a \cdot Tf^*. \end{aligned}$$

For a limit ordinal γ, we have, using the fact that $\mathbf{D}$ is a quantaloid, $f^* \cdot b_\gamma = f^* \cdot \bigvee_{\alpha<\gamma} b_\alpha \approx \bigvee_{\alpha<\gamma} f^* \cdot b_\alpha \approx a \cdot Tf^*$. ∎

3.5.3 Strict and costrict morphisms transfer transitivity

We have seen before that quotients (in $\mathbf{Alg}\,T$) of transitive lax algebras need not be transitive. We will analyze the situation for strict or costrict quotients.

Let $\mathcal{E}$ be a class of covers in $\mathbf{C}$, that is, for every $e \in \mathcal{E}$, $e \cdot e^* \approx 1$ holds in $\mathbf{D}$.

Proposition 3.5.11 *Suppose* T *preserves* $\mathcal{E}^*$*-whiskers. If* $(X,a) \xrightarrow{f} (Y,b)$ *is a costrict morphism in* $\mathcal{E}$ *and* (X,a) *is transitive, then also* (Y,b) *is transitive.*

Proof We have

$$\begin{aligned} b \cdot Tb \cdot m_Y^* &\approx f \cdot f^* \cdot b \cdot Tb \cdot m_Y^* \approx f \cdot a \cdot Tf^* \cdot Tb \cdot m_Y^* \approx f \cdot a \cdot T(f^* \cdot b) \cdot m_Y^* \\ &\approx f \cdot a \cdot T(a \cdot Tf^*) \cdot m_Y^* \approx f \cdot a \cdot Ta \cdot TTf^* \cdot m_Y^* \approx f \cdot a \cdot Ta \cdot m_X^* \cdot Tf^* \\ &\leq f \cdot a \cdot Tf^* \approx b. \end{aligned}$$

∎

Proposition 3.5.12 *Suppose T preserves $\mathcal{E}$-whiskers. If $(X,a) \xrightarrow{f} (Y,b)$ is a strict morphism in $\mathcal{E}$ and (X,a) is transitive, then also (Y,b) is transitive.*

Proof Observe that T sends $\mathcal{E}$-morphisms to covers: $1_{TY} \approx T(f \cdot f^*) \approx Tf \cdot Tf^*$ holds since T preserves $\mathcal{E}$-whiskers. Thus

$$\begin{aligned} b \cdot Tb \cdot m_Y^* &\approx b \cdot Tb \cdot TTf \cdot TTf^* \cdot m_Y^* \approx b \cdot T(b \cdot Tf) \cdot m_X^* \cdot Tf^* \\ &\approx b \cdot T(f \cdot a) \cdot m_X^* \cdot Tf^* \approx b \cdot Tf \cdot Ta \cdot m_X^* \cdot Tf^* \approx f \cdot a \cdot Ta \cdot m_X^* \cdot Tf^* \\ &\leq f \cdot a \cdot Tf^* \approx b. \end{aligned}$$

■

The preceding results were proved in [CH04a] for the special case $\mathbf{D} = \operatorname{Mat}\mathbf{V}$.

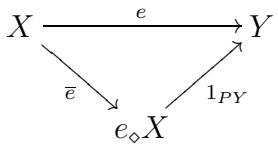
3.6 Quotients and descent

In this section we will study quotient morphisms in detail and sufficient conditions for morphism of lax algebras to be of (effective) descent type.

3.6.1 Quotients and regular epimorphisms of lax algebras

Lemma 3.6.1 *Let $\mathbf{X} \xrightarrow{P} \mathbf{B}$ be a faithful opfibration. Then every extremal epimorphism in $\mathbf{X}$ is final.*

Proof Suppose that $X \xrightarrow{e} Y$ is an extremal epimorphism in $\mathbf{X}$. We can factor e as



Faithfulness of P implies that $e_\diamond X \xrightarrow{1_{PY}} Y$ is a monomorphism in $\mathbf{X}$, and hence an isomorphism. Thus also $X \xrightarrow{e} Y$ is final. ■

Proposition 3.6.2 *Every faithful opfibration preserves and lifts coequalizers. That is, if $\mathbf{X} \xrightarrow{P} \mathbf{B}$ is a faithful opfibration, then a diagram*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{e} Z \tag{3.9}$$

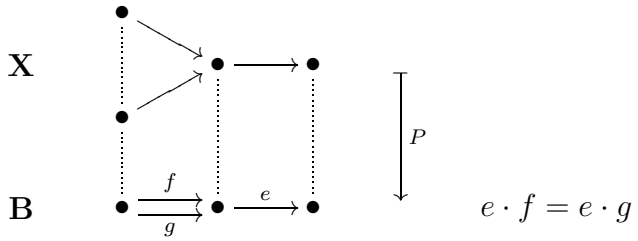
is a coequalizer if and only if its P-image is a coequalizer in $\mathbf{B}$ and $Y \xrightarrow{e} Z$ is final.

Proof Suppose (3.9) is a coequalizer in $\mathbf{X}$. Then e is final by Lemma 3.6.1. Now let $PY \xrightarrow{h} D$ be a $\mathbf{B}$-morphism such that $h \cdot Pf = h \cdot Pg$ holds. We form the final lift $Y \xrightarrow{\overline{h}} h_\diamond Y$, and obtain, by faithfulness of P, $\overline{h} \cdot f = \overline{h} \cdot g$. Thus there exist $Z \xrightarrow{d} h_\diamond Y$

with $d \cdot e = \overline{h}$ and thus $Pd \cdot Pe = h$. Now observe that, whenever $b \cdot Pe = h = P\overline{h}$ for a $\mathbf{B}$-morphism b, b lifts to a $\mathbf{X}$-morphism $Z \xrightarrow{\overline{b}} h_\diamond Y$ with $\overline{b} \cdot e = \overline{h}$ by finality of e. Thus any such b is unique.

The converse follows from Proposition 3.2.9. ■

The situation for regular epimorphisms is slightly more subtle: clearly, if e is a regular epimorphism in $\mathbf{X}$, it is a regular epimorphism in $\mathbf{B}$, but not conversely, as the following example shows:



(Here P is the unique fibering $\mathbf{X} \longrightarrow \mathbf{B}$ which maps objects as shown.) e is a regular epimorphism in $\mathbf{B}$ and final in $\mathbf{X}$, but there is no pair of morphisms in $\mathbf{X}$ which e could coequalize.

Of course, if P is a topological functor this situation can not occur. More general, we have:

Remark 3.6.3 Suppose $\mathbf{X} \xrightarrow{P} \mathbf{B}$ is a faithful opfibration such that for each $B \in \mathbf{B}$ and each $X, X' \in P_B$ there exists a lower bound of $\{X, X'\}$ in P_B. Then $e \in \mathbf{X}$ is a regular epimorphism if and only if e is final and Pe is a regular epimorphism in $\mathbf{B}$. ■

Pullback-stability of quotients in $\mathbf{Alg}\,T$

Proposition 3.6.4 ([CH03]) *Suppose* $(\mathbf{D}, \mathbf{S})$ *satisfies the Frobenius law,* $(\mathbf{D}, T[\mathbf{S}])$ *satisfies the coFrobenius law, and that* $\mathcal{S}$ *and* T *have the (BC)-property. Then final morphisms in* $\mathbf{Alg}\,T$ *are stable under pullback.*

Proof Suppose

$$\begin{array}{ccc} (P,d) & \xrightarrow{q} & (Y,b) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ (X,a) & \xrightarrow[f]{} & (Z,c) \end{array}$$

is a pullback in $\mathbf{Alg}\,T$ such that $c \approx f \cdot a \cdot Tf^*$ holds. Then we have

$$\begin{aligned} q \cdot d \cdot Tq^* &\approx q \cdot p^* \cdot a \cdot Tp \cdot Tq^* \wedge b \\ &\approx g^* \cdot f \cdot a \cdot Tf^* \cdot Tg \wedge b \\ &\approx g^* \cdot c \cdot Tg \wedge b \\ &\approx b, \end{aligned}$$

and thus $(P,d) \xrightarrow{q} (Y,b)$ is final. ∎

Corollary 3.6.5 *Under the conditions of Proposition 3.6.4, regular epimorphisms are stable under pullback in* $\mathbf{Alg}\,T$ *provided they are so in* $\mathbf{S}$.

Proof First observe that under the conditions of Proposition 3.6.4 an $\mathbf{Alg}\,T$-morphism e is a regular epimorphism iff it is final and a regular epimorphism in $\mathbf{S}$. Thus the result follows immediately from Proposition 3.6.4. ∎

Corollary 3.6.6 *Under the conditions of Proposition 3.6.4, regular epimorphisms are stable under pullback in* $\mathbf{Alg}_r\mathsf{T}$ *provided they are so in* $\mathbf{S}$ *and every regular epimorphism in* $\mathbf{S}$ *is a cover in* $\mathbf{D}$. ∎

Of course, these results can not extended to transitive lax algebras: it is well-known that quotients in **Top** are not pullback-stable. Quotients in $\mathbf{Alg}_t\mathsf{T}$ (over the scene $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$) are studied more closely in [Hof05]).

3.6.2 Descent morphisms

We will give sufficient conditions for morphisms of lax algebras to be a (effective) descent. These are straightforward generalizations of the conditions given by Clementino and Hofmann in [CH04a]. We succeed in giving a purely categorical proof.

For more information on the descent problem see, e.g., [Moe89], [ST92], [RT94], and especially [JST04].

Monadic descent

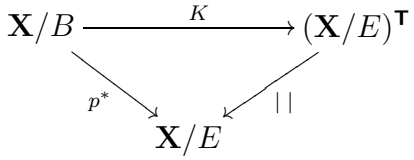
Let $\mathbf{X}$ be a category with (chosen) pullbacks and let $E \xrightarrow{p} B$ be a $\mathbf{X}$-morphism. p induces a change-of-base or re-indexing functor $\mathbf{X}/B \xrightarrow{p^*} \mathbf{X}/E$ by pulling back $\mathbf{X}/B$-objects or "bundles over B" along p:

$$\begin{array}{ccc} E \times_B C & \longrightarrow & C \\ {\scriptstyle p^*(q)}\downarrow & & \downarrow{\scriptstyle q} \\ E & \xrightarrow{\;p\;} & B \end{array}$$

and for a morphism $(C,q) \xrightarrow{f} (C',q')$, we have $p^*f = 1 \times_B f$. p^* has a left adjoint $p_!$ defined by composing with p, thus

$$p_! : (C,q) \longmapsto (C, p \cdot q), \quad f \longmapsto f.$$

As usual, this adjunction induces a monad T on $\mathbf{X}/E$. Thus we can form the Eilenberg–Moore category $(\mathbf{X}/E)^{\mathsf{T}}$ and obtain a canonical comparison functor K



Definition 3.6.7 We say that p is a *descent morphism* if K is full and faithful, and that p is an *effective descent morphism* if K is an equivalence, that is, if p^* is monadic.

The question whether a morphism is a descent morphism is settled easily (see [JT94]):

Proposition 3.6.8 *A morphism p in a category $\mathbf{X}$ with pullbacks is a descent morphism if and only if it is a stably regular epimorphism.* ■

Usually, the class of effective descent morphisms in a category $\mathbf{X}$ is harder to characterize.

Axioms for effective descent We will give axioms for a class $\mathcal{P}$ morphism in a category which ensure that every morphism in $\mathcal{P}$ is an effective descent morphism. These axioms were originally introduced by Moerdijk [Moe89], and slightly improved in [RT94].

Theorem 3.6.9 *Let $\mathbf{X}$ be a category with pullbacks and $\mathcal{P}$ a class of $\mathbf{X}$-morphisms. Every morphisms in $\mathcal{P}$ is an effective descent morphism provided $\mathcal{P}$ satisfies the following conditions* (D1)–(D4)*:*

(D1) *$\mathcal{P}$ contains all isomorphisms and is closed under composition with isomorphisms;*

(D2) *$\mathcal{P}$ is stable under pullback;*

(D3) *$\mathcal{P}$ is contained in the class of regular epimorphisms;*

(D4) *coequalizers of parallel pairs of morphisms in $\mathcal{P}$ exists in $\mathbf{X}$ and are stable under pullback.* ■

The pullback-criterion Whether a morphism is an effective descent morphism depends on the ambient category. Nonetheless, we can make use of the following result from [JT94] (see also [JST04]:

Proposition 3.6.10 (Pullback-Criterion) *Let $\mathbf{X}$ be a category with pullbacks and $\mathbf{Y}$ a full subcategory which is closed under pullbacks in $\mathbf{X}$. The following statements are equivalent for $E \xrightarrow{p} B$ in $\mathbf{Y}$ which is an effective descent morphism in $\mathbf{X}$:*

(1) *p is an effective descent morphism in $\mathbf{Y}$;*

(2) *for every pullback diagram*

$$\begin{array}{ccc} E \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow \\ E & \xrightarrow{p} & B \end{array}$$

in $\mathbf{X}$, if $E \times_B X$ is in $\mathbf{Y}$, so is X. ■

3.6.3 Descent for lax algebras

We will now use the preceding results to obtain results on (effective) descent morphisms in categories of lax algebras. Although not every result in this Section uses all of them, we shall make the following assumptions. Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ be a finitely complete scene with (BC)-property such that $\mathbf{S}$ has coequalizers, is regular, and $(\mathbf{D}, \mathbf{S})$ satisfies the Frobenius law. Let $\mathbf{T} = (T, e, m)$ be an actor monad on $\mathcal{S}$ such that T has the (BC)-property and $(\mathbf{D}, T[\mathbf{S}])$ satisfies the coFrobenius law. Let $\mathcal{E}$ be the class of regular epimorphisms in $\mathbf{S}$. We assume $\mathcal{E} \subset \operatorname{Cov}\mathbf{D}$, and write

$$\mathcal{E}_{\mathrm{F}} = \{\, f \in \mathbf{Alg}\, T \mid f \text{ is final and } f \in \mathcal{E} \,\}.$$

Under this conditions, Remark 3.6.3 applies, and thus $\mathcal{E}_{\mathrm{F}}$ is precisely the class of regular epimorphisms in $\mathbf{Alg}\, T$. Moreover, $\mathcal{E}_{\mathrm{F}}$ satisfies (D1)–(D4): (D1) and (D3) are obvious, (D2) (and hence (D4)) holds by Corollary 3.6.5. This implies immediately that

$$\mathcal{E}_{\mathrm{F}} = \{\, \text{descent morphisms} \,\} = \{\, \text{effective descent morphisms} \,\} \tag{3.10}$$

holds in $\mathbf{Alg}\, T$. Now, using the Pullback-Criterion 3.6.10 and Proposition 3.4.20, we see that the conceptual identity (3.10) holds also in $\mathbf{Alg}_r\mathbf{T}$.

In particular, in **PsTop** every quotient morphism is an effective descent morphism.

Effective descent morphisms in $\mathbf{Alg}_t\mathbf{T}$ Using Propositions 3.5.11 and 3.5.12 together with the Pullback-Criterion 3.6.10, we obtain:

Corollary 3.6.11

(1) *If T preserves $\mathcal{E}^*$-whiskers, then $\mathcal{E}_{\mathrm{C}} \subset \{$ effective descent morphisms $\}$ in $\mathbf{Alg}_t\mathbf{T}$.*

(2) *If T preserves $\mathcal{E}$-whisker, then $\mathcal{E}_{\mathrm{S}} \subset \{$ effective descent morphisms $\}$ in $\mathbf{Alg}_t\mathbf{T}$.* ■

From Corollary 3.6.11 we immediately deduce that every open and every proper surjection in **Top** is an effective descent morphism in **Top**. The same reasoning applies to open resp. proper surjective contractions in **Ap**, and to open resp. closed surjective morphisms in **Clos**.

3.7 Change of theory

As is well-known in the case of Eilenberg–Moore algebras for a monad, every monad morphisms induces a concrete functor between the corresponding categories of algebras.

Definition 3.7.1 Let $(\mathbf{X}, P)$ and $(\mathbf{Y}, Q)$ be concrete categories over $\mathbf{S}$, and let $F : (\mathbf{X}, P) \longrightarrow (\mathbf{Y}, Q)$ and $G : (\mathbf{Y}, Q) \longrightarrow (\mathbf{X}, P)$ be concrete functors. We say (F, G) is a *Galois connection* provided $X \xrightarrow{1_{PX}} GFX$ is an $\mathbf{X}$-morphism for every $X \in \mathbf{X}$ and $FGY \xrightarrow{1_{QY}} Y$ is a $\mathbf{Y}$-morphism for every $Y \in \mathbf{Y}$.

For details on Galois connections see [AHS04] or [Hs90]. Be advised that we do *not* require forgetful functors to be amnestic.

Proposition 3.7.2 *Let T and U be actors on a scene $\mathcal{S}$. A slim transformation $\lambda : T \longrightarrow U$ induces concrete functors*

$$\lambda_\diamond : \mathbf{Alg}\, T \longrightarrow \mathbf{Alg}\, U, \qquad (X, a) \longmapsto (X, a \cdot \lambda_X^*), \qquad f \longmapsto f,$$

and

$$\lambda^\diamond : \mathbf{Alg}\, U \longrightarrow \mathbf{Alg}\, T, \qquad (X, b) \longmapsto (X, b \cdot \lambda_X), \qquad f \longmapsto f.$$

$(\lambda_\diamond, \lambda^\diamond)$ is a Galois connection. $(\ )^\diamond$ and $(\ )_\diamond$ are functorial: we have $1_T^\diamond = 1_{\mathbf{Alg}\, T} = (1_T)_\diamond$ and for any actor S on $\mathcal{S}$ with a slim transformation $\kappa : S \longrightarrow T$, we have $\lambda_\diamond \cdot \kappa_\diamond = (\lambda \cdot \kappa)_\diamond$ and $(\lambda \cdot \kappa)^\diamond = \kappa^\diamond \cdot \lambda^\diamond$. ■

Since for every Galois connection (F, G), F preserves final sinks and G preserves initial sources, we obtain:

Corollary 3.7.3 *$\lambda^\diamond$ preserves initial sources; $\lambda_\diamond$ preserves final sinks.* ■

Proposition 3.7.4 *Suppose $\mathbf{C}$ to be locally skeletal. If λ injects pointwise, then $\lambda_\diamond$ is a full embedding; if λ covers pointwise, then $\lambda^\diamond$ is a full embedding.* ■

Proposition 3.7.5 *Let $\mathsf{T} = (T, e, m)$ and $\mathsf{U} = (U, d, n)$ be actor monads on $\mathcal{S}$, $\lambda : \mathsf{T} \longrightarrow \mathsf{U}$ a monad-morphism.*

(1) *$\lambda^\diamond$ preserves and reflects reflexivity;*

(2) *$\lambda_\diamond$ preserves reflexivity; if λ is pointwise an injection $\lambda_\diamond$ also reflects reflexivity;*

(3) *if λ extends oplax, then $\lambda^\diamond$ preserves unitarity and transitivity;*

(4) *if (λ_X^*) forms an oplax transformation and U is right $\{\lambda_X^*\}$-strict, then $\lambda_\diamond$ preserves transitivity;*

(5) *if λ extends oplax and injects pointwise, then $\lambda_\diamond$ reflects transitivity.*

Proof Let (X, a) be a lax T-algebra and (X, b) a lax U-algebra.

[1] $(X, UX \xrightarrow{b} X)$ is reflexive $\iff d_X^* \leq b \iff e_X^* \cdot \lambda_X^* \leq b \iff e_X^* \leq b \cdot \lambda_X$ $\iff \lambda^\diamond(X, b)$ is reflexive.

[2] $e_X^* \leq a$ implies $d_X^* \approx e_X^* \cdot \lambda_X^* \leq a \cdot \lambda_X^*$, and the converse holds provided $(\ ) \cdot \lambda_X^*$ is fully-faithful, equivalently, if λ is pointwise an injection.

[3] Suppose (X, b) to be unitary. We have

$$\begin{aligned} e_X^* \cdot T(b \cdot \lambda_X) &\leq e_X^* \cdot Tb \cdot T\lambda_X \\ &\leq e_X^* \cdot \lambda_X^* \cdot \lambda_X \cdot Tb \cdot T\lambda_X \\ &\leq d_X^* \cdot Ub \cdot \lambda_{UX} \cdot T\lambda_X && \lambda \text{ oplax} \\ &\leq b \cdot n_X \cdot (\lambda * \lambda)_X && (X, b) \text{ unitary} \\ &= b \cdot \lambda_X \cdot m_X. \end{aligned}$$

If (X, b) is transitive, we have

$$\begin{aligned} b \cdot \lambda_X \cdot T(b \cdot \lambda_X) &\leq b \cdot \lambda_X \cdot Tb \cdot T\lambda_X \\ &\leq b \cdot Ub \cdot \lambda_{UX} \cdot T\lambda_X && \text{since } \lambda \text{ is oplax} \\ &\leq b \cdot n_X \cdot (\lambda * \lambda)_X \\ &\approx b \cdot \lambda_X \cdot m_X, \end{aligned}$$

thus also $\lambda^\diamond(X, b)$ is transitive.

[4] Let (X, a) be transitive and suppose that U is right $\{\lambda_X^*\}$-strict and (λ_X^*) forms an oplax transformation. Then

$$\begin{aligned} a \cdot \lambda_X^* \cdot U(a \cdot \lambda_X^*) \cdot n_X^* &\approx a \cdot \lambda_X^* \cdot Ua \cdot U\lambda_X^* \cdot n_X^* \\ &\leq a \cdot Ta \cdot \lambda_{TX}^* \cdot U\lambda_X^* \cdot n_X^* \\ &\approx a \cdot Ta \cdot m_X^* \cdot \lambda_X^* \\ &\leq a \cdot \lambda_X^*. \end{aligned}$$

[5] If $\lambda_\diamond(X, a)$ is transitive, we have

$$\begin{aligned} a \cdot Ta &\approx a \cdot Ta \cdot \lambda_{TX}^* \cdot \lambda_{TX} \\ &\leq a \cdot \lambda_X^* \cdot Ua \cdot \lambda_{TX} && \text{since } \lambda \text{ is oplax} \\ &\approx a \cdot \lambda_X^* \cdot U(a \cdot \lambda_X^* \cdot \lambda_X) \cdot \lambda_{TX} \\ &\leq a \cdot \lambda_X^* \cdot U(a \cdot \lambda_X^*) \cdot U\lambda_X \cdot \lambda_{TX} \\ &\leq a \cdot \lambda_X^* \cdot n_X \cdot (\lambda * \lambda)_X \\ &\approx a \cdot \lambda_X^* \cdot \lambda_X \cdot m_X \\ &\approx a \cdot m_X \end{aligned}$$

where the last equivalence holds since λ_X is an injection. ■

Proposition 3.7.6 *$\lambda^\diamond$ preserves strictness of morphisms and $\lambda_\diamond$ preserves costrictness.*

If $\mathcal{F}$ is any class of $\mathbf{S}$-morphisms such that λ has the Beck–Chevalley property with respect to $\mathcal{F}$, then $\lambda^\diamond$ preserves costrictness of morphisms over $\mathcal{F}$ and $\lambda_\diamond$ preserves strictness of morphisms over $\mathcal{F}$. ■

Transferring the 2-categorical structure Recall that $\mathbf{Alg}_t\mathsf{T}$ is a (locally thin) 2-category provided e extends oplax, with 2-cells $f \Rightarrow g$ iff $g \leq c \cdot d_Y \cdot f$ for $f, g : (X, a) \longrightarrow (Y, c)$.

Proposition 3.7.7 *If $\lambda_\diamond$, $\lambda^\diamond$ restrict to functors $\mathbf{Alg}_t\mathsf{T} \longrightarrow \mathbf{Alg}_t\mathsf{U}$, $\mathbf{Alg}_t\mathsf{U} \longrightarrow \mathbf{Alg}_t\mathsf{T}$ respectively, then*

(1) *$\lambda_\diamond$ is a 2-functor; $\lambda_\diamond$ is 2-full (full on 2-cells) provided each λ_X is an injection;*

(2) *$\lambda^\diamond$ is a 2-full 2-functor.*

Proof Consider the first statement. We have $d_X \leq \lambda_X^* \cdot \lambda_X \cdot d_X \approx \lambda_X^* \cdot e_X$, with "$\approx$" provided λ_X is an injection. Thus $g \leq c \cdot d_X \cdot f$ implies $g \leq c \cdot \lambda_X^* \cdot e_X \cdot f$, and the converse holds provided λ_X is an injection. This is just the statement we had to prove.

The second statement is proved analogously. ■

3.7.1 Examples

As a first example, consider any monadic actor $\mathsf{T} = (T, e, m)$ on $\mathcal{S}$ which we assume to be non-trivial in the sense that e is pointwise an injection. Moreover we assume e to extend oplax. Denote the identity monadic actor on $\mathcal{S}$ by $\mathbf{1}$; e is a monad-morphism $\mathbf{1} \longrightarrow \mathsf{T}$. By Proposition 3.7.5, we have the following commutative diagram of concrete functors:

$$\begin{array}{ccccccc}
\mathbf{Alg}_t\mathbf{1} & \hookrightarrow & \mathbf{Alg}_u\mathbf{1} & = & \mathbf{Alg}_r\mathbf{1} & \hookrightarrow & \mathbf{Alg}\,\mathbf{1} \\
\uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} \\
\mathbf{Alg}_t\mathsf{T} & \hookrightarrow & \mathbf{Alg}_u\mathsf{T} & \hookrightarrow & \mathbf{Alg}_r\mathsf{T} & \hookrightarrow & \mathbf{Alg}\,\mathsf{T}
\end{array}$$

Closure spaces Let $\mathbb{U}$ be the upper-set monad on $\mathbf{Ord}$. The unit $u = (\ )^\uparrow$ of $\mathbb{U}$ induces a Galois correspondence

$$\mathbf{Bin} \underset{u^\diamond}{\overset{u_\diamond}{\rightleftarrows}} \mathbf{Mop}.$$

By the observations made in Section 2.3.5 and Proposition 3.7.5, *both* $u^\diamond$ and $u_\diamond$ preserve reflexivity and transitivity, hence we have restrictions

$$\mathbf{Ord} \underset{u^\diamond}{\overset{u_\diamond}{\rightleftarrows}} \mathbf{Clos}$$

The full functor $u_\diamond$ assigns an ordered set X its down-closure-operator $\downarrow$. Its right adjoint $u^\diamond$ sends a closure space (X, c) to its *generalization order*, defined by $x \leq y$ if and only if $y \in c(\{x\})$.

Topological spaces For the extension of the ultrafilter monad $\mathbf{U}$ to $(\mathbf{Set}, \mathbf{Rel})$ we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Ord} & \hookrightarrow & \mathbf{RBin} & = & \mathbf{RBin} & \hookrightarrow & \mathbf{Bin} \\
\uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^\diamond} \\
\mathbf{Top} & \hookrightarrow & \mathbf{PrTop} & \hookrightarrow & \mathbf{PsTop} & \hookrightarrow & \mathbf{URS}
\end{array}$$

of concrete functors. $e_\diamond$ assigns an ultrarelational space (X, a) its *generalization relation* $\prec$, defined by $x \prec y$ iff $(e_X(x), y) \in a$. If (X, a) is a topological space, then $\prec$ is its generalization order. The left adjoint $e_\diamond$ of $e^\diamond$ sends a graph (X, a) to the ultrarelational space $(X, a \cdot e_X^\circ)$, that is, $\mathfrak{x}$ convergences to y iff there exists $x \in X$ such that $x\, a\, y$ and $\mathfrak{x} = e_X(x)$. $e_\diamond$ preserves reflexivity, and we obtain a Galois correspondence

$$\mathbf{RBin} \underset{e^\diamond}{\overset{e_\diamond}{\rightleftarrows}} \mathbf{PsTop}.$$

Alas, $e_\diamond$ does not preserve transitivity, nor does it send an ordered set to a pretopological space, as the following example shows:

Let N denote the set of natural numbers and write o for the order-relation $\geq$ on N. We claim that the pseudo-topological space $(N, o \cdot e_N^\circ)$ is not pretopological. Let $\mathfrak{x}$ be a free ultrafilter on N. $\mathfrak{x}$ has no convergence point in $(N, o \cdot e_N^\circ)$. Using the description (3.4) of Uo, we see that $\mathfrak{x}\,(Uo)\,e_N(0)$ holds. Hence we have $(\mathfrak{x}, 0) \in e_N^\circ \cdot Uo = e_N^\circ \cdot U(o \cdot e_N^\circ) \cdot m_N^\circ$, but $(\mathfrak{x}, 0) \notin o \cdot e_N^\circ$, and thus the lax $\mathbf{U}$-algebra $(N, o \cdot e_N^\circ)$ is not unitary, thus not transitive.

Still, we can modify $e_\diamond$ by composing it with the reflector into $\mathbf{Top}$. Formally, we can show:

Proposition 3.7.8 *Let $\mathcal{S}$ be a topological scene, $\mathbf{S} = (S, d, n)$ and $\mathbf{T} = (T, e, m)$ be actor monads on $\mathcal{S}$, $\lambda : \mathbf{S} \longrightarrow \mathbf{T}$ be a monad morphism which extends oplax. Then there exists a concrete functor $\lambda_\diamond^t : \mathbf{Alg}_t\mathbf{S} \longrightarrow \mathbf{Alg}_t\mathbf{T}$ such that $(\lambda_\diamond^t, \lambda^\diamond)$ is a Galois correspondence.*

If in addition d and e extend oplax, then we also have a concrete functor $\lambda_\diamond^u : \mathbf{Alg}_u\mathbf{S} \longrightarrow \mathbf{Alg}_u\mathbf{T}$ such that $(\lambda_\diamond^u, \lambda^\diamond)$ is a Galois correspondence.

Proof Our assumptions imply that there is a diagram

$$\begin{array}{ccc}
\mathbf{Alg}_t\mathbf{S} & \underset{L_S}{\overset{E_S}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} & \mathbf{Alg}_r\mathbf{S} \\
{\scriptstyle \lambda^\diamond}\uparrow & & \downarrow{\scriptstyle \lambda_\diamond} \dashv \uparrow{\scriptstyle \lambda^\diamond} \\
\mathbf{Alg}_t\mathbf{T} & \underset{L_T}{\overset{E_T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} & \mathbf{Alg}_r\mathbf{T}
\end{array}$$

such that $E_S \cdot \lambda_\diamond \approx \lambda_\diamond \cdot E_T$ holds. Define $\lambda^t_\diamond = L_T \cdot \lambda_\diamond \cdot E_S$. Applying (the dual of) Lemma 1.2.12 "fibre-wise" gives the result.

$\lambda^u_\diamond$ is constructed analogously. ∎

The existence of the functor $\lambda^t_\diamond$ was noted in [CHT04] using the fact that $\lambda^\diamond : \mathbf{Alg}_t\mathsf{T} \longrightarrow \mathbf{Alg}_t\mathsf{S}$ preserves initial sources. Clearly Proposition 3.7.8 applies to $\mathbf{1} \xrightarrow{e} \mathbf{U}$, and we obtain an embedding $E : \mathbf{Ord} \longrightarrow \mathbf{Top}$. We are now going to use the construction outlined in the proof of Proposition 3.7.8 to describe E. First we note:

Remark 3.7.9 Let $\mathcal{S}$ be a topological scene, $\mathsf{T} = (T, e, m)$ an actor monad on $\mathcal{S}$ such that e extends oplax and T extends to a lax functor. Take $(X, a) \in \mathbf{Alg}_t\mathbf{1}_\mathcal{S}$. If $e_\diamond(X, a)$ is unitary, then it is transitive.

Proof The assumptions imply $a \cdot e^*_X \cdot T(a \cdot e^*_X) \leq e^*_X \cdot Ta \cdot T(a \cdot e^*_X) \leq e^*_X \cdot T(a \cdot a \cdot e^*_X) \leq e^*_X \cdot T(a \cdot e^*_X)$, thus the result is immediate. ∎

Let us return the situation of topological spaces. Since $\mathbf{U}$ satisfies the premises of Proposition 3.4.18, we see that $e^t_\diamond(X, a) = (X, e^*_X \cdot Ua)$ holds for an ordered set (X, a). The convergence structure $\to := e^\circ_X \cdot Ua$ can be described as follows: we have $\mathfrak{x} \to y$ iff $\mathfrak{x}\,(Ua)\,e_X(y)$ iff $\forall A \in \mathfrak{x}\, \exists x \in A : x\,a\,y$. To establish the last equivalence use (3.4).

We claim that the topology τ_a determined by $\to$ is just $\mathbb{D}(X, a)$. Recall (from, e.g. [HT06]) that $A \subset X$ is in τ_a iff for all $\mathfrak{x} \in UX$ with $\mathfrak{x} \to x$ for some $x \in A$, we have $A \in \mathfrak{x}$. Assume $A \in \tau_a$, $y \in A$, $x\,a\,y$. Since $a \cdot e^\circ_X \subset e^\circ_X \cdot Ua$ holds, we have $e_X(x) \to y$, thus $A \in e_X(x)$ by openness of A, hence $x \in A$, and $A \in \mathbb{D}(X, a)$. Conversely, assume $A \in \mathbb{D}(X, a)$ and $\mathfrak{x} \to x$ with $x \in A$. That is, for all $B \in \mathfrak{x}$ there exists $y \in B$ with $y\,a\,x$. Thus $B \cap A \neq \emptyset$ holds for all $B \in \mathfrak{x}$ by down-closedness of A, and hence $A \in \mathfrak{x}$ since $\mathfrak{x}$ is an ultrafilter. We have shown:

Proposition 3.7.10 $e^t_\diamond : \mathbf{Ord} \longrightarrow \mathbf{Top}$ *assigns an ordered set* $(X, \leq)$ *the* Alexandroff *topological space* $(X, \mathbb{D}(X, \leq))$. ∎

Approach spaces The same reasoning as above applies to the extension $\mathbf{U}_{\overline{\mathbb{R}}_+}$ of $\mathbf{U}$ to $(\mathbf{Set}, \mathrm{Mat}\,\overline{\mathbb{R}}_+)$. To see this, observe that $[0, \infty]^{\mathrm{op}}$ is ccd, and hence $U_{\overline{\mathbb{R}}_+} U_{\overline{\mathbb{R}}_+} \xrightarrow{m} U_{\overline{\mathbb{R}}_+}$ is a natural transformation. Thus, we obtain a Galois correspondence

$$\mathbf{Ap} \underset{e^t_\diamond}{\overset{e^\diamond}{\rightleftarrows}} \mathbf{Met}.$$

$e_\diamond$ transforms an approach space (X, a) into the metric space (X, d) with d defined as $d(x, y) = a(e_X(x), y)$. Again, $e^t_\diamond$ can be described by $(X, d) \longmapsto (X, e^\circ_X \cdot U_{\overline{\mathbb{R}}_+} d)$. An analysis analogously to the one we did for topological spaces shows that $e^t_\diamond(X, d) = (X, a)$, with

$$a(\mathfrak{x}, x) = \inf_{A \in \mathfrak{x}} \sup_{x \in A} d(x, y),$$

where inf and sup refer to the natural order on $[0, \infty]$. See also [Low97] and [CHT04].

3.7.2 Change of base revisited

Let $\mathbf{C}$, $\mathbf{C}'$ be ordered categories. Given pseudo-monads $\mathbf{T} = (T, e, m)$ on $\mathbf{C}$ and $\mathbf{T}' = (T', e', m')$ on $\mathbf{C}'$, a pseudo-functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ is said to be *compatible* with $\mathbf{T}$, $\mathbf{T}'$ provided

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{C}' \\ {\scriptstyle T}\downarrow & \approx & \downarrow{\scriptstyle T'} \\ \mathbf{C} & \xrightarrow[F]{} & \mathbf{C}' \end{array}$$

and $Fe_X \approx e'_{FX}$, $Fm_X \approx m'_{FX}$ hold.

We write **MonActor** for the category whose objects are pairs $(\mathbf{T}, \mathcal{S})$, where $\mathcal{S}$ is a scene and $\mathbf{T}$ is a monadic actor on $\mathcal{S}$, and morphisms are based scenic functors which are compatible with the monadic actors in the sense defined above.

Proposition 3.7.11 *If $F : (\mathbf{T}, \mathcal{S}) \longrightarrow (\mathbf{T}', \mathcal{S}')$ is a morphism in* **MonActor** *then* $\mathbf{Alg}\, F : \mathbf{Alg}\mathbf{T} \longrightarrow \mathbf{Alg}\mathbf{T}'$ *preserves reflexivity. If $T'F \leq FT$ holds, then* $\mathbf{Alg}\, F$ *preserves unitarity. If $F : \mathbf{D} \longrightarrow \mathbf{D}'$ is a lax functor, then* $\mathbf{Alg}\, F$ *preserves transitivity.*

In this case the restriction $\mathbf{Alg}_t\mathbf{T} \longrightarrow \mathbf{Alg}_t\mathbf{T}'$ is a 2-functor. ∎

As an example, consider a monad $\mathbf{T} = (T, e, m)$ on **Set** and let $\mathbf{V}$ and $\mathbf{W}$ be quantales, $\nu : V \longrightarrow W$ a functor which preserves k and $\bot$. The induced based scenic functor $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V}) \xrightarrow{\nu} (\mathbf{Set}, \mathrm{Mat}\,\mathbf{W})$ is compatible with the extensions of $\mathbf{T}$ to $\mathrm{Mat}\,\mathbf{V}$ and $\mathrm{Mat}\,\mathbf{W}$. ν fulfills the additional hypotheses of Proposition 3.7.11 if ν preserves infima and laxly preserves the tensor product (see Propositions 2.3.26 and 1.5.18).

Consider for example the embedding $\iota : \mathbf{2} \longrightarrow \overline{\mathbb{R}}_+$. As we have already seen ι has both a left o and a right adjoint p and we obtain lax functors

$$\iota : \mathbf{Rel} \longrightarrow \mathrm{Mat}\,\overline{\mathbb{R}}_+, \qquad o : \mathrm{Mat}\,\overline{\mathbb{R}}_+ \longrightarrow \mathbf{Rel}, \qquad p : \mathrm{Mat}\,\overline{\mathbb{R}}_+ \longrightarrow \mathbf{Rel}$$

These functors induce Galois correspondences as in the commutative diagram

$$\begin{array}{ccccc} \mathbf{Ord} & \hookrightarrow & \mathbf{RBin} & \hookrightarrow & \mathbf{Bin} \\ {\scriptstyle O}\uparrow\dashv\uparrow{\scriptstyle P} & & {\scriptstyle O}\uparrow\dashv\uparrow{\scriptstyle P} & & {\scriptstyle O}\uparrow\dashv\uparrow{\scriptstyle P} \\ \mathbf{Met} & \hookrightarrow & \mathbf{Alg}_r(\mathbf{1}, \mathrm{Mat}\,\overline{\mathbb{R}}_+) & \hookrightarrow & \mathbf{Alg}(\mathbf{1}, \mathrm{Mat}\,\overline{\mathbb{R}}_+) \end{array}$$

If we try to apply the same reasoning to the ultrafilter-monad $\mathbf{U}$ on **Set**, we see that $O = \mathrm{Mat}\, o$ does neither preserve unitarity nor transitivity of $\mathbf{U}$-algebras since o does not preserve infima (cf. Proposition 2.3.26, see also [CHT04]). Thus, a priori we only obtain

$$\begin{array}{ccccccc} \mathbf{Top} & \hookrightarrow & \mathbf{PrTop} & \hookrightarrow & \mathbf{PsTop} & \hookrightarrow & \mathbf{URS} \\ \downarrow\dashv\uparrow{\scriptstyle P} & & \downarrow\dashv\uparrow{\scriptstyle P} & & {\scriptstyle O}\uparrow\dashv\uparrow{\scriptstyle P} & & {\scriptstyle O}\uparrow\dashv\uparrow{\scriptstyle P} \\ \mathbf{Ap} & \hookrightarrow & \mathbf{PrAp} & \hookrightarrow & \mathbf{PsAp} & \hookrightarrow & \mathbf{Alg}\,\mathbf{U}_{\overline{\mathbb{R}}_+} \end{array}$$

Observe that we obtain concrete left adjoints to the embeddings $\mathbf{Top} \hookrightarrow \mathbf{Ap}$ and $\mathbf{PrTop} \hookrightarrow \mathbf{PrAp}$ by composing $\mathbf{Ap} \hookrightarrow \mathbf{PsAp} \xrightarrow{O} \mathbf{PsTop}$ and $\mathbf{PrAp} \hookrightarrow \mathbf{PsAp} \xrightarrow{O} \mathbf{PsTop}$ with the reflectors $\mathbf{PsTop} \longrightarrow \mathbf{Top}$ and $\mathbf{PsTop} \longrightarrow \mathbf{PrTop}$, respectively, see [CHT04].

4 Further examples

4.1 Comparing Kleisli-scenes and distributors

Suppose we are given a 2-monad $\mathbf{T} = (T, e, m)$ on $\mathbf{Ord}$ such that each Tf has a right adjoint. Proposition 2.3.12 gives an extension of T to an actor on $\mathcal{D} = (\mathbf{Set}, \mathbf{Ord}, \mathbf{Dist})$, which we will simply denote by T. We can now study either

(1) lax algebras for the identity actor on the discrete scene $\mathrm{Klei}\,\mathbf{T} = (\mathbf{Set}, \mathbf{Ord}_{\mathbf{T}})$, or

(2) lax algebras for the actor T on $\mathcal{D}$.

In this section we will compare these lax algebras. As an example, we will study the filter-on-downsets monad $\mathbf{F}$ on $\mathbf{Ord}$. Here (1) leads to the description of topological spaces via neighborhood-filters, while (2) leads to the description of topological spaces via filter-convergence. This example was studied in [HT06].

The embedding

Fix a 2-monad $\mathbf{T} = (T, e, m)$ on $\mathbf{Ord}$ such that TX is skeletal for each ordered set X and Tf has a right adjoint for each f in $\mathbf{Ord}$. Under this condition, we can form the Kleisli-scene $\mathrm{Klei}\,\mathbf{T}$ of $\mathbf{T}$, and define a concrete "comparison" functor

$$\begin{aligned} \mathbf{Alg}(\mathbf{1}, \mathrm{Klei}\,\mathbf{T}) &\xrightarrow{K} \mathbf{Alg}(\mathbf{T}, \mathcal{D}), \\ (X, X \xrightarrow{a} TX) &\longmapsto (X, TX \xrightarrow{a^*\!\!\!\!\!\rightarrow} X). \end{aligned}$$

Here a^* is the right adjoint distributor defined by the (monotone) function a, that is,

$$\mathfrak{x}\, a^* x \iff \mathfrak{x} \leq a(x).$$

Observe that we have, for any $X \xrightarrow{a} TY$, $Y \xrightarrow{b} TZ$ in $\mathbf{Ord}$,

$$(b * a)^* = (m_Z \cdot Tb \cdot a)^* = a^* \cdot Tb^* \cdot m_Z^* = a^* * b^*, \tag{4.1}$$

where the composition on the right denotes Hofmann's coKleisli-composition. Thus the restriction of K to the fibers is an anti-homomorphism with respect to the (Kleisli-) composition and the coKleisli-composition, that is, the coKleisli-composition is just the extension "along $(\ )^*$" of the Kleisli-composition to arbitrary distributors. In particular, using "$*$" to denote either of them should not cause any confusion.

Proposition 4.1.1 *K is a full embedding and each $K(X,a)$ is unitary. $K(X,a)$ is reflexive (transitive) if and only if (X,a) is reflexive (transitive). The restriction of K defines a locally fully-faithful 2-functor $(\mathbf{Alg}_t(\mathbf{1}, \mathrm{Klei}\,\mathbf{T}))^{\mathrm{co}} \longrightarrow \mathbf{Alg}_t(\mathbf{T}, \mathcal{D})$.*

Proof Let (X,a), $(Y,b) \in \mathbf{Alg}(\mathbf{1}, \mathrm{Klei}\,\mathbf{T})$. A function $X \xrightarrow{f} Y$ is a $\mathrm{Klei}\,\mathbf{T}$-functor $(X,a) \longrightarrow (Y,b)$ iff $Tf \cdot a \leq b \cdot f$ holds in **Ord**. This is equivalent to $(Tf \cdot a)^* \leq (b \cdot f)^*$ in **Dist**, and hence to $f_* \cdot a^* \leq b^* \cdot Tf_*$ in **Dist**, that is, to the fact that f is a $\mathbf{Alg}(\mathbf{T}, \mathcal{D})$ morphism $(X, a^*) \longrightarrow (Y, b^*)$.

Since each TX is ordered anti-symmetrically, $(\ )^*$ is an embedding, and thus also K is one. In $\mathbf{Ord}_{\mathbf{T}}$ $a * e_X = a$ holds trivially, and $e_X^* * a^* = a^*$ follows by (4.1). Therefore $K(X,a)$ is unitary.

(X,a) is reflexive iff $e_X \leq a$ holds in **Ord**, iff $e_X^* \leq a^*$ holds in **Dist**, that is, iff $K(X,a)$ is reflexive. Using (4.1), we see that K preserves and reflects transitivity.

Finally, for $f, g : (X,a) \longrightarrow (Y,b)$ in $\mathbf{Alg}_t(\mathbf{1}, \mathrm{Klei}\,\mathbf{T})$, we have $f \Rightarrow g$ iff $e_Y \cdot g \leq b \cdot f$ pointwise in **Ord**, iff $g^* \cdot e_Y^* \leq f^* \cdot b^*$ in **Dist**, iff $f_* \leq b^* \cdot e_Y \cdot g$ in **Dist**, iff $g \Rightarrow f$ in $\mathbf{Alg}_t(\mathbf{T}, \mathcal{D})$. ∎

We will describe the image of K. Observe that any a^* has the property that whenever $x \in X$ and $\mathcal{A} \subset TX$ satisfy $\mathfrak{a}\, a^*\, x$ for all $\mathfrak{a} \in \mathcal{A}$ and $\bigvee \mathcal{A}$ exists in TX, then also $(\bigvee \mathcal{A})\, a^*\, x$. This leads us to the following definition due to Seal [Sea05b]:

Definition 4.1.2 Assume TX to be a complete lattice for all X. We call $(X,r) \in \mathbf{Alg}(\mathbf{T}, \mathcal{O}(\mathbf{V}))$ *continuous* provided

$$r(\bigvee \mathcal{A}, x) = \bigwedge_{\mathfrak{a} \in \mathcal{A}} r(\mathfrak{a}, x) \tag{4.2}$$

holds for all $\mathcal{A} \subset TX$, $x \in X$.

"$\leq$" is always satisfied in (4.2) since r is a distributor. Thus, in case $\mathbf{V} = \mathbf{2}$, (4.2) is in fact equivalent to

$$\mathfrak{a}\, r\, x \quad \text{for all } \mathfrak{a} \in \mathcal{A} \quad \Longrightarrow \quad (\bigvee \mathcal{A})\, r\, x.$$

Lemma 4.1.3 *If TX is complete for all X, then $(X,r) \in \mathbf{Alg}(\mathbf{T}, \mathcal{D})$ is in the image of K if and only if it is continuous. Thus K defines an isomorphism between $\mathbf{Alg}(\mathbf{1}, \mathrm{Klei}\,\mathbf{T})$ and the full subcategory of $\mathbf{Alg}(\mathbf{T}, \mathcal{D})$ spanned by the continuous lax algebras. In particular, every continuous lax algebra is unitary.*

Proof Clearly every $K(X,a)$ is continuous. If (X,r) is continuous, we define a (monotone) function $X \xrightarrow{a} TX$ by $a(x) = \bigvee \{\, \mathfrak{x} \in TX \mid \mathfrak{x}\, r\, x \,\}$. $r = a^*$ follows by continuity of (X,r). ∎

Of course these results are not useful if we have no efficient way to determine the continuous lax algebras. As it turns out, in some cases unitarity is equivalent to continuity.

Proposition 4.1.4 *Suppose that each TX is a complete lattice, and each Tf as well as each m_X preserve suprema. For the extension of $\mathbf{T}$ to a monadic actor on $\mathcal{D}$ every unitary lax algebra is continuous.*

We will first prove a lemma which might be of independent interest. For the rest of this section we will assume that T fulfills the premises of Proposition 4.1.4. Recall that, by Proposition 2.3.12, the extension $\tilde{T}$ of T to **Dist** is defined as $\tilde{T}(r) = (Tr_1)_* \cdot (Tr_0)^*$, where (r_0, r_1) are the "legs" of the relation r. That is, $\mathfrak{x}\,\tilde{T}r\,\mathfrak{y}$ iff there exists $\mathfrak{a} \in Tr$, $\mathfrak{x} \leq Tr_0(\mathfrak{a})$, $Tr_1(\mathfrak{a}) \leq \mathfrak{y}$, where $X \xleftarrow{r_0} r \xrightarrow{r_1} Y$ are the projections. A proof analogous to the one of Proposition 1.5.6 shows that e_X extends to an oplax transformation $1_{\mathbf{Dist}} \longrightarrow \tilde{T}$.

Lemma 4.1.5 *For each distributor $X \xrightarrow{r} Y$ and each $\mathcal{A} \subset TX$, $\mathfrak{y} \in TY$, we have*

$$\bigcap_{\mathfrak{x}\in\mathcal{A}} \tilde{T}r(\mathfrak{x}, \mathfrak{y}) = \tilde{T}r\big(\bigvee \mathcal{A}, \mathfrak{y}\big). \tag{4.3}$$

Proof Since $\tilde{T}r$ is a distributor, we only need to show $\bigcap_{\mathcal{A}} \tilde{T}r(\mathfrak{x}, \mathfrak{y}) \subset \tilde{T}r\big(\bigvee \mathcal{A}, \mathfrak{y}\big)$. Suppose $\mathfrak{x}\,\tilde{T}r\,\mathfrak{y}$ holds for all $\mathfrak{x} \in \mathcal{A}$, that is, for each $\mathfrak{x} \in \mathcal{A}$ there exists $\mathfrak{r}_\mathfrak{x} \in Tr$ with $\mathfrak{x} \leq Tr_0(\mathfrak{r}_\mathfrak{x})$, $Tr_1(\mathfrak{r}_\mathfrak{x}) \leq \mathfrak{y}$. Set $\mathfrak{r} = \bigvee \mathfrak{r}_\mathfrak{x} \in Tr$. Then $\bigvee \mathcal{A} \leq \bigvee Tr_0(\mathfrak{r}_\mathfrak{x}) = Tr_0(\bigvee \mathfrak{r}_\mathfrak{x}) = Tr_0(\mathfrak{r})$ and $Tr_1(\mathfrak{r}) = \bigvee Tr_1(\mathfrak{r}_\mathfrak{x}) \leq \mathfrak{y}$ since Tr_0, Tr_1 preserve suprema, and hence $(\bigvee \mathcal{A})\,\tilde{T}r\,\mathfrak{y}$. ■

Analogous results can be proved for non-empty (or finite, etc.) suprema, in case each Tf preserves these. One possible way to formalize this would be the use of KZ-monads.

Proof (of Proposition 4.1.4) Assume (X, a) to be unitary. Fix any $\mathcal{A} \subset TX$, $x \in X$. Define $\mathfrak{X} = \bigvee_{\mathfrak{x}\in\mathcal{A}} e_{TX}(\mathfrak{x}) \in TTX$. We have $m_X(\mathfrak{X}) = m_X(\bigvee e_{TX}(\mathfrak{x})) = \bigvee m_X \cdot e_{TX}(\mathfrak{x}) = \bigvee \mathcal{A}$. Since e_X extends oplax, we have $a \leq e_X^* \cdot \tilde{T}a \cdot e_{TX}$, and thus

$$\begin{aligned}
\bigwedge_{\mathcal{A}} a(\mathfrak{x}, x) &\leq \bigwedge_{\mathcal{A}} \tilde{T}a(e_{TX}(\mathfrak{x}),\, e_X(x)) \\
&= \tilde{T}a\big(\bigvee\nolimits_{\mathcal{A}} e_{TX}(\mathfrak{x}),\, e_X(x)\big) && \text{Lemma 4.1.5} \\
&= e_X^* \cdot \tilde{T}a(\mathfrak{X}, x) \\
&\leq a \cdot m_X(\mathfrak{X}, x) && (X, a) \text{ unitary} \\
&= a\big(\bigvee \mathcal{A}, x\big),
\end{aligned}$$

where we have used "$\leq$" instead of "$\subset$". The other inequality holds by monotonicity of r. ■

The following result follows immediately from Propositions 4.1.1 and 4.1.4:

Theorem 4.1.6 *If* T *fulfills the conditions of Proposition 4.1.4, then* $\mathbf{Alg}(\mathbf{1}, \mathrm{Klei}\,\mathsf{T})$ *is, via* K*, concretely isomorphic to the full subcategory of* $\mathbf{Alg}(\mathsf{T}, \mathcal{D})$ *spanned by the unitary lax algebras. Moreover,* K *restricts to concrete isomorphisms*

$$\mathbf{Alg}_r(\mathbf{1}, \mathrm{Klei}\,\mathsf{T}) \cong \mathbf{Alg}_u(\mathsf{T}, \mathcal{D}), \qquad \mathbf{Alg}_t(\mathbf{1}, \mathrm{Klei}\,\mathsf{T}) \cong \mathbf{Alg}_t(\mathsf{T}, \mathcal{D}). \qquad \blacksquare$$

A counterexample Let $\mathbb{U}$ be the upper-set monad on **Ord**. The extension of the functor $\mathbb{U}$ to **Dist** is a 2-functor, each $\mathbb{U}X$ is complete, and each $m_X : \mathbb{U}\mathbb{U}X \longrightarrow \mathbb{U}X$ preserves suprema. We have seen (Remark 3.4.11) that every lax $\mathbb{U}$-algebra is unitary.

But Proposition 4.1.4 does not apply. Although every $\mathbb{U}f$ has a *left* adjoint, it does not in general preserve suprema.

On the other hand, a (not necessarily reflexive or transitive) lax algebra for $\mathbb{U}$ on a set X is just a "monotone operator" $c : PX \longrightarrow PX$, and continuity of the algebra means that for every $x \in X$ there is a smallest (w.r.t. $\subset$) $A \subset X$ with $x \in cA$. Thus the category of continuous lax algebras for $\mathbb{U}$ is the category **Bin** of thin graphs.

4.2 Level algebras and tower-extensions

Let $\mathbf{V}$ be a non-trivial quantale and fix a monadic actor $\mathsf{T} = (T, e, m)$ on $\mathcal{D}$. We denote the extension of T to $\mathcal{O}(\mathbf{V})$ defined in Section 2.3.4 by $T_{\mathbf{V}}$.

Level algebras Recall that for a $\mathbf{V}$-matrix $a : X \nrightarrow Y$, we denote by $a_v \subset X \times Y$ the *level*-relation defined by $x\, a_v\, y$ if and only if $v \leq a(x, y)$.

Proposition 4.2.1 *For* $v \in V$ *the assignment* $(X, a) \longmapsto (X, a_v)$ *defines a concrete functor* $L_v : \mathbf{Alg}\, T_{\mathbf{V}} \longrightarrow \mathbf{Alg}\, T$*. We call* $L_v(X, a)$ *the* level-algebra *of* (X, a) *at* v*.*

Proof First recall that a_v is a distributor provided a is an ordered matrix. Suppose that $(X, a) \xrightarrow{f} (Y, b)$ is a morphism in $\mathbf{Alg}\mathsf{T}_{\mathbf{V}}$, that is, $f_* \cdot a \leq b \cdot Tf_*$. Then,

$$\begin{aligned}
f_* \cdot a_v &\subset (f_* \cdot a)_v && \text{Lemma 2.3.13(3)} \\
&\subset (b \cdot Tf_*)_v && \text{assumption and Lemma 2.3.13(1)} \\
&= (b \cdot Tf)_v && \text{since } b \text{ is an ordered matrix} \\
&= b_v \cdot Tf && \text{Lemma 2.3.13(4)} \\
&= b_v \cdot Tf_* && \text{since } b_v \text{ is a distributor.}
\end{aligned}$$

$\blacksquare$

Proposition 4.2.2 *The following statements hold:*

(1) *If each* TX *is complete, then every* L_v *preserves continuity of lax algebras;*

(2) *L_v preserves reflexivity if and only if $v \leq k_{\mathbf{V}}$;*

(3) *(X, a) in $\mathbf{Alg}\mathsf{T}_{\mathbf{V}}$ is unitary if and only if (X, a_v) is unitary for each $v \in V$.*

Proof Let (X, a) be a lax $\mathsf{T}_{\mathbf{V}}$-algebra.

[1] Suppose that (X, a) is continuous and $\mathcal{A} \subset TX$. Assume $\mathfrak{x}\, a_v\, x$ for all $\mathfrak{x} \in \mathcal{A}$. Then $v \leq \bigwedge_{\mathfrak{x}\in\mathcal{A}} a(\mathfrak{x}, x) = a(\bigvee \mathcal{A}, x)$ by continuity.

[2] For $v \leq k_{\mathbf{V}}$ preservation of reflexivity follows from Remark 2.3.14. Conversely, let $X = \{x\}$ and take any $v \in V$. If L_v preserves reflexivity, then in particular $L_v(X, e_X^*)$ is reflexive, and $v \leq e_X^*(e_X(x), x) = k_{\mathbf{V}}$ holds.

[3] "Only if" is obvious. For the converse implication take any $\mathfrak{X} \in TTX$ and $x \in X$. Since a and $T_{\mathbf{V}}a$ are distributors we only need to show that $T_{\mathbf{V}}a(\mathfrak{X}, e_X(x)) \leq a(m_X(\mathfrak{X}), x)$ holds. By definition of $T_{\mathbf{V}}$ it suffices to show $v \leq a(m_X(\mathfrak{X}), x)$ for all $v \in V$ with $\mathfrak{X}\, Ta_v\, e_X(x)$. Take any such v. Since (X, a_v) is unitary by assumption we obtain $m_X(\mathfrak{X})\, a_v\, x$, equivalently, $v \leq a(m_X\mathfrak{X}, x)$. ■

L_v does not need to preserve transitivity. Consider for instance the metric space (X, d) given by $X = \{\, x, y, z \,\}$ and

d	x	y	z
x	0	2	3
y	2	0	2
z	3	2	0

Since (recall the order on $\overline{\mathbb{R}}_+$) $x\, d_r\, y$ iff $d(x, y) \leq r$ for all $r \in \overline{\mathbb{R}}_+$, we have $x\, d_2\, y$, $y\, d_2\, z$, but $x \not d_2\, z$, and thus d_2 is not transitive.

Continuity for actors on $\mathcal{O}(\mathbf{V})$ For this section, additionally assume T to be a 2-monad on **Ord** such that TX is a complete lattice for all ordered sets X and that all Tf and m_X preserve suprema.

Proposition 4.2.3 *Every continuous lax $\mathsf{T}_{\mathbf{V}}$-algebra (X, a) is unitary.*

Proof Suppose (X, a) to be continuous. Then each $L_v(X, a)$ is continuous by Proposition 4.2.2(1). By Proposition 4.1.4, each $L_v(X, a)$ is unitary, and thus, by Proposition 4.2.2(3), also (X, a) is unitary. ■

Lemma 4.2.4 *If V is ccd, we have*

$$\bigwedge_{\mathfrak{x}\in\mathcal{A}} T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y}) = T_{\mathbf{V}}a\big(\bigvee \mathcal{A}, \mathfrak{y}\big)$$

for each distributor $X \xrightarrow{a}\!\!\!\!\rightarrow Y$ and all $\mathcal{A} \subset TX$, $\mathfrak{y} \in TY$.

Proof Since $T_{\mathbf{V}}$ is a distributor, we only have to show "$\leq$". As usual, we write $B \dashv \bigvee : \mathbb{D}V \longrightarrow V$. With $\alpha_{\mathfrak{x}} := T_{\mathbf{V}}a(\mathfrak{x}, \mathfrak{y})$ and $W := \{\, v \in V \mid \bigvee \mathcal{A}\, \tilde{T}a_v\, \mathfrak{y} \,\}$ we need to show $\bigwedge_{\mathfrak{x}\in\mathcal{A}} \alpha_{\mathfrak{x}} \leq \bigvee W$, equivalently $B(\bigwedge \alpha_{\mathfrak{x}}) \subset W$. Take $v \in B(\bigwedge_{\mathcal{A}} \alpha_{\mathfrak{x}}) \subset \bigcap_{\mathcal{A}} B(\alpha_{\mathfrak{x}})$. We have to show $\bigvee \mathcal{A}\, \tilde{T}a_v\, \mathfrak{y}$.

For each $\mathfrak{x} \in \mathcal{A}$, we have $v \leq \alpha_{\mathfrak{x}} = \bigvee\{\, w \in V \mid \mathfrak{x}\, \tilde{T}a_w\, \mathfrak{y} \,\}$, which together with $v \in B(\alpha_{\mathfrak{x}})$ implies $\mathfrak{x}\, \tilde{T}a_v\, \mathfrak{y}$. Since (4.3) holds for $\tilde{T}a_v$, we thus have $v \in W$. ∎

Theorem 4.2.5 *Assume V to be ccd. An algebra in $\mathbf{Alg}(T_{\mathbf{V}}, \mathcal{O}(\mathbf{V}))$ is continuous if and only if it is unitary.*

Proof The proof of Proposition 4.1.4 (with Lemma 4.2.4 instead of Lemma 4.1.5) gives "if". "Only if" follows from Proposition 4.2.3. ∎

Tower extensions

We recall the notion of tower-extension from [Zha00]:

Definition 4.2.6 Let $\mathbf{X} \xrightarrow{P} \mathbf{B}$ be a fiber-small topological functor and V be a complete lattice. The *tower-extension* $\mathbf{X}[V]$ of $\mathbf{X}$ by V is defined as the following category:

- objects are *towers* of $\mathbf{X}$-objects, that is, pairs (B, ϕ), where B is an object of $\mathbf{B}$ and $\phi : V \longrightarrow P_B^{\mathrm{op}}$ preserves suprema;
- a morphism $(B, \phi) \xrightarrow{f} (B', \phi')$ is given by a $\mathbf{B}$-morphism $f : B \longrightarrow B'$ such that $\phi(v) \xrightarrow{f} \phi'(v)$ is an $\mathbf{X}$-morphism for all $v \in V$.

Here, P_B^{op} is the fiber of P over B, ordered by $X \leq X'$ iff $X' \xrightarrow{1_B} X$ is an $\mathbf{X}$-morphism. $\mathbf{X}[V]$ becomes a concrete category by the obvious forgetful functor $\mathbf{X}[V] \longrightarrow \mathbf{B}$. For a tower (X, ϕ), we call $\phi(v)$ the *component* of (X, ϕ) at $v \in V$. Assigning a tower its v-component defines the concrete *component-functors* $C_v : \mathbf{X}[V] \longrightarrow \mathbf{X}$.

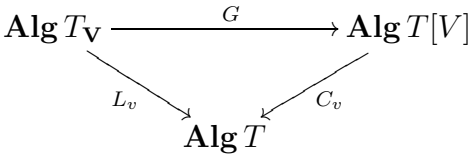
Since the powerset PA is the free **Sup**-lattice on a set A, we can describe $\mathbf{X}[PA]$ as follows: objects are pairs $(B, (X_a)_A)$, where B is a $\mathbf{B}$-object and (X_a) is an A-indexed family of $\mathbf{X}$-objects over B. Morphisms $f : (B, (X_a)) \longrightarrow (B', (X'_a))$ are $\mathbf{B}$-morphisms $f : B \longrightarrow B'$ such that $f : X_a \longrightarrow X'_a$ is an $\mathbf{X}$-morphism for all $a \in A$. In particular, for the two-element chain $\mathbf{2}$, we have a concrete isomorphism $\mathbf{X}[\mathbf{2}] \cong \mathbf{X}$.

Lax algebras are tower extensions Fix a 2-monad $\mathbf{T} = (T, e, m)$ on **Ord**. As usual, $T_{\mathbf{V}}$ denotes the extension of T to $\mathcal{O}(\mathbf{V})$ for a quantale $\mathbf{V}$. Although we will formulate the results in this section for scenes of the type $\mathcal{O}(\mathbf{V})$, all of them remain valid over $(\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$.

We can, by Lemma 2.3.13(1) and Proposition 4.2.1, define a concrete functor

$$G : \mathbf{Alg}\, T_{\mathbf{V}} \longrightarrow \mathbf{Alg}\, T[V], \qquad (X, a) \longmapsto (X, (v \mapsto a_v)).$$

Proposition 4.2.7 *G is an isomorphism and for each $v \in V$ the following diagram commutes:*



Proof G^{-1} sends (X, ϕ) to $(X, \Gamma\phi)$, where $\Gamma\phi(\mathfrak{x}, x) = \bigvee\{\, v \in V \mid \mathfrak{x}\, \phi(v)\, x \,\}$. Clearly G^{-1} is functorial.

We have $\Gamma(v \mapsto a_v)(\mathfrak{x}, x) = \bigvee\{\, w \in V \mid \mathfrak{x}\, a_w\, x \,\} = a(\mathfrak{x}, x)$ for each $\mathbf{V}$-matrix a, thus $G^{-1}G = 1$. Now let $\phi : V \longrightarrow (\mathbf{Dist}(TX, T))^{\mathrm{op}}$ preserve suprema. We need to show $(\Gamma\phi)_v = \phi(v)$ for each $v \in V$. Clearly, $\phi(v) \subset (\Gamma\phi)_v$ holds. Take $x \in X$, $\mathfrak{x} \in TX$ and write $W = \{\, v \in V \mid \mathfrak{x}\, \phi(v)\, x \,\}$. W is a downset, and since ϕ preserves suprema we have $(\mathfrak{x}, x) \in \phi(\bigvee W)$, thus $\bigvee W \in W$. Hence, $v \leq \bigvee W$ implies $v \in W$, and $(\Gamma\phi)_v \subset \phi(v)$ follows. ∎

Proposition 4.2.8 *For $(X, \phi) \in \mathbf{Alg}\, T[V]$, we have:*

(1) *$G^{-1}(X, \phi) \in \mathbf{Alg}\, T_{\mathbf{V}}$ is reflexive if and only if $(X, (\phi(k)) \in \mathbf{Alg}\, T$ is reflexive.*

(2) *$G^{-1}(X, \phi) \in \mathbf{Alg}\, T_{\mathbf{V}}$ is unitary if and only if for each $v \in V$, $(X, \phi(v)) \in \mathbf{Alg}\, T$ is unitary.*

Proof Follows from 4.2.2(2) and (3). ∎

Corollary 4.2.9 *Let $\mathbf{V}$ be integral. G and G^{-1} restrict to concrete isomorphisms*

$$\mathbf{Alg}_r T_{\mathbf{V}} \cong \mathbf{Alg}_r T[\mathbf{V}] \qquad \textit{and} \qquad \mathbf{Alg}_u T_{\mathbf{V}} \cong \mathbf{Alg}_u T[\mathbf{V}].$$ ∎

Proposition 4.2.10 *Let $(X, a) \in \mathbf{Alg}\, T_{\mathbf{V}}$ be transitive. For each $u, v \in V$ we have $a_u * a_v \subset a_{u \otimes v}$, where $*$ denotes the coKleisli-composition.*

Proof Recall that $T(a_v) \subset (T_{\mathbf{V}} a)_v$ holds for $v \in V$. Hence, it suffices to show $a_u \cdot (T_{\mathbf{V}} a)_v \subset a_{u \otimes v} \cdot m_X$. Take any $\mathfrak{X} \in TTX$, $x \in X$, and assume $\mathfrak{X}\, (a_u \cdot (T_{\mathbf{V}} a)_v)\, x$. Thus, there exists $\mathfrak{x} \in TX$ with $u \leq a(\mathfrak{x}, x)$ and $v \leq T_{\mathbf{V}} a(\mathfrak{X}, \mathfrak{x})$. In particular,

$$u \otimes v \leq \bigvee_{\mathfrak{x} \in TX} a(\mathfrak{x}, x) \otimes T_{\mathbf{V}} a(\mathfrak{X}, \mathfrak{x}) = a \cdot T_{\mathbf{V}} a(\mathfrak{X}, x) \leq a \cdot m_X(\mathfrak{X}, x).$$

Since $(a \cdot m_X)_{u \otimes v} = a_{u \otimes v} \cdot m_X$ holds, we are done. ∎

The two preceding results motivate the following definition:

Definition 4.2.11 Let us call $(X, \phi) \in \mathbf{Alg}\, T[V]$

- *coherent* if $\phi(u) * \phi(v) \subset \phi(u \otimes v)$ holds for all $u, v \in V$, and
- *normal* if $e_X^* \subset \phi(k_{\mathbf{V}})$ holds.

Thus (X, ϕ) is normal and coherent iff $\phi : \mathbf{V} \longrightarrow \mathbf{Dist}(TX, X)^{\mathrm{op}}$ is a lax monoidal functor, where $\mathbf{Dist}(TX, X)$ has the (not necessarily associative) coKleisli-composition as "tensor". Normality of (X, ϕ) is equivalent to reflexivity of $(X, \phi(v))$ for all $v \leq k_{\mathbf{V}}$.

Observe that if (X, ϕ) is coherent then $(X, \phi(u))$ is transitive for all *idempotent* $u \in V$. In particular, $(X, \phi(k))$ is transitive. In case $\mathbf{V}$ is a frame, we even have:

Lemma 4.2.12 *Assume that* $\mathbf{V}$ *is a frame, that is,* $\otimes = \wedge$ *holds. Then* $(X, \phi) \in \mathbf{Alg}\, T[V]$ *is coherent if and only if each* $(X, \phi(v))$ *is transitive.*

Proof "Only if" is obvious. To prove "if" take any $u, v \in V$. Write $w = u \wedge v$, thus $\phi(u) \subset \phi(w)$ and $\phi(v) \subset \phi(w)$ hold. This implies $\phi(u) * \phi(v) \subset \phi(w) * \phi(w) \subset \phi(w) = \phi(u \wedge v)$. ∎

Proposition 4.2.13 *A lax algebra* (X, a) *in* $\mathbf{Alg}\, T_{\mathbf{V}}$ *is transitive if and only if* $G(X, a)$ *is coherent. Thus* G *and* G^{-1} *restrict to isomorphisms between the full subcategory of* $\mathbf{Alg}\, T_{\mathbf{V}}$ *spanned by the transitive lax algebras and the full subcategory of* $\mathbf{Alg}\, T[V]$ *spanned by the coherent towers.*

Proof Assume $G(X, a) = (X, (a_v)_V)$ to be coherent. Clearly, it suffices to show $a(\mathfrak{x}, x) \otimes T_{\mathbf{V}} a(\mathfrak{X}, \mathfrak{x}) \leq a(m_X(\mathfrak{X}), x)$ for all $\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$, $x \in X$. Let us fix such $\mathfrak{X}$, $\mathfrak{x}$, x, and write $u = a(\mathfrak{x}, x)$, $W = \{\, w \in V \mid \mathfrak{X}\, T a_w\, \mathfrak{x} \,\}$. We have

$$a(\mathfrak{x}, x) \otimes T_{\mathbf{V}} a(\mathfrak{X}, \mathfrak{x}) = u \otimes \bigvee W = \bigvee_{w \in W} u \otimes w.$$

But for any $w \in W$, we have $(\mathfrak{X}, x) \in a_u \cdot T a_w = \phi(u) \cdot T\phi(w) \subset \phi(u \otimes w) \cdot m_X = a_{u \otimes v} \cdot m_X$, thus $u \otimes w \leq a(m_X(\mathfrak{X}), x)$, and the claim is proved.

The converse implication follows from Proposition 4.2.10. ∎

We summarize our results in the following theorem, which characterizes reflexive and transitive lax $T_{\mathbf{V}}$-algebras in terms of towers:

Theorem 4.2.14 $\mathbf{Alg}_t T_{\mathbf{V}}$ *is concretely isomorphic to the full subcategory of* $\mathbf{Alg}\, T[V]$ *spanned by the normal and coherent towers. If* $\mathbf{V}$ *is integral, then* $\mathbf{Alg}_t T_{\mathbf{V}}$ *is concretely isomorphic to the full subcategory of* $\mathbf{Alg}_r T[V]$ *spanned by the coherent towers.* ∎

In particular, if $(X, \phi) \in \mathbf{Alg}\, T[V]$ is normal and coherent, then each $(X, \phi(v))$ is unitary.

Corollary 4.2.15 *If* $\otimes$ *idempotent, then each* L_v *preserves transitivity. Moreover,* $\otimes = \wedge$ *implies* $\mathbf{Alg}_t \mathbf{T}[V] \cong \mathbf{Alg}_t \mathbf{T}_{\mathbf{V}}$.

Proof Follows immediately from Lemma 4.2.12. ∎

The result of this section were — in a slightly different setting — proved independently by Seal in [Sea05b], under the additional hypothesis that V is ccd. Seal was first to recognize the importance of what we choose to call *normal* towers.

4.3 An example: topological spaces via filter convergence

The goal of this section is to show how topological spaces can be described as lax algebras using filter-convergence instead of ultrafilter-convergence, thereby avoiding the use of the Axiom of Choice. For other approaches to the same characterization, see [Wyl91], [Pis99], [HT06], [Sea05a].

Let $\mathsf{F} = (F, e, m)$ be the filter-on-downsets monad on **Ord**. We have seen in Section 1.3.6 that F satisfies the premises of Theorem 4.1.6, which gives concrete isomorphisms $\mathbf{Alg}_r(\mathbf{1}, \operatorname{Klei}\mathsf{F}) \cong \mathbf{Alg}_u(\mathsf{F}, \mathcal{D})$ and $\mathbf{Alg}_t(\mathbf{1}, \operatorname{Klei}\mathsf{F}) \cong \mathbf{Alg}_t(\mathsf{F}, \mathcal{D})$. Since, by Proposition 1.6.15, we have $\mathbf{Alg}_t(\mathbf{1}, \operatorname{Klei}\mathsf{F}) \cong \mathbf{Top}$, and moreover $\mathbf{Alg}_r(\mathbf{1}, \operatorname{Klei}\mathsf{F}) \cong \mathbf{PrTop}$ (cf. Section 3.4.2), we have shown

Theorem 4.3.1 *We have concrete isomorphisms* $\mathbf{Alg}_u(\mathsf{F}, \mathcal{D}) \cong \mathbf{PrTop}$, $\mathbf{Alg}_t(\mathsf{F}, \mathcal{D}) \cong \mathbf{Top}$. ∎

Clearly, $\mathbf{Alg}_r(\mathsf{F}, \mathcal{D})$ is just the category **Conv** of *(filter) convergence spaces* (in the sense of, e.g., [Wyl91]). Thus our typical string (3.3) of lax algebras can for F be described as:

$$\mathbf{Top} \hookrightarrow \mathbf{PrTop} \hookrightarrow \mathbf{Conv} \hookrightarrow \mathbf{Alg}\,\mathsf{F}.$$

4.3.1 Extending the filter-functor to Dist

Although we did not need an explicit description of the extension of F to **Dist** to apply Theorem 4.1.6, we will now give one for further reference. As it turns out, this extension is a 2-functor.

Given a relation $r \subset X \times Y$ and $B \subset Y$, we write $r^{\triangleleft}B$ for $r^{\circ}[B] = \{\, x \in X \mid \exists b \in B : x\, r\, b \,\}$. Observe that $r^{\triangleleft}B$ is a downset for any subset $B \subset Y$ if r is a distributor. In particular, $\leq^{\triangleleft} B = B^{\downarrow}$ is the down-closure of B, and if $X \xrightarrow{f} Y$ is monotone and $B \in \mathbb{D}Y$, then $f_*^{\triangleleft}B = f^{-1}[B]$.

We define $F : \mathbf{Dist} \longrightarrow \mathbf{Dist}$ by

$$\mathfrak{x}\, Fr\, \mathfrak{y} \iff \forall B \in \mathfrak{y} : r^{\triangleleft}B \in \mathfrak{x}, \tag{4.4}$$

for any distributor $X \xrightarrow{r} Y$.

Lemma 4.3.2 *F is a 2-functor* $\mathbf{Dist} \longrightarrow \mathbf{Dist}$ *and the following diagram commutes.*

$$\begin{array}{ccc} \mathbf{Ord} & \xrightarrow{(\,)_*} & \mathbf{Dist} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{Ord} & \xrightarrow{(\,)_*} & \mathbf{Dist} \end{array} \tag{4.5}$$

Proof The identity on X is given by the order-relation $\leq_X$ and we have $\mathfrak{x}\, F(\leq_X)\, \mathfrak{y} \iff \forall B \in \mathfrak{y} : B \in \mathfrak{x} \iff \mathfrak{x} \leq \mathfrak{y}$ for all $\mathfrak{x}, \mathfrak{y} \in FX$, thus F preserves identities.

To show that F preserves composition, let $X \xrightarrow{r\ } Y \xrightarrow{\ s} Z$ be distributors and $\mathfrak{x} \in FX$, $\mathfrak{z} \in FZ$. Then

$$\mathfrak{x}\, F(s \cdot r)\, \mathfrak{z} \iff \forall C \in \mathfrak{z} : r^{\triangleleft}(s^{\triangleleft} C) \in \mathfrak{x}, \tag{4.6}$$

and

$$\begin{aligned} \mathfrak{x}\, (Fs \cdot Fr)\, \mathfrak{z} &\iff \exists \mathfrak{y} \in FY : \mathfrak{x}\, Fr\, \mathfrak{y},\ \mathfrak{y}\, Fs\, \mathfrak{z} \\ &\iff \exists \mathfrak{y} \in FY\, \forall B \in \mathfrak{y}\, \forall C \in \mathfrak{z}\, r^{\triangleleft} B \in \mathfrak{x},\ s^{\triangleleft} C \in \mathfrak{y}. \end{aligned}$$

Thus clearly $Fs \cdot Fr \subset F(s \cdot r)$ holds. To show the converse, suppose that for all $C \in \mathfrak{z}$, $r^{\triangleleft}(s^{\triangleleft} C) \in \mathfrak{x}$. Define $\mathfrak{b} \subset \mathbb{D}Y$ as $\mathfrak{b} = \{\, B \mid \exists C \in \mathfrak{z} : B = s^{\triangleleft} C \,\}$. Clearly $\mathfrak{b}$ is filtered, and hence $\mathfrak{y} = \mathfrak{b}^{\uparrow} \in FY$. For each $C \in \mathfrak{z}$, $s^{\triangleleft} C \in \mathfrak{y}$ holds by construction. On the other hand, for each $B \in \mathfrak{y}$, there exists $C \in \mathfrak{z}$ with $s^{\triangleleft} C \subset B$, thus $r^{\triangleleft}(s^{\triangleleft} C) \subset r^{\triangleleft} B$, and hence $r^{\triangleleft} B \in \mathfrak{x}$ by assumption. We have proved $F(s \cdot r) \subset Fs \cdot Fr$.

That F is locally monotone and makes (4.5) commutative is obvious. ∎

4.3.2 Other topological constructs via filter convergence

For a quantale $\mathbf{V}$ let $\mathsf{F}_{\mathbf{V}}$ denote the extension of F to a monadic actor on $\mathcal{O}(\mathbf{V})$, formed by extending $\mathbf{Dist} \xrightarrow{F} \mathbf{Dist}$ to an actor on $\mathcal{O}(\mathbf{V})$ according to the recipe of Section 2.3.4. We are going to describe the corresponding categories of lax algebras.

Approach spaces

Take $\mathbf{V} = \overline{\mathbb{R}}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ (cf. Examples 1.5.8(3)). Observe that $[0,\infty]$ (and hence $[0,\infty]^{\mathrm{op}}$) is ccd: the left adjoint B of $\bigvee$ is given by $B(a) = \{\, b \in [0,\infty] \mid b < a \,\}$.

Proposition 4.3.3 *We have concrete isomorphisms*

$$\mathbf{Alg}_u \mathsf{F}_{\overline{\mathbb{R}}_+} \cong \mathbf{PrAp} \qquad \textit{and} \qquad \mathbf{Alg}_t \mathsf{F}_{\overline{\mathbb{R}}_+} \cong \mathbf{Ap},$$

where $\mathbf{PrAp}$ *is the category of* pre-approach spaces *and* contractions *and* $\mathbf{Ap}$ *is the full subcategory of* approach spaces.

Proof We have $\mathbf{Alg}_u \mathsf{F}_{\overline{\mathbb{R}}_+} \cong \mathbf{Alg}_u \mathsf{F}[\overline{\mathbb{R}}_+] \cong \mathbf{PrTop}[\overline{\mathbb{R}}_+]$. By a result of Brock and Kent [BK97, Proposition 10] (see also [Zha00]) $\mathbf{PrTop}[\overline{\mathbb{R}}_+]$ is isomorphic to $\mathbf{PrAp}$.

Observe that by Theorem 4.2.14, $(X, a) \in \mathbf{Alg}_t \mathsf{F}_{\overline{\mathbb{R}}_+}$ is completely determined by a $[0,\infty]$-indexed family of pretopologies, which we describe as interior-operators i_ϵ, on X such that i_∞ is the discrete pretopology, we have $i_\epsilon = \bigcap_{\gamma > \epsilon} i_\gamma$ (here we make use of Remark 1.3.8) and $i_\epsilon \cdot i_\gamma \subset i_{\epsilon+\gamma}$ (here we use Lemma 1.6.16). But this data constitutes precisely the data used in [Low97] to define an approach space. ∎

Remark 4.3.4 Using Theorem 4.2.5 we can give a slightly different proof of the first isomorphism in Proposition 4.3.3: simply observe that, via Theorem 4.2.5 for a lax $\mathsf{F}_{\overline{\mathbb{R}}_+}$-algebra (X, a) unitarity is equivalent to continuity, and continuity is precisely condition (PRAL) of [LL89]. ■

Remark 4.3.5 Almost the same reasoning as in the proof of Proposition 4.3.3 can be employed to prove statements (2) and (3) of Theorem 3.4.5. In place of Lemma 1.6.16 one has to use [HT06, Theorem D]. ■

Other examples

Using the results of Zhang [Zha00] on tower extensions and our description of lax algebras over the scenes $\mathcal{O}(\mathbf{V})$ and $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ we obtain descriptions of various categories of lax algebras. For instance, we have for $\mathbf{V} = ([0,1], \wedge, 1)$

- $\mathbf{Alg}_r\mathsf{U}_{[0,1]}$ is concretely isomorphic to the category of *probabilistic pseudotopological spaces* of [BK97];
- $\mathbf{Alg}_u\mathsf{F}_{[0,1]}$ and $\mathbf{Alg}_u\mathsf{U}_{[0,1]}$ are concretely isomorphic to the category of *probabilistic pretopological spaces* of [BK97];
- $\mathbf{Alg}_t\mathsf{F}_{[0,1]}$ and $\mathbf{Alg}_t\mathsf{U}_{[0,1]}$ are concretely isomorphic to the category of *probabilistic topological spaces* of [BK97].

For more examples see [Zha00]. In all cases the proofs are routine using the methods already employed for the corresponding results concerning approach spaces. For an example which is not a chain, consider $\mathbf{V} = \mathbf{2^2} \cong P\mathbf{2}$. We have $\mathbf{Alg}_t\mathsf{F}_{\mathbf{2^2}} \cong \mathbf{Alg}_t\mathsf{U}_{\mathbf{2^2}} \cong \mathbf{BiTop}$, the category of bitopological spaces. This was first observed by G. Seal (personal communication, September 2006).

4.4 Back to Kleisli

We have seen in Section 4.1 that for a 2-monad T on $\mathbf{Ord}$ there is a concrete isomorphism between $\operatorname{Cat}\operatorname{Klei}\mathsf{T}$ and the category and reflexive and transitive lax algebras for the canonical extension of T to an actor on $\mathcal{D}$, provided each TX is complete and, crucially, each Tf and each m_X preserve suprema. Thus suitable completeness properties allow us to switch from a "neighborhood"-point of view (Kleisli-scenes) to a "convergence"-point of view (distributors), and vice versa. This correspondence breaks down if Tf does not preserve suprema, as we have seen for the upper-set monad $\mathbb{U}$.

We will now present a construction which allows us to complete a monad, so that the above-mentioned correspondence holds. In particular, we get a "neighborhood"-version of closure spaces. Our method is inspired by Seal's [Sea05b] use of up-closed families of subsets.

The setting Fix a 2-monad $\mathbf{D} = (D, d, n)$ on $\mathbf{Ord}$ such that each Df has a right adjoint D^*f and each n_X has a right adjoint n_X^*. We form the Kleisli-scene $\mathcal{K}(\mathbf{D}) = (\mathbf{Set}, \mathbf{Ord}, \mathbf{Ord}_{\mathbf{D}}^{\mathrm{op}})$. For simplicity, we write $\mathbf{D} = \mathbf{Ord}_{\mathbf{D}}^{\mathrm{op}}$. Let $\mathbf{T} = (T, e, m)$ be any 2-monad on $\mathbf{Ord}$ which extends to a 2-monad along $(\ )^* = F_{\mathbf{D}} : \mathbf{Ord} \longrightarrow \mathbf{D}$. That is, there exists a 2-functor $\tilde{T} : \mathbf{D} \longrightarrow \mathbf{D}$ such that $\tilde{\mathbf{T}} = (\tilde{T}, (e_X^*)_X, (m_X^*)_X)$ is a 2-monad on $\mathbf{Ord}_{\mathbf{D}}$ and $\tilde{T} \cdot F_{\mathbf{D}} = F_{\mathbf{D}} \cdot T$ holds.

The construction By Proposition 2.3.33 there exists a distributive law λ of $\mathbf{T}$ over $\mathbf{D}$ such that $\tilde{\mathbf{T}} = \mathbf{T}_\lambda$. Using λ, we form the composite monad $\mathbf{S} = (S, s, r) = (\mathbf{DT})_\lambda$.

Lemma 4.4.1 *Each Sf and each r_X has a right adjoint.*

Proof A right adjoint of $Sf = DTf$ is given by $D^*(Tf)$, and a right adjoint of $r_X = Dm_X \cdot n_{TTX} \cdot D\lambda_{TX}$ is given by $r_X^* = D^*(\lambda_{TX}) \cdot n_{TTX}^* \cdot D^*(m_X)$. ∎

Hence we can form the Kleisli-scene $\mathcal{K}(\mathbf{S})$. For this, we obtain:

Theorem 4.4.2 *There are concrete isomorphisms*

$$\mathbf{Alg}_u(\mathbf{S}, \mathcal{D}) \cong \mathbf{Alg}_r(\mathbf{1}, \mathcal{K}(\mathbf{S})) \cong \mathbf{Alg}_u(\mathbf{T}, \mathcal{K}(\mathbf{D})),$$
$$\mathbf{Alg}_t(\mathbf{S}, \mathcal{D}) \cong \mathbf{Alg}_t(\mathbf{1}, \mathcal{K}(\mathbf{S})) \cong \mathbf{Alg}_t(\mathbf{T}, \mathcal{K}(\mathbf{D})).$$

Proof The isomorphisms on the left-hand side follow from Theorem 4.1.6. By Lemma 2.3.34 we have $\mathbf{Ord}_{\mathbf{S}} \cong (\mathbf{Ord}_{\mathbf{D}})_{\mathbf{T}_\lambda} \cong (\mathbf{Ord}_{\mathbf{D}})_{\tilde{\mathbf{T}}}$, from which the other isomorphisms follow immediately. ∎

4.4.1 An example: closure spaces via neighborhoods

We will apply this method to $\mathbf{D} = \mathbb{D}$, thus $\mathcal{K}(\mathbf{D}) \cong \mathcal{D}$, and $\mathbf{T} = \mathbb{U}$. The upper-set monad $\mathbb{U}$ has an extension to a 2-monad on $\mathbf{Ord}_{\mathbb{D}}$, which was constructed using the distributive law r. With the help of r, we define the monad $\mathbb{DU} = (\mathbb{DU})_r$, which was already studied in Section 1.3.1. We recall that $\mathbf{Alg}_t(\mathbb{U}, \mathcal{D})$ is the category $\mathbf{Clos}$ of closure spaces. Hence Theorem 4.4.2 implies

$$\mathbf{Clos} \cong \mathrm{Cat}\,\mathrm{Klei}\,\mathbb{DU}.$$

We would like to point out that the results of Section 4.2 allow us deduce Seal's description [Sea05b] of $\mathbf{Alg}_t\mathbb{U}_{\mathbf{V}}$ as $\mathbf{V}$-graded family of closure operators for *any* (not necessarily ccd) quantale $\mathbf{V}$. Indeed, Seal's result follows immediately from Theorem 4.2.14 using Lemma 3.4.8 and Remark 1.3.8.

In the special case of $\mathbf{V}$ being a frame and the underlying lattice V being ccd, $\mathbb{U}$ extends also to a 2-monad on $\mathbf{Ord}_{D\mathbf{V}} \cong \mathrm{OrdMat}(\mathbf{V})^{\mathrm{op}}$. Thus we obtain a composite monad $D_{\mathbf{V}}\mathbb{U}$ and concrete isomorphisms

$$\mathrm{Cat}\,\mathrm{Klei}\,D_{\mathbf{V}}\mathbb{U} \cong \mathbf{Alg}_t(\mathbb{U}, \mathcal{O}(\mathbf{V})) \cong \mathbf{Alg}_t(\mathbb{U}, \mathcal{D})[V] \cong \mathbf{Clos}(\mathbf{V}),$$

where the last category denotes Seal's **V**-graded closure spaces; see [Sea05b]. Hence, not only the category of closure spaces can be described by "neighborhoods", but also the category of **V**-valued closure spaces, equivalently, of **V**-graded closure operators [Sea05b], in case **V** is a ccd frame.

4.4.2 Algebraic closure spaces

Let $\mathbb{U}_\omega$ be the finitely generated upper-set KZ-monad on **Ord**, and $i : \mathbb{U}_\omega \longrightarrow \mathbb{U}$ the natural inclusion. The restriction $\mathbb{U}_\omega\mathbb{D} \xrightarrow{r^\omega} \mathbb{D}\mathbb{U}_\omega$ provides us with an extension $\tilde{\mathbb{U}}_\omega$ of $\mathbb{U}_\omega$ to **Dist**. We will describe the lax algebras for it.

The natural embedding $i : \mathbb{U}_\omega \hookrightarrow \mathbb{U}$ extends by Remark 2.3.35 to a natural transformation $i^* : \tilde{\mathbb{U}} \longrightarrow \tilde{\mathbb{U}}_\omega$ and hence an induces a full embedding

$$\mathbf{Alg}\,\tilde{\mathbb{U}}_\omega \xrightarrow{i_\diamond} \mathbf{Alg}\,\tilde{\mathbb{U}},$$

which by Proposition 3.7.5 preserves and reflects both reflexivity and transitivity. Thus we obtain the following commutative diagram of full embeddings

$$\begin{array}{ccccc} \mathbf{Alg}_t\tilde{\mathbb{U}}_\omega & \longrightarrow & \mathbf{Alg}_r\tilde{\mathbb{U}}_\omega & \longrightarrow & \mathbf{Alg}\,\tilde{\mathbb{U}}_\omega \\ \downarrow{\scriptstyle i_\diamond} & & \downarrow{\scriptstyle i_\diamond} & & \downarrow{\scriptstyle i_\diamond} \\ \mathbf{Clos} & \longrightarrow & \mathbf{Alg}_r\tilde{\mathbb{U}} & \longrightarrow & \mathbf{Alg}\,\tilde{\mathbb{U}} \end{array}$$

Thus, to describe $\mathbf{Alg}_t\tilde{\mathbb{U}}_\omega$, we just need to characterize those closure spaces which form the image of $i_\diamond : \mathbf{Alg}_t\tilde{\mathbb{U}}_\omega \longrightarrow \mathbf{Clos}$.

This is done easily: $(X, PX \xrightarrow{r} X)$ is in the image iff r factors as $r = r' \cdot i_X^*$. Any such factorization is unique. Thus (X, r) is in the image iff whenever $A\,r\,x$, there exists a finite subset $F \subset A$ with $F\,r'\,x$. Then clearly $F\,r\,x$. Thus, in the language of closure operators, with $c_r(A) = \{\, x \in X \mid A\,r\,x \,\}$:

$$x \in c_r(A) \iff \exists F \subset A,\ F \text{ finite},\ x \in c_r(F).$$

That is, $\mathbf{Alg}_t\tilde{\mathbb{U}}_\omega$ is precisely the category **AClos** of *algebraic closure operators*.

Corollary 4.4.3 $I : \mathbf{AClos} \hookrightarrow \mathbf{Clos}$ *is concretely coreflective.*

Proof A concrete right adjoint of $I = i_\diamond$ is given by $i^\diamond$. By Proposition 3.7.5 $i^\diamond$ preserves reflexivity and transitivity. ∎

The distributive law r^ω allows us to construct the composite monad $\mathbb{D}\mathbb{U}_\omega$, and Theorem 4.4.2 gives

$$\mathrm{Cat}\,\mathrm{Klei}\,\mathbb{D}\mathbb{U}_\omega \cong \mathbf{AClos},$$

that is, also algebraic closure spaces may be described by a suitable type of "neighborhoods".

We note that $\mathbf{Ord}^{\mathbb{D}\mathbb{U}_\omega}$ is just the category of frames [MRW02, 5.7].

4.4.3 Grounded closure spaces

We can repeat the above discussion for the inclusion $j : \mathbb{U}_0 \longrightarrow \mathbb{U}$ of the non-empty uppersets into $\mathbb{U}$. As above, j induces a full embedding $j_\diamond : \mathbf{Alg}\,\tilde{\mathbb{U}}_0 \longrightarrow \mathbf{Alg}\,\tilde{\mathbb{U}}$ which preserves reflexivity and transitivity, and we end up with the problem of describing the image of the restriction $i_\diamond : \mathbf{Alg}_t\tilde{\mathbb{U}}_0 \longrightarrow \mathbf{Clos}$.

Since (X, r) is in the image precisely when r factors as $r = r' \cdot j_X^*$, we see that $A\,r\,x$ iff there exists a non-empty $B \subset A$ with $B\,r\,x$. Thus, (X, r) is in the image precisely when $c_r(\emptyset) = \emptyset$ holds.

Let us call a closure space (X, c) *grounded* if $c\emptyset = \emptyset$ holds, and denote the full subcategory of **Clos** spanned by the grounded closure spaces by **GClos**. We have, by the above considerations:

Proposition 4.4.4 **GClos** *concretely isomorphic to* $\mathbf{Alg}_t\tilde{\mathbb{U}}_0$. *Thus the full embedding* **GClos** $\longhookrightarrow$ **Clos** *is concretely coreflective.* ∎

For the composite monad $\mathbb{D}\mathbb{U}_0$, we obtain $\mathbf{GClos} \cong \mathrm{Cat}\,\mathrm{Klei}\,\mathbb{D}\mathbb{U}_0$.

Since $\mathbb{U}_0$ as well as $\mathbb{U}_\omega$ extend also to 2-comonads on $\mathrm{OrdMat}(\mathbf{V})$ for any ccd frame $\mathbf{V}$, one can also describe frame valued grounded (or algebraic) closure spaces via neighborhoods. We refrain from spelling out the details in full.

4.5 The filter construction

Let $\mathbb{F}$ be the filter coKZ-monad on **Ord**. For a locally small, locally finitely complete ordered category $\mathbf{O}$ we define the *filter-category* $\mathrm{Fil}\,\mathbf{O}$ as follows (see also [CH03]):

- $\mathrm{obj}\,\mathrm{Fil}\,\mathbf{O} = \mathrm{obj}\,\mathbf{O}$;
- $\mathrm{Fil}\,\mathbf{O}(A, B) = \mathbb{F}(\mathbf{O}(A, B))$; thus $F \leq F'$ iff $F \supset F'$;
- the identity on A is given by $1_A^\uparrow = \{\, f \in \mathbf{O}(A, A) \mid 1_A \leq f \,\}$;
- the composite of G and F is defined to be the filter generated by $\{\, g \cdot f \mid f \in F, g \in G \,\}$.

For any $f \in \mathbf{O}(A, B)$, let $f^\uparrow = \{\, h \in \mathbf{O}(A, B) \mid f \leq h \,\}$.

Remark 4.5.1 $G \cdot f^\uparrow = \{\, g \cdot f \mid g \in G \,\}^\uparrow$ and $h^\uparrow \cdot G = \{\, h \cdot g \mid g \in G \,\}^\uparrow$ hold whenever the composites are defined. ∎

Hence $(\ )^\uparrow : \mathbf{O} \longrightarrow \mathrm{Fil}\,\mathbf{O}$ is functorial, moreover

$$f \leq g \iff g^\uparrow \subset f^\uparrow \iff f^\uparrow \leq g^\uparrow$$

holds, so $(\ )^\uparrow$ is locally fully-faithful, in particular a pseudo-functor and hence preserves adjoint situations: $(f^\uparrow)^* \approx (f^*)^\uparrow$ holds for any $f \in \mathrm{Map}\,\mathbf{O}$.

The construction of $\mathrm{Fil}\,\mathbf{O}$ is closely related to the *local completion* of $\mathbf{O}$ constructed in [FS90, 2.221]. Indeed, the local completion, which is the free quantaloid of a locally small ordered category, is defined as above, using $\mathbb{D}$ in place of $\mathbb{F}$. The author does not know whether (the embedding of $\mathbf{O}$ into) $\mathrm{Fil}\,\mathbf{O}$ satisfies any universal property.

Remark 4.5.2 $F \leq G \iff G \subset F \iff \forall g \in G\, \exists f \in F : f \leq g$ holds. ∎

Lemma 4.5.3 $\mathrm{Fil}\,\mathbf{O}$ *is locally complete. Given* $(F_i)_I \subset \mathrm{Fil}\,\mathbf{O}(A,B)$*, the infimum is the filter generated by* $\bigcup_I F_i$ *and their supremum is the intersection* $\bigcap_I F_i$*. In particular, we have*

$$f \in (F_1 \wedge F_2) \iff \exists f_1 \in F_1, f_2 \in F_2 : f_1 \wedge f_2 \leq f, \tag{4.7}$$

hence $(\ )^\uparrow : \mathbf{O} \longrightarrow \mathrm{Fil}\,\mathbf{O}$ *locally preserves finite infima and arbitrary suprema.* ∎

Let $\mathbf{O}$ and $\mathbf{O}'$ be locally finitely complete, locally small ordered categories. For any 2-graph morphism $T : \mathbf{O} \longrightarrow \mathbf{O}'$, we define a 2-graph morphism $\mathrm{Fil}\,T : \mathrm{Fil}\,\mathbf{O} \longrightarrow \mathrm{Fil}\,\mathbf{O}'$ by

$$\mathrm{Fil}\,T(X) = T(X), \qquad \mathrm{Fil}\,T(F) = \{\, Tf \mid f \in F \,\}^\uparrow.$$

Proposition 4.5.4 Fil *is 2-functorial in the following sense: We have* $\mathrm{Fil}(1_{\mathbf{O}}) = 1_{\mathrm{Fil}\,\mathbf{O}}$*,* $\mathrm{Fil}(TS) = \mathrm{Fil}\,T \cdot \mathrm{Fil}\,S$ *for any 2-graph morphism* T*,* S*, and*

$$\begin{array}{ccc} \mathbf{O} & \xrightarrow{(\ )^\uparrow} & \mathrm{Fil}\,\mathbf{O} \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathrm{Fil}\,T} \\ \mathbf{O}' & \xrightarrow[(\ )^\uparrow]{} & \mathrm{Fil}\,\mathbf{O}' \end{array}$$

commutes. If T *is a lax functor (an oplax functor, a pseudo-functor, respectively), then* $\mathrm{Fil}\,T$ *is a lax functor (an oplax functor, a 2-functor, respectively). If* $T \xrightarrow{\lambda} T'$ *is a lax (oplax, pseudo-natural, respectively) transformation, then* $(\lambda^\uparrow_X)_X$ *forms a lax (oplax, pseudo-natural, respectively) transformation* $\mathrm{Fil}\,T \longrightarrow \mathrm{Fil}\,T'$*.* ∎

In particular, if $\mathbf{O}$ has an involution $(\ )^\circ$ we can define an involution, also denoted by $(\ )^\circ$, on $\mathrm{Fil}\,\mathbf{O}$ by $F^\circ = \{f^\circ \mid f \in F\}$. Using this notion of involution, we can prove:

Lemma 4.5.5 $\mathrm{Fil}\,\mathbf{O}$ *is an allegory provided* $\mathbf{O}$ *is an allegory.* ∎

Although $(\ )^\uparrow : \mathbf{O} \longrightarrow \mathrm{Fil}\,\mathbf{O}$ is only an embedding if $\mathbf{O}$ is locally separated, we identify an $\mathbf{O}$-morphism with its $(\ )^\uparrow$-image.

Lemma 4.5.6 *Let* M *be a class of maps in* $\mathbf{O}$*. The (co-)Frobenius law holds in* $(\mathbf{O}, M)$ *if and only if the (co-)Frobenius law holds in* $(\mathrm{Fil}\,\mathbf{O}, M)$*.* ∎

In case $\mathbf{O}$ is even locally complete, we define a 2-graph morphism $\bigwedge : \operatorname{Fil}\mathbf{O} \longrightarrow \mathbf{O}$ be $X \longmapsto X$, $F \longmapsto \bigwedge F$. Clearly, $\bigwedge \cdot (\ )^{\uparrow} \approx 1_{\mathbf{O}}$ and $(\ )^{\uparrow} \cdot \bigwedge \leq 1_{\operatorname{Fil}\mathbf{O}}$ hold. For $F \in \operatorname{Fil}\mathbf{O}(X,Y)$, $G \in \operatorname{Fil}\mathbf{O}(Y,Z)$, $t \in \mathbf{O}(X,Z)$, we have

$$\begin{aligned} t \leq \bigwedge G \cdot \bigwedge F &\implies \forall g \in G, f \in F : t \leq g \cdot f \\ &\implies \forall h \in (G \cdot F) : t \leq h \\ &\implies t \leq \bigwedge G \cdot F, \end{aligned}$$

thus $\bigwedge : \operatorname{Fil}\mathbf{O} \longrightarrow \mathbf{O}$ is a normally lax functor. Moreover, for any *map* $f : X \longrightarrow Y$ in $\mathbf{O}$, we have

$$(\bigwedge G) \cdot f = \bigwedge \{\, g \cdot f \mid g \in G \,\} = \bigwedge \{\, g \cdot f \mid g \in G \,\}^{\uparrow} = \bigwedge (G \cdot f^{\uparrow}) \tag{4.8}$$

since $(\ ) \cdot f$ preserves infima.

Observe that for any 2-graph morphism $T : \mathbf{O} \longrightarrow \mathbf{O}'$ with $\mathbf{O}$, $\mathbf{O}'$ locally complete we have $T \cdot \bigwedge \leq \bigwedge \cdot \operatorname{Fil} T$, that is, $\bigwedge$ is "lax natural".

Filter-scenes Let $\mathcal{S} = \left(\mathbf{S} \longrightarrow \mathbf{C} \underset{(\ )^*}{\overset{(\ )_*}{\rightrightarrows}} \mathbf{D} \right)$ be a scene with $\mathbf{D}$ locally small and locally finitely complete. We define the *scene* $\operatorname{Fil}\mathcal{S}$ by the following diagram:

$$\mathbf{S} \longrightarrow \mathbf{C} \underset{(\ )^{\uparrow}\cdot(\ )^*}{\overset{(\ )^{\uparrow}\cdot(\ )_*}{\rightrightarrows}} \operatorname{Fil}\mathbf{D}$$

Then $(1_{\mathbf{S}}, 1_{\mathbf{C}}, (\ )^{\uparrow}) : \mathcal{S} \longrightarrow \operatorname{Fil}\mathcal{S}$ is a based scenic functor by construction, for which we will just write $(\ )^{\uparrow}$. If $\mathbf{D}$ is locally complete, then $(1_{\mathbf{S}}, 1_{\mathbf{C}}, \bigwedge)$ defines a based scenic functor $\operatorname{Fil}\mathcal{S} \longrightarrow \mathcal{S}$, which we will simply denote by $\bigwedge$.

Extending scenic functors The following result is immediate from Proposition 4.5.4:

Proposition 4.5.7 *Let $\mathcal{S}$, $\mathcal{S}'$ such that $\operatorname{Fil}\mathcal{S}$ and $\operatorname{Fil}\mathcal{S}'$ are defined. For any scenic functor $(T,U) : \mathcal{S} \longrightarrow \mathcal{S}'$, $(T, \operatorname{Fil} U)$ defines a scenic functor $\operatorname{Fil}\mathcal{S} \longrightarrow \operatorname{Fil}\mathcal{S}'$.* ∎

Fix a monadic actor $\mathsf{T} = ((T,U),e,m)$ on $\mathcal{S}$. We define $\operatorname{Fil}\mathsf{T} = ((T, \operatorname{Fil} U), e, m)$. We may rephrase the results obtained so far by saying that $(\ )^{\uparrow} : (\mathsf{T}, \mathcal{S}) \longrightarrow (\operatorname{Fil}\mathsf{T}, \operatorname{Fil}\mathcal{S})$ and $\bigwedge : (\operatorname{Fil}\mathsf{T}, \operatorname{Fil}\mathcal{S}) \longrightarrow (\mathsf{T}, \mathcal{S})$ are morphisms in **MonAct**. Thus we obtain concrete functors $E = \mathbf{Alg}(\ )^{\uparrow} : \mathbf{Alg}(\mathsf{T}, \mathcal{S}) \longrightarrow \mathbf{Alg}(\operatorname{Fil}\mathsf{T}, \operatorname{Fil}\mathcal{S})$ and $M = \mathbf{Alg}\bigwedge : \mathbf{Alg}(\operatorname{Fil}\mathsf{T}, \operatorname{Fil}\mathcal{S}) \longrightarrow \mathbf{Alg}(\mathsf{T}, \mathcal{S})$. We summarize their properties in the following remark:

Remark 4.5.8 We have $M \cdot E \approx 1_{\mathbf{Alg}(\mathsf{T},\mathcal{S})}$ and $E \cdot M \leq 1_{\mathbf{Alg}(\operatorname{Fil}\mathsf{T},\operatorname{Fil}\mathcal{S})}$, thus (E,M) is a Galois correspondence. Moreover, both E and M preserve reflexivity, unitarity, and transitivity.

Proof By Proposition 3.7.11. ∎

We conclude by remarking that the Fil-construction preserves (finite) completeness and topologicity of scenes.

4.5.1 Quasi-uniform spaces

Clementino and Hofmann proved in [CH03]:

Proposition 4.5.9 $\mathbf{Alg}_t(\mathbf{1}, \operatorname{Fil}\mathbf{Rel}) \cong \operatorname{Cat}(\mathbf{Set}, \operatorname{Fil}\mathbf{Rel})$ *is concretely isomorphic to the category* **QUnif** *of quasi-uniform spaces.* ∎

Since $\operatorname{Fil}\mathbf{Rel}$ has an involution, given by $F^\circ = \{\, r^\circ \mid r \in F \,\}$, we may also speak about groupoids in $(\mathbf{Set}, \operatorname{Fil}\mathbf{Rel})$.

Corollary 4.5.10 ([Sch02]) $\operatorname{Grpd}(\mathbf{Set}, \operatorname{Fil}\mathbf{Rel})$ *is concretely isomorphic to the category* **Unif** *of uniform spaces.* ∎

Strict and costrict morphisms Let (X, F) and (Y, G) be uniform spaces. James [Jam90, 1.12] calls a morphism $(X, F) \xrightarrow{f} (Y, G)$ *uniformly open* if for every $r \in F$ there exists $s \in G$ such that for all $x \in X$, $s[f(x)] \subset f[r[x]]$ holds. Since the definition applies also to quasi-uniform spaces, we may state:

Proposition 4.5.11 *A* $\operatorname{Cat}(\mathbf{Set}, \operatorname{Fil}\mathbf{Rel})$*-morphism* $(X, F) \xrightarrow{f} (Y, G)$ *is strict if and only if it is uniformly open.*

Proof We have:

$$\begin{aligned} G \cdot f^\uparrow \leq f^\uparrow \cdot F &\iff f^\uparrow \cdot F \subset G \cdot f^\uparrow \\ &\iff \forall r \in F : f \cdot r \in (G \cdot f^\uparrow) \\ &\iff \forall r \in F\, \exists s \in G : s \cdot f \leq f \cdot r \end{aligned}$$

and $s \cdot f \leq f \cdot r$ in $\mathbf{Rel}$ is obviously equivalent to $s[f(x)] \subset f[r[x]]$ for all $x \in X$. ∎

To describe costrict morphisms of quasi-uniform spaces, observe that in the present context $(X, F) \xrightarrow{f} (Y, G)$ is costrict if and only if $(X, F^\circ) \xrightarrow{f} (Y, G^\circ)$ is strict. Thus we obtain: A morphism $(X, F) \xrightarrow{f} (Y, G)$ of uniform spaces is a costrict $\operatorname{Cat}(\mathbf{Set}, \operatorname{Fil}\mathbf{Rel})$-morphism if and only if $\forall r \in F\, \exists s \in G\, s^\circ \cdot f \leq f \cdot r^\circ$. Let us call such a morphism *uniformly coopen*. Clearly, in the context of uniform spaces, a morphism is uniformly open iff it is uniformly coopen.

Corollary 4.5.12 *The classes of open morphisms and of coopen morphisms of quasi-uniform spaces are stable under pullback, closed under composition, and contain all isomorphisms.*

Proof The (co-)Frobenius law as well as the (BC)-property hold in (Fil **Rel**, **Set**). ∎

Proposition 4.5.13 *Every surjective open and every surjective coopen morphism of quasi-uniform spaces is an effective descent morphism in* **QUnif**.

Proof Clearly the reasoning of Section 3.6.3 applies. ∎

Strict diagonals For a morphism $X \xrightarrow{f} Y$ in a finitely complete category, write $\operatorname{Ker} f$ for the following pullback:

$$\begin{array}{ccc} \operatorname{Ker} f & \xrightarrow{f_1} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

and define the *diagonal* $\delta_f : X \longrightarrow \operatorname{Ker} f$ *of* f by $f_1 \cdot \delta_f = 1_X = f_0 \cdot \delta_f$.

Proposition 4.5.14 *Let* $(X, F) \xrightarrow{f} (X, G)$ *by a morphism of uniform spaces. Then the diagonal* δ_f *is uniformly open if and only if* f *is* transverse to F *in the sense of [Jam90], that is, there exists* $r \in F$ *such that* $1_X = r \cap \{\, (x, x') \in X^2 \mid f(x) = f(x') \,\}$.

Proof We will first give a description of the uniform space $\operatorname{Ker} f$. First observe that we may assume $\operatorname{Ker} f$ to be canonically constructed, that is, the underlying set of $\operatorname{Ker} f$ is the set in the statement of the proposition.

Let us denote the uniform structure on $\operatorname{Ker} f$ by H. From the construction of initial structures, we see that for $s \subset (\operatorname{Ker} f)^2$ we have, by Remark 4.5.1 and the fact that F is a filter,

$$s \in H \iff \exists r \in F : f_0^\circ \cdot r \cdot f_0 \cap f_1^\circ \cdot r \cdot f_1 \subset s.$$

Suppose now that δ_f is open. Since $(X \times X) \in F$, we obtain $s \in H$ with $s[\delta_f(x)] \subset \delta_f[(X \times X)[x]] = 1_X$ for all $x \in X$, that is, whenever $(x, x)\, s\, (x_0, x_1)$, then $x_0 = x_1$. By construction of H, there exists some $r \in F$ with $f_0^\circ \cdot r \cdot f_0 \cap f_1^\circ \cdot r \cdot f_1 \subset s$. We claim that this r satisfies $r \cap \operatorname{Ker} f = 1_X$. Indeed, whenever we have $x_0\, (r \cap \operatorname{Ker} f)\, x_1$, then since $x_0\, r\, x_0$ by $1_X \subset r$, we obtain

$$(x_0, x_0)\, (f_0^\circ \cdot r \cdot f_0)\, (x_0, x_1) \quad \text{and} \quad (x_0, x_0)\, (f_1^\circ \cdot r \cdot f_1)\, (x_0, x_1),$$

thus $(x_0, x_0)\, s\, (x_0, x_1)$, and hence $x_0 = x_1$.

Suppose that f is transverse to F, that is, there exists $r_0 \in F$ with $r_0 \cap \operatorname{Ker} f = 1_X$. For any $r \in F$, set $r_1 = r \cap r_0 \in F$. We clearly have $r_1 \cap \operatorname{Ker} f = 1_X$. Define $s = f_0^\circ \cdot r_1 \cdot f_0 \cap f_1^\circ \cdot r_1 \cdot f_1$. We claim that $s[\delta_f(x)] \subset \delta_f[r[x]]$ holds for all $x \in X$. Fix $x \in X$, and take any $(x_0, x_1) \in \operatorname{Ker} f$ with $(x, x)\, s\, (x_0, x_1)$, thus, $x\, r_1\, x_0$ and $x\, r_1\, x_1$. Since $r_1 \in F$, and (X, F) is an uniform spaces, we obtain $x_0\, r_1\, x_1$. Thus, $x_0 = x_1$ since $r_1 \cap \operatorname{Ker} f = 1_X$. Moreover, $(x, x_0) \in r_1 \subset r$ gives $(x_0, x_1) \in \delta_f[r[x]]$. ∎

Let $\mathcal{S}$ be a finitely complete and locally small scene, $\mathbf{T} = (T, e, m)$ an actor monad on $\mathcal{S}$ such that e extends oplax. Using Remark 4.5.1 and (4.8), we see that in

$$\begin{array}{ccccc}
\mathbf{Alg}(T,\mathcal{S}) & \xrightarrow{(\,)^\uparrow} & \mathbf{Alg}(\mathrm{Fil}\,T, \mathrm{Fil}\,\mathcal{S}) & \xrightarrow{\bigwedge} & \mathbf{Alg}(T,\mathcal{S}) \\
\downarrow{\scriptstyle e^\diamond} & & \downarrow{\scriptstyle e^\diamond} & & \downarrow{\scriptstyle e^\diamond} \\
\mathbf{Alg}(1,\mathcal{S}) & \xrightarrow[(\,)^\uparrow]{} & \mathbf{Alg}(1, \mathrm{Fil}\,\mathcal{S}) & \xrightarrow[\bigwedge]{} & \mathbf{Alg}(1,\mathcal{S})
\end{array}$$

the square on the left-hand side commutes and the square on the right-hand side commutes "up to $\approx$". All functors in the above diagram preserve transitivity. Thus we obtain the following diagram by restriction:

$$\begin{array}{ccccc}
\mathbf{Alg}_t(T,\mathcal{S}) & \xrightarrow{(\,)^\uparrow} & \mathbf{Alg}_t(\mathrm{Fil}\,T, \mathrm{Fil}\,\mathcal{S}) & \xrightarrow{\bigwedge} & \mathbf{Alg}_t(T,\mathcal{S}) \\
\downarrow{\scriptstyle e^\diamond} & & \uparrow{\scriptstyle e^t_\diamond} \dashv \downarrow{\scriptstyle e^\diamond} & & \downarrow{\scriptstyle e^\diamond} \\
\mathrm{Cat}\,\mathcal{S} & \xrightarrow[(\,)^\uparrow]{} & \mathrm{Cat}\,\mathrm{Fil}\,\mathcal{S} & \xrightarrow[\bigwedge]{} & \mathrm{Cat}\,\mathcal{S}.
\end{array}$$

The functor $e^t_\diamond : \mathrm{Cat}\,\mathrm{Fil}\,\mathcal{S} \longrightarrow \mathbf{Alg}_t(\mathrm{Fil}\,T, \mathrm{Fil}\,\mathcal{S})$ exists since $\mathrm{Fil}\,\mathcal{S}$ is locally small and locally complete by assumption. Specializing to $\mathcal{S} = (\mathbf{Set}, \mathbf{Rel})$ and $\mathbf{T}$ the ultrafilter-monad, we obtain:

$$\begin{array}{ccccc}
\mathbf{Top} & \longrightarrow & \mathbf{Alg}_t\,\mathrm{Fil}\,\mathbf{U} & \longrightarrow & \mathbf{Top} \\
\downarrow & & \uparrow \downarrow & & \downarrow \\
\mathbf{Ord} & \longrightarrow & \mathbf{QUnif} & \longrightarrow & \mathbf{Ord}
\end{array}$$

In [CHT04] it is shown (using the essentially equivalent concept of *proalgebra*) that the composite functor $\mathbf{QUnif} \longrightarrow \mathbf{Top}$ in the above diagram is precisely the usual forgetful functor. It is well-known that this functor has a left-adjoint. It would be interesting to give a lax-algebraic construction of this left adjoint.

5 Partial morphisms

In this chapter we are going to study whether partial morphism in categories of lax algebras are representable. We will give some applications to T_0-objects and the exponentiability of monomorphisms in categories of lax algebras.

5.1 Generalities

First we will review the notion of partial morphism; we follow quite closely [AHS04, Section 28]. Let $\mathbf{X}$ be a category, $\mathcal{M}$ be a class of $\mathbf{X}$-morphisms. A *$\mathcal{M}$-partial morphism* from A to B is a span

$$A \xleftarrow{m} \bullet \xrightarrow{f} B \tag{5.1}$$

with $m \in \mathcal{M}$.

We say that $B \xrightarrow{\eta_B} B^+$ *represents* (or is a *representer* for) $\mathcal{M}$-partial morphisms into B provided that

1. $\eta_B \in \mathcal{M}$,
2. for every morphism $A \xrightarrow{f} B^+$ there exists a pullback

$$\begin{array}{ccc} P & \xrightarrow{m} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\eta_B]{} & B^+ \end{array}$$

   with $m \in \mathcal{M}$, and
3. for each $\mathcal{M}$-partial morphism $A \xleftarrow{m} X \xrightarrow{f} B$ there exists a unique $A \xrightarrow{\overline{f}} B^+$ such that

$$\begin{array}{ccc} X & \xrightarrow{m} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ B & \xrightarrow[\eta_B]{} & B^+ \end{array}$$

   is a pullback.

We say that $\mathbf{X}$ has *representable $\mathcal{M}$-partial morphisms* if for every $B \in \mathbf{X}$ there exists $B \xrightarrow{\eta_B} B^+$ which represents $\mathcal{M}$-partial morphisms into B.

Remark 5.1.1 If $\mathbf{X}$ has representable $\mathcal{M}$-partial morphisms then we obtain a pointed endofunctor $((\ )^+, \eta)$. Moreover, η is cartesian. ■

Lemma 5.1.2 *If* $\mathbf{X}$ *has representable* $\mathcal{M}$*-partial morphisms, then*

(1) *pullbacks of* $\mathcal{M}$*-morphisms exists and belong to* $\mathcal{M}$*,*

(2) *every isomorphism belongs to* $\mathcal{M}$ *and* $\mathcal{M}$ *consists of regular monomorphisms.*

Proof See [AHS04, Proposition 28.3] ∎

If $\mathbf{X}$ has representable $\mathcal{M}$-partial morphisms, $\mathcal{M}$ need not be closed under composition. For an example, see [AHS04, Remark 28.4]. In case $\mathcal{M}$ is closed under composition there exists a natural transformation $\mu_X : (\ )^{++} \longrightarrow (\ )^{+}$ such that $((\ )^{+}, \eta, \mu)$ is a monad on $\mathbf{X}$. Indeed, μ_X is defined to be the unique morphism which turns

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & X^{+} & \xrightarrow{\eta_{X^{+}}} & X^{++} \\ \| & & & & \vdots\,{\scriptstyle \mu_X} \\ X & & \xrightarrow{\quad\eta_X\quad} & & X^{+} \end{array}$$

into a pullback-diagram. The fact that μ is a natural transformation and the monad laws follow by pasting pullbacks and using uniqueness of the induced morphism.

Examples

1. **Set** has representable Mono-partial morphisms. Following [Tay99] we define, for any set B,

$$B^{+} = \{\, A \subset B \mid \forall a, a' \in A : a = a' \,\}, \qquad \eta_B : b \longmapsto \{b\}. \tag{5.2}$$

   We can identify B^{+} with the set $B + \{\emptyset\}$ and η_B with the coproduct injection. Given a partial morphism $A \xleftarrow{m} P \xrightarrow{f} B$, we define $A \xrightarrow{\overline{f}} B^{+}$ by

$$a \longmapsto \{\, b \in B \mid \exists p \in P : m(p) = a \text{ and } f(p) = b \,\}. \tag{5.3}$$

2. Every topos has representable Mono-partial morphisms. The representers can be constructed by interpreting (5.2) and (5.3) in the internal language of the topos. A more "diagrammatic" construction can be found in [Joh02, 2.4.7].

3. The category **Bin** of binary endorelations has representable $\mathcal{M}$-partial morphisms, for $\mathcal{M}$ the class of initial monomorphisms. Given (B, r), B a set and $r \subset B \times B$, we can define a relation r^{+} on $B^{+} = B + 1$ by the following "block-matrix"

$$\left(\begin{array}{ccc|c} & & & 1 \\ & r & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 1 \end{array}\right) \tag{5.4}$$

   Then it is easy to see that $(B, r) \xrightarrow{\eta_B} (B^{+}, r^{+})$ represents $\mathcal{M}$-partial morphisms.

Using the last example, we can explain out general strategy for constructing representers in categories of lax algebras: η_B defines a function

$$\eta_B^\diamond = \eta_B^* \cdot (\ ) \cdot \eta_B : \mathbf{Rel}(B^+, B^+) \longrightarrow \mathbf{Rel}(B, B), \tag{5.5}$$

and r^+ is just the maximal element of $\mathbf{Rel}(B^+, B^+)$ such that $\eta_B^\diamond(r^+) \leq r$. That is, the assignment $r \longmapsto r^+$ determines a right-adjoint of $\eta_B^\diamond$. This observation allows us to give a purely categorical proof of the existence of representers for suitable classes of morphisms in categories of lax algebras.

5.2 Representing initial morphisms

Proposition 5.2.1 *Let $P : \mathbf{X} \longrightarrow \mathbf{B}$ be a faithful fibration and let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be $\mathbf{X}$-morphisms with f initial. If*

$$\begin{array}{ccc} D & \xrightarrow{p} & PY \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Pg} \\ PX & \xrightarrow[Pf]{} & PZ \end{array}$$

is a pullback in $\mathbf{B}$, then the pullback of f along g exists in $\mathbf{X}$ and is initial. ∎

If P is a faithful fibration and $\mathcal{M}$ is a class of $\mathbf{B}$-morphisms we write, as before,

$$\mathcal{M}_{\mathrm{I}} := \{\, f \in \mathbf{X} \mid f \text{ is initial and } Pf \in \mathcal{M} \,\}.$$

Proposition 5.2.2 *Let $P : \mathbf{X} \longrightarrow \mathbf{B}$ be a faithful fibration such that P has a right and a left adjoint, that is, $(\mathbf{X}, P)$ has discrete and indiscrete objects.*

If $\mathcal{M}$ is a class of $\mathbf{B}$-morphisms such that $\mathcal{M}_{\mathrm{I}}$-partial morphisms are representable in $\mathbf{X}$, then $\mathcal{M}$-partial morphisms are representable in $\mathbf{B}$.

Moreover, P preserves representers if and only if $\mathcal{M}_{\mathrm{I}}$-subobjects of discrete objects are discrete.

Proof The proof of Proposition 28.12 in [AHS04] applies. ∎

Observe that $\mathcal{M}_{\mathrm{I}}$-subalgebras of discrete algebras in $\mathbf{Alg}\, T$ are discrete provided composition preserves the bottom-element, and $\mathcal{M}_{\mathrm{I}}$-subalgebras of discrete algebras in $\mathbf{Alg}_r\mathsf{T}$, $\mathbf{Alg}_u\mathsf{T}$, and $\mathbf{Alg}_t\mathsf{T}$ are discrete if T sends $\mathcal{M}$-morphisms into injections.

Since most categories of lax algebras of interest satisfy the additional requirements of the above proposition, we will only study the problem of lifting a given $\mathcal{M}$-partial morphism representer.

Blanket assumptions We fix a scene $\mathcal{S}$ which satisfies the Beck–Chevalley property and a class $\mathcal{M} \subset \operatorname{mor}\mathbf{S}$ such that $((\ )^+, \eta)$ represents $\mathcal{M}$-partial morphisms in $\mathbf{S}$. Moreover, we fix an actor T on $\mathcal{S}$ such that T has the (BC)-property with respect to $\mathcal{M}$. These assumptions are fulfilled, for example, for $\mathcal{S} = (\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ and any (extension of a) taut **Set**-functor T.

Lemma 5.2.3 *The blanket assumptions imply the following statements:*

(1) *every $\mathcal{M}$-morphism is an injection;*

(2) *every T-image of a $\mathcal{M}$-morphism is an injection.*

Proof Every $(A \xrightarrow{m} X) \in \mathcal{M}$ is a monomorphism, hence

$$\begin{array}{ccc} A & = & A \\ \| & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[m]{} & X \end{array}$$

is a pullback, and Beck–Chevalley implies $m^* \cdot m \approx 1_A$ and $Tm^* \cdot Tm \approx 1_{TA}$. ∎

We aim at the following theorem:

Theorem 5.2.4 *The following statements are equivalent for an $\mathbf{S}$-object B:*

(1) *$\psi = \eta_B^\diamond : \mathbf{D}(T(B^+), B^+) \longrightarrow \mathbf{D}(TB, B)$, $a \longmapsto \eta_B^* \cdot a \cdot T\eta_B$ has a right adjoint ϕ;*

(2) *for every lax algebra (B, b) there is a lift (B^+, b^+) of B^+ such that $(B, b) \xrightarrow{\eta_B} (B^+, b^+)$ represents $\mathcal{M}_{\mathrm{I}}$-partial morphisms into (B, b).*

In this case $b^+ = \phi(b)$.

Some observations: ψ always has a fully-faithful *left* adjoint $\eta_{B\diamond}$, given by $c \longmapsto \eta_B \cdot c \cdot T\eta_B^*$. Hence, by Lemma 1.2.11, any right adjoint ϕ which might exist is forced to be fully-faithful.

Proof (2) implies (1): Define $\phi(b) = b^+$. First observe that ϕ is monotone. Indeed, if $b \leq c$, then $(B, b) \xrightarrow{1_B} (B, c)$ is a morphism and the upper-left corner in

$$\begin{array}{ccc} (B, b) & \xrightarrow{\eta_B} & (B^+, b^+) \\ \downarrow{\scriptstyle 1_B} & & \vdots{\scriptstyle f} \\ (B, c) & \xrightarrow[\eta_B]{} & (B^+, c^+) \end{array} \tag{5.6}$$

is a $\mathcal{M}_{\mathrm{I}}$-partial morphism into (B, c), hence there exists a unique f such that (5.6) is a pullback. Clearly, $f = 1_B$, hence $b^+ \leq c^+$.

Initiality of η_B gives $\psi \cdot \phi \approx 1$. It remains to show $1 \leq \phi \cdot \psi$, i.e. $a \leq (\psi a)^+$ for every $a \in \mathbf{D}(TB^+, B^+)$. Consider

$$\begin{array}{ccc} (B, \psi a) & \xrightarrow{\eta_B} & (B^+, a) \\ \| & & \vdots f \\ (B, \psi a) & \xrightarrow[\eta_B]{} & (B^+, (\psi a)^+). \end{array}$$

The upper-left corner is a $\mathcal{M}_{\mathrm{I}}$-partial morphism by definition of ψa, hence there is a unique f as shown such that the diagram is a pullback, clearly $f = 1_{B^+}$ and the claim follows.

Initiality of η_B follows from $\psi \cdot \phi \approx 1$.

(1) implies (2): $\psi \cdot \phi \approx 1$ is equivalent to the initiality of $(B, b) \xrightarrow{\eta_B} (B^+, \phi(b))$. Suppose $(B, b) \xleftarrow{f} (Y, a) \xrightarrow{m} (Z, c)$ is an $\mathcal{M}_{\mathrm{I}}$-partial morphism. Then there exists a unique $\overline{f}$ such that

$$\begin{array}{ccc} Y & \xrightarrow{m} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ B & \xrightarrow[\eta_B]{} & B^+ \end{array} \tag{5.7}$$

is a pullback in $\mathbf{S}$. Since the above as well as its T-image is a (BC)-square,

$$\psi(\overline{f} \cdot c \cdot T\overline{f}^*) \approx \eta_B^* \cdot \overline{f} \cdot c \cdot T\overline{f}^* \cdot T\eta_B \approx f \cdot m^* \cdot c \cdot Tm \cdot Tf^* \approx f \cdot a \cdot Tf^* \leq b$$

holds, which, by $\psi \dashv \phi$ implies that $(Z, c) \xrightarrow{\overline{f}} (B^+, b^+)$ is a morphism. Since m is initial, the above diagram becomes a pullback in $\mathbf{Alg}\, T$. ∎

Representations in categories of reflexive and unitary algebras

Proposition 5.2.5 *(B, b) is reflexive if and only if (B^+, b^+) is reflexive.*

Proof Suppose (B, b) is reflexive. It suffices to show $\psi(e_{B^+}^*) \leq b$, which holds since $\eta_B^* \cdot e_{B^+}^* \cdot T\eta_B \approx e_B^* \cdot T\eta_B^* \cdot T\eta_B \approx e_B^* \leq b$.

The converse follows from the initiality of $(B, b) \xrightarrow{\eta_B} (B^+, b^+)$. ∎

Proposition 5.2.6 *If m has the (BC)-property with respect to $\mathcal{M}$, then (B, b) is unitary if and only if (B^+, b^+) is unitary.*

Proof Since $(B,b) \xrightarrow{\eta_B} (B^+,b^+)$ is initial, (B,b) is unitary provided (B^+,b^+) is so. So show the converse implication, consider

$$\begin{aligned}
\psi(e^*_{B^+} \cdot Tb^+ \cdot m^*_{B^+}) &\approx \eta^*_B \cdot e^*_{B^+} \cdot Tb^+ \cdot m^*_{B^+} \cdot T\eta_B \\
&\approx e^*_B \cdot T\eta^*_B \cdot Tb^+ \cdot TT\eta_B \cdot m^*_B \qquad \text{since } m \text{ is } \mathcal{M}\text{-BC} \\
&\leq e^*_B \cdot T(\eta^*_B \cdot b^+ \cdot T\eta_B) \cdot m^*_B \\
&\approx e^*_B \cdot Tb \cdot m^*_B \\
&\leq b,
\end{aligned}$$

from which $e^*_{B^+} \cdot Tb^+ \cdot m^*_{B^+} \leq b^+$ follows by $\psi \dashv \phi$. ∎

Unfortunately, the above construction does not extend to transitive lax algebras, i.e., if (B,b) is (reflexive and) transitive, (B^+,b^+) need not be so. For a simple counterexample, consider

$$b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ which leads to } b^+ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

which is clearly not transitive. Also it is well-known that **Top** has no representable Embedding-partial maps, see [Her88]. On the other hand, if we restrict our attention to open or closed embeddings, the corresponding partial morphisms become representable in **Top**.

5.3 Representing strict and costrict morphisms

Additional assumptions In this section we assume $\mathbf{D}$ to be locally finitely complete.

We recall the following notation from Section 3.5:

$$\mathcal{M}_{\mathrm{S}} = \{f \in \mathbf{Alg}\,T \mid f \text{ strict}, |f| \in \mathcal{M}\}, \qquad \mathcal{M}_{\mathrm{C}} = \{f \in \mathbf{Alg}\,T \mid f \text{ costrict}, |f| \in \mathcal{M}\},$$

and $\mathcal{M}_{\mathrm{B}} = \mathcal{M}_{\mathrm{S}} \cap \mathcal{M}_{\mathrm{C}}$.

Lemma 5.3.1 *Our blanket assumptions imply:*

(1) $\mathcal{M}_{\mathrm{S}} \subset \mathcal{M}_{\mathrm{I}}$ *and* $\mathcal{M}_{\mathrm{C}} \subset \mathcal{M}_{\mathrm{I}}$.

(2) *Pullbacks of* $\mathcal{M}_{\mathrm{S}}$*-morphisms exist. If* $(\mathbf{D},\mathcal{M})$ *satisfies the Frobenius law, then* $\mathcal{M}_{\mathrm{S}}$ *is stable under pullback.*

(3) *Pullbacks of* $\mathcal{M}_{\mathrm{C}}$*-morphisms exist. If* $(\mathbf{D},T[\mathcal{M}])$ *satisfies the coFrobenius law, then* $\mathcal{M}_{\mathrm{C}}$ *is stable under pullback.*

Proof ad (1): $(f : (X, a) \longrightarrow (Y, b)) \in \mathcal{M}_{\mathrm{S}}$ implies $f^* \cdot b \cdot Tf \approx f^* \cdot f \cdot a \approx a$, hence $f \in \mathcal{M}_{\mathrm{I}}$. Analogously one shows $\mathcal{M}_{\mathrm{C}} \subset \mathcal{M}_{\mathrm{I}}$.

ad (2): Pullbacks of $\mathcal{M}_{\mathrm{S}}$-morphisms exist by (1), Proposition 5.2.1, and the fact that pullbacks of $\mathcal{M}$-morphisms exist in $\mathbf{S}$. The proof of Proposition 3.5.8 shows that under our blanket assumptions $\mathcal{M}_{\mathrm{S}}$ is stable under pullback provided $(\mathbf{D}, \mathcal{M})$ satisfies the Frobenius law.

(3) is proved analogously to (2), using Proposition 3.5.9. ■

5.3.1 Costrict morphisms

We assume that $(\mathbf{D}, T[\mathcal{M}])$ satisfies the coFrobenius law. For a $\mathbf{S}$-object B define $\sigma_B : \mathbf{D}(T(B^+), B^+) \longrightarrow \mathbf{D}(T(B^+), B)$ by $c \longmapsto \eta_B^* \cdot c$. Observe that σ_B has a fully-faithful left adjoint, given by $a \longmapsto \eta_B \cdot a$.

Theorem 5.3.2 *η_B lifts to a $\mathcal{M}_{\mathrm{C}}$-partial morphism representer if σ_B has a right adjoint ρ. In this case the extension of (B, b) is given by (B^+, b^c), with $b^c = \rho(b \cdot T\eta_B^*)$.*

Proof Suppose a right adjoint ρ of σ_B exists. Since σ_B has a fully-faithful left adjoint, ρ will be fully-faithful as well, that is, $\sigma_B \cdot \rho \approx 1$. Thus $\eta_B^* \cdot b^c \approx \sigma_B \cdot \rho(b \cdot T\eta_B^*) \approx b \cdot T\eta_B^*$ holds, hence $(B, b) \xrightarrow{\eta_B} (B^+, b^c)$ is costrict. Now suppose we have $(B, b) \xleftarrow{f} (Y, a) \xrightarrow{m} (Z, c)$ with $m \in \mathcal{M}_{\mathrm{C}}$. Then there is a unique $\mathbf{S}$-morphism $Z \xrightarrow{\overline{f}} B^+$ such that (5.7) is a pullback in $\mathbf{S}$. We need to show that $(Z, c) \xrightarrow{\overline{f}} (B^+, b^c)$ is a morphism. From

$$\begin{aligned}
\sigma_B(\overline{f} \cdot c \cdot T\overline{f}^*) &\approx \eta_B^* \cdot \overline{f} \cdot c \cdot T\overline{f}^* \\
&\approx f \cdot m^* \cdot c \cdot T\overline{f}^* \\
&\approx f \cdot a \cdot Tm^* \cdot T\overline{f}^* \\
&\approx f \cdot a \cdot Tf^* \cdot T\eta_B^* \\
&\leq b \cdot T\eta_B^*
\end{aligned}$$

follows $\overline{f} \cdot c \cdot T\overline{f}^* \leq \rho(b \cdot T\eta_B^*) = b^c$, hence $\overline{f}$ is a morphism $(Z, c) \longrightarrow (B^+, b^c)$. Since $(Y, a) \xrightarrow{m} (Z, c)$ is initial by Lemma 5.3.1(1), the pullback lifts to a pullback in $\mathbf{Alg}\, T$. ■

Proposition 5.3.3 *Suppose that σ_B has a right adjoint ρ. Then*

(1) *(B^+, b^c) is reflexive if and only if (B, b) is reflexive.*

(2) *If T preserves $\{\eta_B\}^*$-whiskers, then (B^+, b^c) is unitary (transitive) if and only if (B, b) is unitary (transitive).*

Proof ad (1): First suppose (B, b) is reflexive. Then $\sigma_B(e_{B^+}^*) = \eta_B^* \cdot e_{B^+}^* = e_B^* \cdot T\eta_B^* \leq b \cdot T\eta_B^*$ holds, which implies $e_{B^+}^* \leq b^c$ by $\sigma_B \dashv \rho$. The converse implication follows from initiality of $(B, b) \xrightarrow{\eta_B} (B^+, b^c)$.

ad (2): Strictness of T implies

$$T\eta_B^* \cdot Tb^c \cdot m_{B^+}^* \approx T(\eta_B^* \cdot b^c) \cdot m_{B^+}^* \approx T(b \cdot T\eta_B^*) \cdot m_{B^+}^* \approx Tb \cdot TT\eta_B^* \cdot m_{B^+}^* \approx Tb \cdot m_B^* \cdot T\eta_B^*.$$

Suppose (B, b) is unitary. Using the above equation, we obtain

$$\begin{aligned} \sigma_B(e_{B^+}^* \cdot Tb^c \cdot m_{B^+}^*) &\approx \eta_B^* \cdot e_{B^+}^* \cdot Tb^c \cdot m_{B^+}^* \\ &\approx e_B^* \cdot T\eta_B^* \cdot Tb^c \cdot m_{B^+}^* \\ &\approx e_B^* \cdot Tb \cdot m_B^* \cdot T\eta_B^* \\ &\leq b \cdot T\eta_B^*. \end{aligned}$$

In case (B, b) is transitive, we obtain:

$$\begin{aligned} \sigma_B(b^c \cdot Tb^c \cdot m_{B^+}^*) &\approx \eta_B^* \cdot b^c \cdot Tb^c \cdot m_{B^+}^* \\ &\approx b \cdot T\eta_B^* \cdot Tb^c \cdot m_{B^+}^* \\ &\approx b \cdot Tb \cdot m_B^* \cdot T\eta_B^* \\ & b \cdot T\eta_B^*. \end{aligned}$$

Therefore, unitarity (transitivity) of (B, b) implies unitarity (transitivity) of (B^+, b^c) by $\sigma_B \dashv \rho$. The converse implications follow from initiality of $(B, b) \xrightarrow{\eta_B} (B^+, b^c)$. ∎

5.3.2 Strict morphisms

We assume that $(\mathbf{D}, \mathcal{M})$ satisfies the Frobenius law.

Let B be an $\mathbf{S}$-object and define $\tau_B : \mathbf{D}(T(B^+), B^+) \longrightarrow \mathbf{D}(TB, B^+)$ by $c \longmapsto c \cdot T\eta_B$. Observe that τ_B has a fully-faithful left adjoint, given by $a \longmapsto a \cdot T\eta_B^*$.

Theorem 5.3.4 *η_B lifts to a $\mathcal{M}_{\mathrm{S}}$-partial morphism representer if τ_B has a right adjoint υ. In this case the extension of (B, b) is given by (B^+, b^s) with $b^s = \upsilon(\eta_B \cdot b)$.*

Proof Any right adjoint υ will be fully-faithful, that is, will satisfy $\tau_B \cdot \upsilon \approx 1$.

$(B, b) \xrightarrow{\eta_B} (B^+, b^+)$ is strict since $b^s \cdot T\eta_B \approx \tau_B(\upsilon(\eta_B \cdot b)) \approx \eta_B \cdot b$. Now suppose we have $(B, b) \xleftarrow{f} (Y, a) \xrightarrow{m} (Z, c)$ with $m \in \mathcal{M}_{\mathrm{S}}$. Then there is a unique $\mathbf{S}$-morphism $Z \xrightarrow{\overline{f}} B^+$ such that (5.7) is a pullback in $\mathbf{S}$. We need to show that $(Z, c) \xrightarrow{\overline{f}} (B^+, b^s)$ is a morphism. From

$$\begin{aligned} \tau_B(\overline{f} \cdot c \cdot T\overline{f}^*) &\approx \overline{f} \cdot c \cdot T\overline{f}^* \cdot T\eta_B \\ &\approx \overline{f} \cdot c \cdot Tm \cdot Tf^* \\ &\approx \overline{f} \cdot m \cdot a \cdot Tf^* \\ &\approx \eta_B \cdot f \cdot a \cdot Tf^* \\ &\leq \eta_B \cdot b \end{aligned}$$

follows $\overline{f} \cdot c \cdot T\overline{f}^* \leq \upsilon(\eta_B \cdot b) = b^s$, thus $(Z, c) \xrightarrow{\overline{f}} (B^+, b^s)$ is a morphism. Since m is initial, the pullback (5.7) lifts to a pullback in $\mathbf{Alg}\, T$. ∎

Proposition 5.3.5 *Suppose that τ_B has a right adjoint υ. Then*

(1) *If e has the (BC)-property with respect to $\mathcal{M}$, then (B^+, b^s) is reflexive if and only if (B, b) is reflexive.*

(2) *If T preserves $\{\eta_B\}$-whiskers and e, m have the (BC)-property with respect to $\mathcal{M}$, then (B^+, b^s) is transitive if and only if (B, b) is transitive.*

(3) *If T preserves $\{\eta_B\}$-whiskers, and m has the (BC)-property with respect to $\mathcal{M}$, then (B^+, b^s) is transitive if and only if (B, b) is transitive.*

Proof Suppose (B, b) is reflexive. Then $\tau_B(e^*_{B^+}) = e^*_{B^+} \cdot T\eta_B \approx \eta_B \cdot e^*_B \leq \eta_B \cdot b$ holds, which, using $\tau_B \dashv \upsilon$, implies reflexivity of (B^+, b^s).

Strictness of T and the fact that m has the (BC)-property with respect to $\mathcal{M}$ imply

$$Tb^s \cdot m^*_{B^+} \cdot T\eta_B \approx Tb^s \cdot TT\eta_B \cdot m^*_B \approx T(b^s \cdot T\eta_B) \cdot m^*_B \approx T(\eta_B \cdot b) \cdot m^*_B \approx T\eta_B \cdot Tb \cdot m^*_B.$$

Suppose (B, b) is unitary. We obtain:

$$\begin{aligned} \tau_B(e^*_{B^+} \cdot Tb^s \cdot m^*_{B^+}) &\approx e^*_{B^+} \cdot Tb^s \cdot m^*_{B^+} \cdot T\eta_B \\ &\approx e^*_{B^+} \cdot T\eta_B \cdot Tb \cdot m^*_B \\ &\approx \eta_B \cdot e^*_B \cdot Tb \cdot m^*_B \\ &\leq \eta_B \cdot b. \end{aligned}$$

In case (B, b) is transitive, we obtain:

$$\begin{aligned} \tau_B(b^s \cdot Tb^s \cdot m^*_{B^+}) &\approx b^s \cdot Tb^s \cdot m^*_{B^+} \cdot T\eta_B \\ &\approx b^s \cdot T\eta_B \cdot Tb \cdot m^*_B \\ &\approx \eta_B \cdot b \cdot Tb \cdot m^*_B \\ &\leq \eta_B \cdot b. \end{aligned}$$

Hence, unitarity (transitivity) or (B^+, b^s) follows from unitarity (transitivity) of (B, b) by $\tau_B \dashv \upsilon$. The converse implications follow by initiality of $(B, b) \xrightarrow{\eta_B} (B^+, b^s)$. ∎

5.4 Examples

Over $\mathrm{Mat}\,\mathbf{V}$ Let us first consider the case $\mathcal{S} = (\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ for an arbitrary quantale $\mathbf{V}$. Let $\mathcal{M}$ denote the class of injective functions. A functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ maps inverse images to (BC)-squares if and only if T is taut. A natural transformation has the (BC)-property with respect to $\mathcal{M}$ if and only if it is taut. Finally, $\mathrm{Mat}\,\mathbf{V}$ has division since it is a quantaloid, thus all the required right adjoints exist. Thus we have:

Proposition 5.4.1 *For any actor T on $\mathcal{S} = (\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ with $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ taut, partial morphisms are representable in $\mathbf{Alg}\,T$ and the forgetful functor $\mathbf{Alg}\,T \longrightarrow \mathbf{Set}$ preserves representations.*

If $\mathbf{T} = (T, e, m)$ is a monadic actor on $\mathcal{S}$ with T taut, then also $\mathbf{Alg}_r\mathbf{T}$ has concretely representable partial morphisms. If m is a taut transformation, then also $\mathbf{Alg}_u\mathbf{T}$ has concretely representable partial morphisms.

Since in the present context, $(\operatorname{Mat}\mathbf{V}, \mathcal{M})$ satisfies both the Frobenius and the coFrobenius law, and for any taut **Set**-functor its extension to $\operatorname{Mat}\mathbf{V}$ preserves {Mono}- and {Mono}*-whiskers, we also have:

Proposition 5.4.2 *For any monadic actor $\mathbf{T}$ on $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ with T taut, $\mathbf{Alg}\,T$, $\mathbf{Alg}_r\mathbf{T}$, $\mathbf{Alg}_u\mathbf{T}$, and $\mathbf{Alg}_t\mathbf{T}$ have concretely representable costrict-embedding partial morphisms.*

*In case (T, e, m) is a taut **Set**-monad, $\mathbf{Alg}\,T$, $\mathbf{Alg}_r\mathbf{T}$, $\mathbf{Alg}_u\mathbf{T}$, and $\mathbf{Alg}_t\mathbf{T}$ have concretely representable strict-embedding partial morphisms.*

Thus in particular, **PsTop**, **PrTop**, **PsAp**, **PrAp**, **Bin**, **RBin** have concretely representable partial morphisms. **Top** and **Ap** have concrete representations for open- and closed-embedding partial morphisms. **Ord** has concrete representations for up- and down-closed embedding-partial morphisms.

The same reasoning applies to **Clos** (recall that the extension of $\mathbb{U}$ to **Dist** is a 2-functor). Thus we obtain that closed-embedding partial morphisms are representable in **Clos**.

All the above-mentioned representations are done by one-point extensions.

Over $\operatorname{Rel}\mathbf{C}$ Proposition 5.4.2 remains true in for any scene of the form $(\mathbf{C}, \operatorname{Rel}\mathbf{C})$, provided $\mathbf{C}$ is a logos. To wit:

Proposition 5.4.3 *Let $\mathbf{C}$ be a logos, $\mathcal{M} \subset \operatorname{mor}\mathbf{C}$ a class of morphisms such that $\mathbf{C}$ has representable $\mathcal{M}$-partial morphisms, T a taut functor $\mathbf{C} \longrightarrow \mathbf{C}$. $\mathbf{Alg}\,T$ has representable $\mathcal{M}_{\mathrm{I}}$-partial morphisms and the forgetful functor preserves them.*

In case $\mathbf{T} = (T, e, m)$ is a monad on $\mathbf{C}$ with T taut, also $\mathbf{Alg}_r\mathbf{T}$ has concretely representable $\mathcal{M}_{\mathrm{I}}$-partial morphism. Furthermore, $\mathbf{Alg}\,T$, $\mathbf{Alg}_r\mathbf{T}$, $\mathbf{Alg}_u\mathbf{T}$, and $\mathbf{Alg}_t\mathbf{T}$ have concretely representable $\mathcal{M}_{\mathrm{C}}$-partial morphisms.

If moreover m is taut, then also $\mathbf{Alg}_r\mathbf{T}$ has concretely representable $\mathcal{M}_{\mathrm{I}}$-partial morphisms.

If $\mathbf{T}$ is a taut monad on $\mathbf{C}$, then $\mathbf{Alg}\,T$, $\mathbf{Alg}_r\mathbf{T}$, $\mathbf{Alg}_u\mathbf{T}$, and $\mathbf{Alg}_t\mathbf{T}$ have concretely representable $\mathcal{M}_{\mathrm{S}}$-partial morphisms.

5.5 Some applications

We are now giving some applications of the results in this Chapter. To make the discussion more understandable, we leave out some (easily established) technical details.

5.5.1 Subobject classifier and T_0-objects

Recall that, for a class $\mathcal{M}$ of (mono-)morphisms in a category $\mathbf{X}$ with a terminal object 1, a *$\mathcal{M}$-subject classifier* is a morphism $1 \xrightarrow{t} \Omega$ such that for every $A \xrightarrow{m} X$ in $\mathcal{M}$, there exists a unique morphism $X \xrightarrow{\chi_A} \Omega$ such that

$$\begin{array}{ccc} A & \xrightarrow{m} & X \\ \downarrow & & \downarrow{\scriptstyle \chi_A} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

is a pullback. In case $\mathcal{M}$-partial morphisms are representable in $\mathbf{X}$, clearly $1 \xrightarrow{\eta_1} 1^+$ is $\mathcal{M}$-subobject classifier. In general, representability of $\mathcal{M}$-partial morphisms is a stronger condition than the existence of $\mathcal{M}$-subobject classifiers, cf. Lemma 5.5.3.

In particular, **Top** has a subobject classifier $1 \xrightarrow{t} S$ for open-embeddings. Here S is the Sierpinski-space and t "is" the "open point" of S.

Of course, this definition can now be mimicked in every category of lax algebras with representable $\mathcal{M}_C$-partial morphisms ($\mathcal{M}$ a suitable class of monomorphisms), leading to a notion of *Sierpinski-object* for the class of lax algebras under consideration.

Mindful of the fact that a topological space is T_0 if and only if it is a subspace of a power of S, we can now define a lax algebra to be T_0 if it lies in the full subcategory cogenerated by the respective Sierpinski-object.

In fact this has been done by Manes [Man02] for reflexive and transitive lax algebras over the scene $(\mathbf{Set}, \mathbf{Rel})$ for an actor defined by a taut **Set**-functor. He uses an explicit construction of the Sierpinski-object.

We end up with the following examples:

- T_0-objects in $\mathbf{Top} \cong \mathbf{Alg}_t\mathbf{T}$ are precisely the T_0-topological spaces;
- T_0-objects in $\mathbf{Ord} \cong \mathbf{Alg}_t\mathbf{1}$ are precisely the anti-symmetric ordered sets.

5.5.2 Exponentiability

We will now apply the results obtained so far to give examples of exponentiable morphisms in various categories of lax algebras.

Let $\mathbf{X}$ denote a finitely complete category. A morphism $f : X \longrightarrow Y$ in $\mathbf{X}$ is called *exponentiable* if $(\ ) \times f : \mathbf{X}/Y \longrightarrow \mathbf{X}/Y$ has a right adjoint. We note two well-known results on exponentiable morphisms. For proofs we refer to [Nie82].

Lemma 5.5.1 *The following statements are equivalent for $f : X \longrightarrow Y$ in $\mathbf{X}$:*

(1) *f is exponentiable;*

(2) *the pullback functor $f^* : \mathbf{X}/Y \longrightarrow \mathbf{X}/X$ has a right adjoint.* ∎

Proposition 5.5.2 *The class of exponentiable morphisms in* $\mathbf{X}$ *contains all isomorphisms, is closed under composition, and stable under pullback.* ■

Exponentiability and the topic of this chapter are connected by the following result.

Lemma 5.5.3 ([DT87]) *The following statements are equivalent for a pullback-stable class* $\mathcal{M} \subset \operatorname{mor} \mathbf{X}$*:*

(1) $\mathcal{M}$*-subobjects are representable;*

(2) $\mathcal{M}$ *consists of exponentiable monomorphisms and* $\mathbf{X}$ *has a* $\mathcal{M}$*-subobject classifier.* ■

This immediately implies that in **PrTop** and in **PrAp** every embedding is exponentiable, that every open embedding and every closed embedding is exponentiable in **Top**, that each up- and each down-closed embedding is exponentiable in **Ord**, that every closed embedding is exponentiable in **Clos**, etc.

We can even say a little bit more:

Exponentiability in Top: We call a subspace embedding $A \hookrightarrow X$ in **Top** *locally closed* if $A = O \cap C$ for O open in X and C closed in X.

Proposition 5.5.4 *Every locally closed embedding is exponentiable in* **Top**.

Proof Every open embedding and every closed embedding is exponentiable in **Top**. So, by Proposition 5.5.2, is every intersection of an open and a closed embedding. But these are just the locally closed embeddings. ■

Proposition 5.5.4 was established by Niefield [Nie82]; see also [CGT04]. In fact, she showed that an embedding is exponentiable in **Top** if and only if it is locally closed.

The definition of locally closed embedding can of course be generalized to any category of lax algebras for a finitely complete scene. Then the results from Sections 5.3.1 and 5.3.2 together with Proposition 5.5.2 and Lemma 5.5.3 immediately provide us with a large number of exponentiable morphisms. We refrain from writing down the necessary definitions in full, and just point out the following example.

Example 5.5.5 Besides the down-closed and up-closed embeddings, every inclusion of a convex subset is exponentiable in **Ord**. Here we call $A \subset X$ convex if $a \leq x \leq a'$ for arbitrary $x \in X$, $a, a' \in A$ implies $x \in A$. Indeed, A is convex precisely when $A = A^{\uparrow} \cap A^{\downarrow}$.

We remark that, although we are far from providing a characterization of exponentiable embeddings in any category of lax algebras, the example of topological spaces shows that we are actually able to obtain the strongest result possible using our general approach.

6 More on limits and colimits

6.1 Coproducts

The material presented in this section is mostly an extension of [MST06].

In this section, we will fix a scene $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ and say that the *(finite) basic assumptions* are satisfied provided

(1) $\mathbf{D}$ is locally small, has locally (finite) suprema, and these are preserved by composition,

(2) $\mathbf{S}$ has (finite) coproducts, and the embedding $\mathbf{S} \xrightarrow{(\,)_* \cdot E} \mathbf{D}$ sends them to biproducts.

We write $\mathcal{C}$ for the class of all coproduct-injections in $\mathbf{S}$.

Examples of scenes which satisfy the (finite) basic are:

- $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ for any quantale $\mathbf{V}$.
- Let $\mathbf{C}$ be a well-powered regular category. $(\mathbf{C}, \operatorname{Rel}\mathbf{C})$ satisfies the (finite) basic assumptions provided $\mathbf{C}$ has (finite) coproducts and is (finitely) extensive; see [FS90].
- If $\mathcal{S}$ satisfies the (finite) basic assumption, so does $\operatorname{Dist}(\mathcal{S})$. This is shown in Lemma 1.6.5 and Proposition 1.6.10.
- $\mathcal{O}(\mathbf{V}) = (\mathbf{Set}, \mathbf{Ord}, \operatorname{OrdMat}(\mathbf{V}))$ satisfies the basic assumptions. To see this, observe that the proof of Proposition 1.6.10 shows that $\operatorname{OrdMat}(\mathbf{V})$ is closed under biproducts in $\operatorname{Dist}(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$.

6.1.1 Coproducts in categories of lax algebras

Since $\mathbf{Alg}\,T \xrightarrow{|\,|} \mathbf{S}$ is a fibering $\mathbf{Alg}\,T$ has (finite) concrete coproducts precisely when $\mathbf{S}$ has (finite) coproducts and the fibers of $|\,|$ have (finite) coproducts, that is, $\mathbf{D}(TB, B)$ has (finite) suprema for all $B \in \mathbf{S}$.

To study more refined properties of coproducts in $\mathbf{Alg}\,T$, we will now assume that the (finite) basic assumptions are satisfied. In particular, the forgetful functor $\mathbf{Alg}\,T \longrightarrow \mathbf{S}$ has a left adjoint, and thus preserves all limits which exist in $\mathbf{Alg}\,T$.

Proposition 6.1.1 *$\mathbf{Alg}\,T$ has (finite) concrete coproducts. The following statements are equivalent for a (finite) family $(X_i, a_i)_I$ in $\mathbf{Alg}\,T$ and a coproduct $(X_i \xrightarrow{t_i} X)_I$ in $\mathbf{S}$:*

(1) $\big((X_i, a_i) \xrightarrow{t_i} (X, a)\big)_I$ *is a coproduct in* $\mathbf{Alg}\, T$*;*

(2) *each* $(X_i, a_i) \xrightarrow{t_i} (X, a)$ *is costrict.*

Proof Existence and concreteness of coproducts follow from the above remarks. Suppose $\big((X_i, a_i) \xrightarrow{t_i} (X, a)\big)_I$ is a coproduct in $\mathbf{Alg}\, T$. By essential uniqueness of final lifts, we have $a \approx \bigvee_I t_i \cdot a_i \cdot Tt_i^*$, and hence, using the fact that $(X_i \xrightarrow{t_i} X)_I$ is a biproduct in $\mathbf{D}$:

$$t_j^* \cdot a \approx t_j^* \cdot \bigvee t_i \cdot a_i \cdot Tt_i^* \approx \bigvee t_j^* \cdot t_i \cdot a_i \cdot Tt_i^* \approx a_j \cdot Tt_j^*,$$

that is, each $(X_j, a_j) \xrightarrow{t_j} (X, a)$ is costrict. On the other hand if each t_j is costrict, we obtain $\bigvee t_i \cdot a_i \cdot Tt_i^* \approx \bigvee t_i \cdot t_i^* \cdot a \approx a$, thus (X, a) is a final lax algebra with respect to the coproduct sink, and hence a coproduct. ■

Since by the basic assumptions all coproduct-sinks are jointly covering in $\mathbf{D}$, $\mathbf{Alg}_r\mathbf{T}$ is closed under coproducts in $\mathbf{Alg}\, T$ by Proposition 3.4.20.

Proposition 6.1.2 *Assume* $\mathbf{D}$ *to be a quantaloid and* T *to preserve* $\mathcal{C}^*$*-whiskers. Given a (finite) family* $(X_i, a_i)_I$ *in* $\mathbf{Alg}_t\mathbf{T}$*, a coproduct* $(X_i \xrightarrow{t_i} X)_I$ *in* $\mathbf{S}$*, and a lax algebra* (X, a)*, the following statements are equivalent:*

(1) $\big((X_i, a_i) \xrightarrow{t_i} (X, a)\big)_I$ *is a coproduct in* $\mathbf{Alg}\, T$*;*

(2) (X, a) *is in* $\mathbf{Alg}_t\mathbf{T}$ *and* $\big((X_i, a_i) \xrightarrow{t_i} (X, a)\big)_I$ *is a coproduct in* $\mathbf{Alg}_t\mathbf{T}$*;*

(3) *each* $t_i : (X_i, a_i) \longrightarrow (X, a)$ *is costrict.*

In particular, $\mathbf{Alg}_t\mathbf{T}$ *is closed under coproducts in* $\mathbf{Alg}\, T$*. The statements hold true with* $\mathbf{Alg}_u\mathbf{T}$ *in place of* $\mathbf{Alg}_t\mathbf{T}$*.*

Proof Observe first that under the conditions of the propositions, $\mathbf{Alg}_t\mathbf{T}$ is reflective in $\mathbf{Alg}\, T$, and thus coproducts are formed by applying the reflector to the coproduct formed in $\mathbf{Alg}\, T$.

[(1) ⇒ (2)]: Let (X, c) be a coproduct of the family $(X_i, a_i)_I$ in $\mathbf{Alg}_t\mathbf{T}$. By Proposition 3.5.10, each $(X_i, a_i) \xrightarrow{t_i} (X, c)$ is open. Thus $c \approx \bigvee t_i \cdot t_i^* \cdot c \approx \bigvee t_i \cdot a_i \cdot Tt_i^* \approx a$, that is, c is just the final structure with respect to the coproduct embeddings, and thus the coproduct of the family $(X_i, a_i)_I$ in $\mathbf{Alg}\, T$.

[(2) ⇒ (3)]: Proposition 3.5.10.

[(3) ⇒ (1)]: Proposition 6.1.1. ■

Although we formulate the following results only for $\mathbf{Alg}\,T$, they are valid in $\mathbf{Alg}_r\mathbf{T}$, $\mathbf{Alg}_u\mathbf{T}$, and $\mathbf{Alg}_t\mathbf{T}$, provided the latter are closed under coproducts in $\mathbf{Alg}\,T$, in particular, if Proposition 6.1.2 holds.

Proposition 6.1.3 *Given a (finite) family* $\big((X_i, a_i) \xrightarrow{g_i} (Y, b)\big)_I$ *of morphisms, the following statements are equivalent:*

(1) *every* g_i *is costrict;*

(2) *the induced morphism* $g = [g_i] : \sum_I (X_i, a_i) \longrightarrow (Y, b)$ *is costrict.*

Proof If g is costrict, so is every g_i since costrict morphisms compose. To show the converse implication, observe that Proposition 1.4.6 implies $g^* \approx \bigvee t_i \cdot g_i^*$, where t_j denotes the jth coproduct injection ($j \in I$). Thus we obtain:

$$(\bigvee t_i \cdot a_i \cdot Tt_i^*) \cdot Tg^* \approx \bigvee t_i \cdot a_i \cdot Tt_i^* \cdot Tg^* \approx \bigvee t_i \cdot a_i \cdot Tg_i^* \approx \bigvee t_i \cdot g_i^* \cdot b \approx g^* \cdot b. \quad \blacksquare$$

Corollary 6.1.4 *Given a (finite) family* $\big((X_i, a_i) \xrightarrow{f_i} (Y_i, b_i)\big)_I$ *of costrict* $\mathbf{Alg}\,T$*-morphisms,* $\sum_I f_i$ *is costrict as well.*

Proof $\sum_I f_i = [s_i \cdot f_i]$, where each s_i is a coproduct injection and hence costrict. $\blacksquare$

We recall the following result (see Proposition 3.5.7):

Remark 6.1.5 If $T[\mathcal{C}] \subset \operatorname{Inj}\mathbf{D}$, then every coproduct injection in $\mathbf{Alg}\,T$ is initial, hence an embedding. $\blacksquare$

Proposition 6.1.6 *If* $T[\mathcal{C}] \subset \operatorname{Inj}\mathbf{D}$*, then for a (finite) family* $\big((X_i, a_i) \xrightarrow{f_i} (Y_i, b_i)\big)_I$ *of* $\mathbf{Alg}\,T$*-morphisms,* $\sum_I f_i$ *is costrict if and only if every* f_i *is costrict.* $\blacksquare$

Strict morphisms and coproducts

Lemma 6.1.7 *Let* T *locally preserve* $\bot$*. Then coproduct injections in* $\mathbf{Alg}\,T$ *are strict.* $\blacksquare$

Proposition 6.1.8 *If* T *is left- and right* $\mathcal{C}^*$*-strict, then the class of strict morphisms in* $\mathbf{Alg}\,T$ *is closed under (finite) coproducts.*

Proof Let

$$\begin{array}{ccc} (X, a) & \xrightarrow{f} & (Y, b) \\ {\scriptstyle t_i}\uparrow & & \uparrow{\scriptstyle s_i} \\ (X_i, a_i) & \xrightarrow{f_i} & (Y_i, b_i) \end{array} \qquad (i \in I)$$

be commutative with columns coproducts in $\mathbf{Alg}\,T$. By Lemma 1.4.10, $f_i \cdot t_i^* \approx s_i^* \cdot f$ holds, which implies $Tf_i \cdot Tt_i^* \approx Ts_i^* \cdot Tf$ by assumption. Thus we obtain:

$$f \cdot a \approx \bigvee f \cdot t_i \cdot a_i \cdot Tt_i^* \approx \bigvee s_i \cdot f_i \cdot a_i \cdot Tt_i^* \approx \bigvee s_i \cdot b_i \cdot Tf_i \cdot Tt_i^* \approx b \cdot Tf. \quad \blacksquare$$

Proposition 6.1.8 does *not* imply that given a family $((X_i, a_i) \longrightarrow (Y, b))$ of strict morphisms, also the induced morphism $\coprod(X_i, a_i) \longrightarrow (Y, b)$ is strict. In fact, this is false. For a counterexample, consider the family $\left(\{*\} \xrightarrow{q} \mathbb{R}\right)_{q\in\mathbb{Q}}$ in $\mathbf{Top} \cong \mathbf{Alg}_t\mathbf{U}$ defined by $q(*) = q$. Every q is a closed embedding, thus strict, but the induced morphism is not closed, thus not strict.

Proposition 6.1.9 *Suppose we are given a diagram*

$$\begin{array}{ccc} (X,a) & \xrightarrow{\ f\ } & (Y,b) \\ {\scriptstyle t_i}\uparrow & \nearrow_{f_i} & \\ (X_i,a_i) & & \end{array} \qquad (i \in I)$$

in $\mathbf{Alg}\,T$ *such that the column is a coproduct and* $(Tt_i)_I$ *is jointly covering. Then* f *is strict provided each* f_i *is strict.* ∎

Clearly the above proposition applies in particular when T preserves I-fold coproducts.

6.1.2 The frame of a lax algebra

We assume the (*infinite*) basic assumptions, and additionally let $(\mathcal{E}, \mathcal{M})$ be a splitable (with respect to $\mathcal{S}$) factorization system on $\mathbf{S}$ such that $\mathbf{S}$ is $\mathcal{M}$-wellpowered. As usual, we write $\mathcal{M}_{\mathrm{C}}$ for the class of costrict $\mathbf{Alg}\,T$-morphisms whose underlying $\mathbf{S}$-morphism is in $\mathcal{M}$.

For each X in $\mathbf{S}$, the ordered set $\mathrm{Sub}_{\mathcal{M}}X$ of $\mathcal{M}$-subobjects is a complete lattice. Indeed, given a (set-indexed) family $(A_i \xrightarrow{m_i} X)_I$ of $\mathcal{M}$-subobjects, its supremum is given by the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of the induced morphism $[m_i] : \coprod A_i \longrightarrow X$.

Lemma 6.1.10 *The set of costrict* $\mathcal{M}_{\mathrm{I}}$*-subobjects of a lax algebra* (X, a) *is closed under suprema in the complete lattice of* $\mathcal{M}_{\mathrm{I}}$*-subobjects.*

Proof We may use the construction as well as the notation from above. If each m_i is costrict, then also $[m_i]$ is costrict by Proposition 6.1.3. Now if we $(\mathcal{E}, \mathcal{M})$-factor $[m_i]$ as $[m_i] = m \cdot e$, e is a cover, and hence m is costrict as well, which proves our claim. ∎

Now Proposition 3.5.9 gives closedness of $\mathcal{M}_{\mathrm{C}}$-subobjects under binary intersections. Since each isomorphism is strict, we obtain:

Proposition 6.1.11 *Suppose* $\mathcal{S}$ *to be finitely complete such that* $\mathrm{Sub}_{\mathcal{M}}X$ *is a frame for each* $\mathbf{S}$*-object* X*,* T *has the (BC)-property with respect to* $\mathcal{M}$*, and* $(\mathbf{D}, T[\mathcal{M}])$ *satisfies the coFrobenius law. Then the* $\mathcal{M}_{\mathrm{C}}$*-subobjects of a lax algebra* (X, a) *form a subframe of* $\mathrm{Sub}_{\mathcal{M}}X$*, which we denote by* $\mathcal{T}(X, a)$*.* ∎

But Proposition 3.5.9 actually says more: for any $\mathbf{Alg}\,T$-morphism $(X,a) \xrightarrow{f} (Y,b)$, the pullback-functor $f^{-1} : \mathrm{Sub}_{\mathcal{M}}Y \longrightarrow \mathrm{Sub}_{\mathcal{M}}X$ restricts to $\mathcal{T}(Y,b) \longrightarrow \mathcal{T}(X,a)$. Hence we have:

Proposition 6.1.12 *Assume the conditions of Proposition 6.1.11 and, additionally, that for each $f : X \longrightarrow Y$ in $\mathbf{S}$, the pullback-functor $f^{-1} : \mathrm{Sub}_{\mathcal{M}}Y \longrightarrow \mathrm{Sub}_{\mathcal{M}}X$ is a frame morphism. Then $\mathcal{T}$ defines a functor $\mathbf{Alg}\,T \longrightarrow \mathbf{Loc}$ into the category of locales.*

Proof Recall that the category **Loc** of locales is the dual of the category of frames. Hence we can define $\mathcal{T}f$ to be f^{-1}. ■

Over Set In case $\mathbf{S} = \mathbf{Set}$ with $\mathcal{E} = \mathrm{Epi}$ and $\mathcal{M} = \mathrm{Mono}$ we can improve our results some more. Since $\mathrm{Sub}_{\mathcal{M}}X$ is isomorphic to the powerset PX, we obtain $\mathcal{T}(X,a)$ as a subframe of PX, that is, as a topology on X. Thus $\mathcal{T}$ defines a *concrete* functor $\mathcal{T}' : \mathbf{Alg}\,T \longrightarrow \mathbf{Top}$.

Proposition 6.1.13 $\mathcal{T}' : \mathbf{Alg}\,T \longrightarrow \mathbf{Top}$ *preserves coproducts and sends costrict embeddings to open embeddings.*

Proof By construction $\mathcal{T}'$ sends costrict embeddings to coproducts. This, together with concreteness of $\mathcal{T}'$ implies preservation of coproducts. Indeed, coproducts in $\mathbf{Alg}\,T$ and in **Top** are characterized by the fact that all embeddings are costrict. ■

We find the preceding results remarkable in that they apply to $\mathbf{Alg}\,T$ as well as $\mathbf{Alg}_r\mathbf{T}$ and $\mathbf{Alg}_t\mathbf{T}$.

6.1.3 Extensivity

In this section we assume, in addition to the blanket assumptions, that $T[\mathcal{C}] \subset \mathrm{Inj}\,\mathbf{D}$ holds.

Remark 6.1.14 If coproducts in $\mathbf{S}$ are disjoint, then coproducts in $\mathbf{Alg}\,T$ are disjoint as well.

Proof Clearly coproduct injections in $\mathbf{Alg}\,T$ are monomorphisms. Moreover the blanket assumptions imply that there is a unique lax algebra structure on the initial object of $\mathbf{S}$. The claim follows since pullbacks in $\mathbf{Alg}\,T$ are concrete. ■

Theorem 6.1.15 *The following statements are equivalent:*

(1) $\mathbf{Alg}\,T$ *is (finitely) extensive;*

(2) $\mathbf{S}$ *is (finitely) extensive and pullbacks of coproduct injections in* $\mathbf{Alg}\,T$ *(exist and) are costrict.*

Proof [(2) ⇒ (1)]: Since every coproduct injection is initial, existence of pullbacks of coproduct injections follows from Proposition 5.2.1. Remark 6.1.14 gives disjointness of coproducts in $\mathbf{Alg}\,T$. Hence we only need to show that (finite) coproducts in $\mathbf{Alg}\,T$ are universal. Let $\big((Y_i, b_i) \xrightarrow{t_i} (Y, b)\big)_I$ be a (finite) coproduct in $\mathbf{Alg}\,T$ and $(X, a) \xrightarrow{f} (Y, b)$ an arbitrary $\mathbf{Alg}\,T$-morphism. Forming pullbacks, we obtain

$$\begin{array}{ccc} (X_i, a_i) & \xrightarrow{s_i} & (X, a) \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle f} \\ (Y_i, b_i) & \xrightarrow[t_i]{} & (Y, b) \end{array}$$

Since the forgetful functor $\mathbf{Alg}\,T \longrightarrow \mathbf{S}$ preserves limits and (finite) coproducts, $(Y_i \xrightarrow{t_i} Y)_I$ is a coproduct, and thus, by extensivity of $\mathbf{S}$, also $(X_i \xrightarrow{s_i} X)_I$ is a coproduct. Moreover, each $(X_i, a_i) \xrightarrow{s_i} (X, a)$ is costrict by hypothesis, and thus $\big((X_i, a_i) \xrightarrow{s_i} (X, a)\big)_I$ is a coproduct in $\mathbf{Alg}\,T$.

[(1) ⇒ (2)]: If $\mathbf{Alg}\,T$ is extensive, then pullbacks of coproduct injections exist and are coproduct injections, thus costrict. Extensivity of $\mathbf{S}$ follows from the fact that $\mathbf{S}$ is isomorphic to the full subcategory of $\mathbf{Alg}\,T$ determined by the discrete lax algebras, which is closed in $\mathbf{Alg}\,T$ under coproducts and embeddings. ■

Corollary 6.1.16 $\mathbf{S}$ *is (finitely) extensive and pullbacks of coproduct injections in* $\mathbf{Alg}\,T$ *exist and are costrict if and only if* $\mathbf{Alg}_r\mathsf{T}$ *is (finitely) extensive.*

Proof The first claim follows from the fact that under the blanket assumptions $\mathbf{Alg}_r\mathsf{T}$ is closed under coproducts and pullbacks in $\mathbf{Alg}\,T$. For the converse implication, we need to show that (complemented) subalgebras of discrete $\mathbf{Alg}_r\mathsf{T}$ objects are discrete. This follows from $t^* \cdot e_X^* \cdot Tt \approx e_A^* \cdot Tt^* \cdot Tt \approx e_A^*$, where the last equivalence holds since, by our general assumption, T maps coproduct injections to injections. ■

Corollary 6.1.17 *If* $\mathbf{D}$ *is a quantaloid and* T *preserves* $\mathcal{C}^*$*-whiskers, then* $\mathbf{Alg}_t\mathsf{T}$ *is (finitely) extensive if and only if* $\mathbf{S}$ *is (finitely) extensive and pullbacks of coproduct injections are costrict in* $\mathbf{Alg}\,T$.

Proof Proposition 6.1.2 applies. Thus $\mathbf{Alg}_t\mathsf{T}$ is closed under coproducts (and pullbacks) in $\mathbf{Alg}\,T$, and the result follows with Theorem 6.1.15. ■

Examples From the results that we have proved we can deduce that **Bin**, **RBin**, **Ord**, **Met**, **PsTop**, **PrTop**, **Top**, **PsAp**, **PrAp**, **Ap**, and **BiTop** are (infinitely) extensive.

Moreover, we see that proper continuous functions are closed under coproducts in **Top**, and analogously for proper contractions in **Ap**.

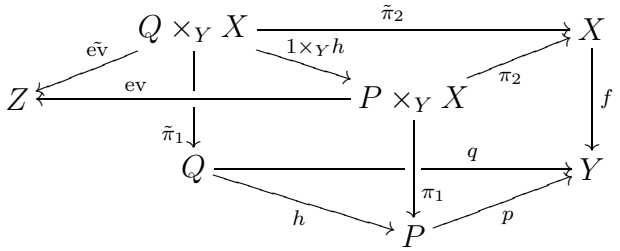
6.2 Local cartesian closedness

In this section we assume $\mathcal{S}$ to be finitely complete, with (BC)-property, and satisfying the Frobenius law with respect to $\mathbf{S}$ and the coFrobenius law with respect to $T[\mathbf{S}]$. Moreover, we assume that T has the (BC)-property.

These basic assumption are fulfilled for $\operatorname{Rel}\mathbf{S}$, $\mathbf{S}$ any regular category, and $T : \mathbf{S} \longrightarrow \mathbf{S}$ nearly pullback preserving, and for $(\mathbf{Set}, \operatorname{Mat}\mathbf{V})$ provided V is cartesian closed, and $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ nearly pullback preserving.

Partial products and exponentiable morphisms Let $\mathbf{X}$ denote a finitely complete category.

Definition 6.2.1 ([DT87]) For $X \xrightarrow{f} Y$ and Z in $\mathbf{X}$ the *partial product* $P(f, Z)$ of f and Z is given by an object P together with morphisms $P \xrightarrow{p} Z$, $P \times_Y X \xrightarrow{\mathrm{ev}} Z$ such that for any other $Q \xrightarrow{q} Y$, $Q \times_Y X \xrightarrow{\tilde{\mathrm{ev}}} Z$ there exists a unique $Q \xrightarrow{h} P$ such that



(6.1)

commutes.

Taking $Y = 1$, we obtain $P(!_X, Z) = Z^X$, furthermore $P(1_X, Z) = X \times Z$, thus partial products subsume cartesian products and exponentials; moreover:

Proposition 6.2.2 *For an* $\mathbf{X}$*-morphism* $f : X \longrightarrow Y$ *the partial product exists for all* Z *in* $\mathbf{X}$ *if and only if* f *is exponentiable in* $\mathbf{X}$*.*

Proof See [Joh02, Proposition 1.5.7]. ∎

Proposition 6.2.2 implies in particular that **Set** (and any topos) has partial products for all f and Z. Taking $Q = 1$ in diagram (6.1), we see that the elements of P correspond to pairs (y, α), where $y \in Y$ and $\alpha : f^{-1}y \longrightarrow Z$, and $p : P \longrightarrow Y$ is the projection. Thus $P \times_Y X$ is isomorphic to the set $\{\, (x, \alpha) \mid x \in X,\ f^{-1}f(x) \xrightarrow{\alpha} Z \,\}$, and ev is the evaluation, $\mathrm{ev}(x, \alpha) = \alpha(x)$.

Partial products in $\mathbf{Alg}\,T$ Take (Z, c) and $(X, a) \xrightarrow{f} (Y, b)$ in $\mathbf{Alg}\,T$ and let $P \xrightarrow{p} Y$ and $R = P \times_Y X \xrightarrow{\mathrm{ev}} Z$ be the partial product of the underlying $\mathbf{S}$-data, which we assume to exist. We have to find $d \in \mathbf{D}(TP, P)$ such that (P, d) is a partial product in $\mathbf{Alg}\,T$. This should be the "optimal" (= greatest) structure $x \in \mathbf{D}(TP, P)$ such that

$x \leq p^* \cdot b \cdot Tp$ and $\psi(x) \leq \mathrm{ev}^* \cdot c \cdot T\mathrm{ev}$; here $\psi(x)$ is the initial lift of the pullback projections with respect to a and x. That is,

$$\begin{aligned} \mathbf{D}(TP, P) &\xrightarrow{\psi} \mathbf{D}(TR, R) \\ x &\longmapsto \pi_1^* \cdot x \cdot T\pi_1 \wedge \pi_2^* \cdot a \cdot T\pi_2. \end{aligned}$$

Proposition 6.2.3 *Assume ψ to have a right adjoint φ. Then*

$$d = p^* \cdot b \cdot Tp \wedge \varphi(\mathrm{ev}^* \cdot c \cdot T\mathrm{ev})$$

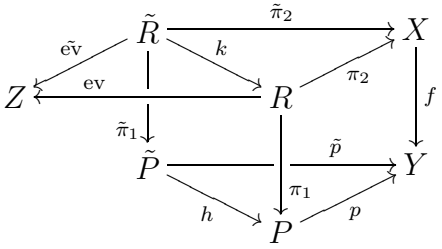
turns (P, d) into a partial product of (Z, c) and $(X, a) \xrightarrow{f} (Y, b)$ in $\mathbf{Alg}\, T$.

Proof By construction we have $d \leq p^* \cdot b \cdot Tp$ and $\psi(d) \leq \psi(\varphi(\mathrm{ev}^* \cdot c \cdot T\mathrm{ev})) \leq \mathrm{ev}^* \cdot c \cdot T\mathrm{ev}$, that is, d as above turns $(P, d) \xrightarrow{p} (Y, b)$ and $(R, \psi(d)) \xrightarrow{\mathrm{ev}} (Z, c)$ into morphisms.

To establish the universal property, assume we are given another diagram

$$\begin{array}{ccccc} (Z, c) & \xleftarrow{\tilde{\mathrm{ev}}} & (\tilde{R}, q) & \xrightarrow{\tilde{\pi}_2} & (X, a) \\ & & \downarrow{\scriptstyle \tilde{\pi}_1} & & \downarrow{\scriptstyle f} \\ & & (\tilde{P}, \tilde{d}) & \xrightarrow{\tilde{p}} & (Y, b) \end{array}$$

where the square is a pullback. We must show that the unique $\mathbf{S}$-morphism $\tilde{P} \xrightarrow{h} P$ which makes



$$k = h \times 1_X \tag{6.2}$$

commute, lifts to a morphism $(\tilde{P}, \tilde{d}) \xrightarrow{h} (P, d)$. Observe that all three vertical faces in (6.2) are pullbacks. Clearly it is sufficient to show that

(i) $h \cdot \tilde{d} \cdot Th^* \leq p^* \cdot b \cdot Tp$ and

(ii) $h \cdot \tilde{d} \cdot Th^* \leq \varphi(\mathrm{ev}^* \cdot c \cdot T\mathrm{ev})$

hold. We have $p \cdot h \cdot \tilde{d} \cdot Th^* \cdot Tp^* \approx \tilde{p} \cdot \tilde{d} \cdot T\tilde{p}^* \leq b$, which is equivalent to (i). To show (ii), first observe

$$\begin{aligned} \psi(h \cdot \tilde{d} \cdot Th^*) &\approx \pi_1^* \cdot h \cdot \tilde{d} \cdot Th^* \cdot T\pi_1 \wedge \pi_2^* \cdot a \cdot T\pi_2 \\ &\approx k \cdot \tilde{\pi}_1^* \cdot \tilde{d} \cdot T\tilde{\pi}_1 \cdot Tk^* \wedge \pi_2^* \cdot a \cdot T\pi_2 && \text{(BC)-property} \\ &\approx k \cdot (\tilde{\pi}_1^* \cdot \tilde{d} \cdot T\tilde{\pi}_1 \wedge k^* \cdot \pi_2^* \cdot a \cdot T\pi_2 \cdot Tk) \cdot Tk^* && \text{(co)Frobenius} \\ &\approx k \cdot (\tilde{\pi}_1^* \cdot \tilde{d} \cdot T\tilde{\pi}_1 \wedge \tilde{\pi}_2^* \cdot a \cdot T\tilde{\pi}_2) \cdot Tk^* \\ &\approx k \cdot q \cdot Tk^*. \end{aligned}$$

This implies $\mathrm{ev} \cdot \psi(h \cdot \tilde{d} \cdot Th^*) \cdot T\mathrm{ev}^* \approx \mathrm{ev} \cdot k \cdot q \cdot Tk^* \cdot T\mathrm{ev}^* \approx \tilde{\mathrm{ev}} \cdot q \cdot T\tilde{\mathrm{ev}}^* \leq c$. This in turn implies $\psi(h \cdot \tilde{d} \cdot Th^*) \leq \mathrm{ev}^* \cdot c \cdot T\mathrm{ev}$, which is equivalent to (ii) by $\psi \dashv \varphi$. ∎

Observe that $\psi : \mathbf{D}(TP, P) \longrightarrow \mathbf{D}(TR, R)$ factors as

$$\begin{array}{ccc} \mathbf{D}(TP,P) & \xrightarrow{\quad\psi\quad} & \mathbf{D}(TR,R) \\ & {\scriptstyle \pi_1^* \cdot (\,) \cdot T\pi_1} \searrow \quad \nearrow {\scriptstyle (\,) \wedge \pi_2^* \cdot a \cdot T\pi_2} & \\ & \mathbf{D}(TR,R) & \end{array}$$

Thus ψ has a right adjoint provided $\mathbf{D}$ has division and is locally cartesian closed, in the sense that each $\mathbf{D}(TR, R)$ is cartesian closed. We have proved:

Theorem 6.2.4 *Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ be a finitely complete scene with (BC)-property, T an actor on $\mathcal{S}$ which has the (BC)-property, such that $(\mathbf{D}, \mathbf{S})$ satisfies the Frobenius law, $(\mathbf{D}, T[\mathbf{S}])$ satisfies the coFrobenius law, $\mathbf{D}$ has division and for all $B \in \mathbf{S}$ the ordered set $\mathbf{D}(TB, B)$ is cartesian closed.*

If $\mathbf{S}$ is locally cartesian closed, so is $\mathbf{Alg}\, T$, and moreover the forgetful functor preserves exponentials. ∎

Surely, Theorem 6.2.4 is applicable to $\mathbf{Alg}(T, (\mathbf{Set}, \mathrm{Mat}\, \mathbf{V}))$ for $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ nearly pullback preserving and V cartesian closed.

Corollary 6.2.5 *Let $\mathbf{V} = (V, \otimes, k)$ be a quantale such that V is cartesian closed. For any functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ which nearly preserves pullbacks, $\mathbf{Alg}(T, (\mathbf{Set}, \mathrm{Mat}\, \mathbf{V}))$ is concretely locally cartesian closed.* ∎

Hence **Bin** and **URS** are concretely locally cartesian closed over **Set**.[1] Combining the above results with the fact that embedding-partial morphisms are representable in **Bin** and **URS** (and in fact in any category of the form $\mathbf{Alg}(T, (\mathbf{Set}, \mathrm{Mat}\, \mathbf{V})$ for T a taut set-functor), we see, using [AHS04, Theorem 28.18]:

Corollary 6.2.6 *Let $\mathbf{V}$ be a quantale such that V is cartesian closed. For any nearly pullback preserving functor $T : \mathbf{Set} \longrightarrow \mathbf{Set}$, $\mathbf{Alg}(T, (\mathbf{Set}, \mathrm{Mat}\, \mathbf{V}))$ is universally topological.* ∎

Corollary 6.2.7 *Let $\mathbf{E}$ be a topos, $T : \mathbf{E} \longrightarrow \mathbf{E}$ a functor which nearly preserves pullbacks. Then $\mathbf{Alg}(T, (\mathbf{E}, \mathrm{Rel}\, \mathbf{E}))$ is concretely locally cartesian closed.* ∎

[1] Observe that the claim in [CHT03a, footnote (1), page 147] is wrong. **URS** is, contrary to **PsTop**, *concretely* locally cartesian closed. Thus the construction in [CHT03a] cannot be used.

7 Compactness

In this chapter we are going to study notions of compactness for lax algebras. There are several approaches to this. One, used in [Kam74], [Möb81], and independently in [Sch02], uses only the "convergence relation" a of a lax algebra (X, a) over a scene with involution $(\ )^\circ$: we call (X, a) compact if $1_{TX} \leq a^\circ \cdot a$ holds.

Instead of following this approach, we are going to use the Clementino–Giuli–Tholen theory of "topology in a category"; see [CGT04] and [Tho03]. This works as follows: in a (finitely) complete category $\mathbf{X}$ we single out a class $\mathcal{P}$ of morphisms, satisfying certain properties which model the behavior of proper maps of topological spaces. We then call an $\mathbf{X}$-object X ($\mathcal{P}$-)compact if the unique morphism $X \xrightarrow{!_X} 1$ into a (chosen) terminal object is in $\mathcal{P}$.

In case $\mathbf{X}$ is a category of lax algebras, we will always use for $\mathcal{P}$ the class of strict morphisms. The advantage of this notion is that it is stable under slicing, and thus also works for the corresponding slice-categories (or categories of *bundles*) $\mathbf{X}/X$. Moreover, we have a sort of dual theory, using the class of costrict morphisms of lax algebras instead. This leads to a theory of "local homeomorphisms" of lax algebras.

We would like to remark that both approaches cannot only handle compactness, but also provide notions for Hausdorff-separation, regularity, local compactness, etc. Alas, we will not treat this in the present work. For more information on these topics see the above-cited references.

7.1 Topology in a (finitely) complete category

Fix a finitely complete category $\mathbf{X}$.

Definition 7.1.1 ([Tho03]) A *topology* on $\mathbf{X}$ is a class $\mathcal{P}$ of $\mathbf{X}$-morphisms such that

1. every isomorphism is in $\mathcal{P}$;
2. $\mathcal{P}$ is closed under composition;
3. $\mathcal{P}$ is stable under pullback.

If $\mathcal{E}$ is any class of morphisms in $\mathbf{X}$, a topology $\mathcal{P}$ is called an *$\mathcal{E}$-topology* provided that $\mathcal{P}$ is right-cancellable with respect to $\mathcal{E}$, that is, $f \cdot e \in \mathcal{P}$ and $e \in \mathcal{E}$ implies $f \in \mathcal{P}$.

Examples of topologies abound. For instance, the classes of open and of proper morphisms are topologies on **Top**. We give a generic example:

Example 7.1.2 Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{S})$ be a finitely complete scene with (BCP) such that $(\mathbf{D}, \mathbf{S})$ satisfies the Frobenius law. For any actor T on $\mathcal{S}$, $\mathbf{Alg}\,T$ has finite concrete limits and the class $(\operatorname{mor}\mathbf{S})_{\mathrm{S}}$ of all strict $\mathbf{Alg}\,T$-morphisms is a topology on $\mathbf{Alg}\,T$.

Moreover, strict morphisms also form a topology on each of $\mathbf{Alg}_r\mathsf{T}$, $\mathbf{Alg}_u\mathsf{T}$, and $\mathbf{Alg}_t\mathsf{T}$ for any monadic actor T on $\mathcal{S}$.

These topologies are $\mathcal{E}$-topologies for any $\mathcal{E}$ such that $T[\mathcal{E}] \subset \operatorname{Cov}\mathbf{D}$.

Thus, we obtain the following examples of topologies:

- **Clos** with all closed morphisms;
- **Unif** with all uniformly open morphisms;
- **Ap** with all proper contractions.

Proposition 7.1.3 *Let* $\mathbf{X} \xrightarrow{P} \mathbf{B}$ *be a fibration such that* $\mathbf{B}$ *and* $\mathbf{X}$ *are finitely complete and* P *preserves pullbacks. For any topology* $\mathcal{P}$ *in* $\mathbf{B}$*,*

$$\mathcal{P}_P := \{\, f \in \mathbf{X} \mid f \text{ is cartesian and } Pf \in \mathcal{P} \,\}$$

is a topology on $\mathbf{X}$*.*

Proof Since an $\mathbf{X}$-morphism f is an $\mathbf{X}$-isomorphism if and only if it is cartesian and Pf is an isomorphism every isomorphism is in $\mathcal{P}_P$. Closedness under composition follows from the fact that the composition of cartesian morphisms is cartesian. To show pullback-stability, observe first that pullbacks of cartesian morphisms are cartesian. Hence pullback-stability of $\mathcal{P}_P$ follows from the pullback-stability of $\mathcal{P}$. ∎

Proposition 7.1.3 implies in particular that the notion of topology is stable under slicing:

Corollary 7.1.4 *For every finitely complete category* $\mathbf{B}$ *with a topology* $\mathcal{P}$ *and* $B \in \operatorname{obj}\mathbf{B}$*,* $\mathcal{P}$ *can be considered as a topology on* $\mathbf{B}/B$*.*

Proof The domain functor $\partial_0 : \mathbf{B}/B \longrightarrow \mathbf{B}$ is a pullback-preserving fibration, with respect to which every morphism in $\mathbf{B}/B$ is cartesian. ∎

Compactness Fix a finitely complete category $\mathbf{X}$ equipped with a topology $\mathcal{P}$. For convenience, fix a class $\mathcal{E} \subset \operatorname{mor}\mathbf{X}$ such that $\mathcal{P}$ is an $\mathcal{E}$-topology. This is no restriction since we could choose $\mathcal{E} = \{\, 1_X \mid X \in \mathbf{X} \,\}$.

Definition 7.1.5 We call an $\mathbf{X}$-object X *($\mathcal{P}$-)compact* if $!_X : X \longrightarrow 1$ is in $\mathcal{P}$. We call a morphism $X \xrightarrow{f} Y$ *($\mathcal{P}$-)compact* if it is $\mathcal{P}_Y$-compact as an object of $\mathbf{X}/Y$, equivalently, if $f \in \mathcal{P}$.

In our generic example $\mathbf{X} = \mathbf{Alg}\, T$, $\mathcal{P} = (\mathrm{mor}\, \mathbf{S})_{\mathrm{S}}$, a lax algebra (X, a) is compact if and only if $!_X \cdot a \approx \top \cdot T!_X \approx \top$ holds.

Thus, a topological space X, considered as a lax $\mathbf{U}$-algebra is compact in our sense if and only if every ultrafilter on X converges to some point, while an approach space is compact if and only if it is 0-compact.

A closure space (X, c) is compact if and only if $cA \neq \emptyset$ for every $A \subset X$. Then in particular, $c\emptyset \neq \emptyset$, and thus by monotonicity (X, c) is compact if and only if there exists $x \in X$ with $x \in cA$ for all $A \subset X$.

Every ordered set and every (generalized) metric space is compact.

Basic properties of $\mathcal{P}$-compact objects in any finitely complete category $\mathbf{X}$ are:

Proposition 7.1.6 ([Tho03], [CGT04]) (1) *Terminal objects are compact;*

(2) *if $X \xrightarrow{e} Y$ is in $\mathcal{E}$ and X is compact then Y is compact;*

(3) *if $X \xrightarrow{f} Y$ is in $\mathcal{P}$ and Y is compact then X is compact;*

(4) *the class of compact objects is closed under finite products in $\mathbf{X}$;*

(5) *the class of compact morphisms is closed under finite products in $\mathbf{X}$.*

7.2 A Tychonoff Theorem

As we have seen, the class of $\mathcal{P}$-compact objects is closed under finite products. When we want closedness under arbitrary products, and moreover stability under slicing, we are led to the following definition:

Definition 7.2.1 Let $\mathbf{X}$ be a complete category. A topology $\mathcal{P}$ on $\mathbf{X}$ is called *stable* provided $\mathcal{P}$ is closed under wide pullbacks.

The notion of stable topology is just a "sliced Tychonoff Theorem" in disguise:

Proposition 7.2.2 *Let $\mathbf{X}$ be complete. The following statements are equivalent for a topology $\mathcal{P}$ on $\mathbf{X}$:*

(1) *$\mathcal{P}$ is stable;*

(2) *for every $\mathbf{X}$-object B, $\mathcal{P}$-compact objects in $\mathbf{X}/B$ are closed under products.* ■

A well-known example of a stable topology is given by the class of proper maps in **Top**.

Proposition 7.2.3 (Frolík Theorem for stable topologies)
If $\mathcal{P}$ is a stable topology on a complete category $\mathbf{X}$, then $\mathcal{P}$ is closed under products.

Proof This follows with a standard categorical argument: In any complete category we may form the product of a family $(f_i : A_i \longrightarrow B_i)_I$ of morphisms by forming the following pullbacks

$$\begin{array}{ccc} P_i & \xrightarrow{\overline{q}_i} & A_i \\ {\scriptstyle \overline{f}_i}\downarrow & & \downarrow{\scriptstyle f_i} \\ B & \xrightarrow[q_i]{} & B_i \end{array}$$

where $(B, q_i)_I$ is the chosen product of the B_i. Then $f = \prod f_i$ is the diagonal of a wide pullback of the family $(\overline{f}_i)$.

Now, if every f_i is in $\mathcal{P}$, then also every $\overline{f}_i$ is in $\mathcal{P}$ by pullback-stability. Hence, $f = \prod f_i$ is in $\mathcal{P}$ by closedness of $\mathcal{P}$ under wide pullbacks. ■

7.2.1 Stable topologies for lax algebras

Mindful of the result that proper maps of topological spaces are closed under wide pullback, one may ask whether also strict morphisms in other categories of lax algebras form stable topologies. In this section, we will investigate conditions under which the class of strict morphisms is closed under wide pullback. By Proposition 7.2.2, we will actually prove a variety of Tychonoff-Theorems, one for each slice.

Definition 7.2.4 Let $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ denote a complete scene and $\mathcal{M} \subset \operatorname{mor} \mathbf{S}$ a class of morphisms. We say that $\mathcal{S}$ is *pullback-modular with respect to* $\mathcal{M}$, or *$\mathcal{M}$-pullback modular*, if for each diagram

$$\begin{array}{ccccc} P & \xrightarrow{p_i} & A_i & \xleftarrow{r_i} & X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_i} & & \\ & & A & & \end{array} \qquad (i \in I) \tag{7.1}$$

in $\mathbf{D}$ where $(P, (f, p_i))$ is a wide pullback of the family $(f_i)_I$ in $\mathbf{S}$ and each f_i is in $\mathcal{M}$,

$$f \cdot \bigwedge_I p_i^* \cdot r_i = \bigwedge_I f_i \cdot r_i \tag{7.2}$$

holds in $\mathbf{D}$. In case $\mathcal{M} = \operatorname{mor} \mathbf{S}$ we just say that $\mathcal{S}$ is pullback-modular.

Proposition 7.2.5 ([Sch05]) *If $\mathcal{S}$ is $\mathcal{M}$-pullback modular, then the class $\mathcal{M}_{\mathrm{S}}$ of strict $\mathcal{M}$-morphisms is closed under wide pullbacks in* $\mathbf{Alg}\, T$.

Proof Suppose

$$\begin{array}{ccc} (P, d) & \xrightarrow{p_i} & (X_i, a_i) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_i} \\ & & (X, a) \end{array} \qquad (i \in I)$$

is a wide pullback diagram in $\mathbf{Alg}\,T$ and $f_i : (X_i, a_i) \longrightarrow (X, a)$ is strict for all $i \in I$. We need to show that $f = f_i \cdot p_i : (P, d) \longrightarrow (X, a)$ is strict.

Strictness of f_i implies $f_i \cdot a_i \cdot Tp_i = a \cdot Tf_i \cdot Tp_i = a \cdot Tf$. By construction of limits in $\mathbf{Alg}\,T$, (P, d) is the initial lift of the source (p_i), hence $d = \bigwedge p_i^* \cdot a_i \cdot Tp_i$, and we obtain, using (7.2),

$$f \cdot d = f \cdot \bigwedge p_i^* \cdot a_i \cdot Tp_i = \bigwedge f_i \cdot a_i \cdot Tp_i = \bigwedge a \cdot Tf = a \cdot Tf,$$

hence $f : (P, d) \longrightarrow (X, a)$ is strict. ∎

Remark 7.2.6 If $\mathcal{S}$ is $\mathcal{M}$-pullback modular with $\mathcal{M}$ containing all identities, then $(\mathbf{D}, \mathcal{M})$ satisfies the Frobenius law.

Proof In (7.1) specify $I = \{0, 1\}$, $r_0 = a$, $r_1 = b$, $f_0 = f$, $f_1 = 1_A$.[1] ∎

7.2.2 Pullback-modular scenes

V-matrices We recall, e.g., from [GHK+03]:

Definition 7.2.7 A complete lattice V is *completely distributive* if

$$\bigwedge_{i \in I} \bigvee_{a \in A_i} x_{i,a} = \bigvee_{a \in \left(\prod_{i \in I} A_i\right)} \bigwedge_{i \in I} x_{i,a_i} \tag{7.3}$$

holds for all sets I, $(A_i)_{i \in I}$ and $\{x_{i,a} \mid i \in I, a \in A_i\} \subset V$.

For a discussion of the relationship between completely distributive lattices and constructive completely distributive (ccd) lattices as used so far we refer to [Woo04]. We just remark that in (ZF) set theory, every completely distributive lattice is ccd. But even for the simplest lattices complete distributivity relies on the axiom of choice. To wit:

Proposition 7.2.8 *In (ZF) set theory, the following statements are equivalent:*

(1) The axiom of choice;

(2) $\mathbf{2}$ *is completely distributive;*

(3) every complete chain is completely distributive;

(4) for each set X *the powerset* PX *is completely distributive.* ∎

Proposition 7.2.9 ([Sch05]) *Let* $\mathbf{V} = (V, \otimes, k)$ *be a quantale. The following statements are equivalent:*

[1] This argument was provided by an anonymous referee of [Sch05].

(1) *V is completely distributive;*

(2) $\mathrm{Mat}\,\mathbf{V}$ *is pullback-modular.*

For any $\mathbf{V}$, $\mathrm{Mat}\,\mathbf{V}$ *is pullback-modular with respect to the class of injective functions.*

Proof Given a diagram as in (7.1), we may assume that the pullback is canonically constructed. By definition of the composition in $\mathrm{Mat}\,\mathbf{V}$, we have

$$\left(f \cdot \bigwedge_I p_i^* \cdot r_i\right)(x,a) = \bigvee_{p \in f^{-1}\{a\}} \bigwedge_I p_i^* \cdot r_i(x,p) = \bigvee_{p \in f^{-1}\{a\}} \bigwedge_I r_i(x, p_i(p)), \tag{7.4}$$

and

$$\bigwedge_I f_i \cdot r_i(x,a) = \bigwedge_I \bigvee_{a_i \in f_i^{-1}\{a\}} r_i(x,a_i). \tag{7.5}$$

By the canonical construction of wide pullbacks, $\prod_I f_i^{-1}\{a\} = f^{-1}\{a\}$ holds, hence (7.3) implies the equality of (7.4) and (7.5).

To prove complete distributivity from (2), with given data as in (7.3), set $X = A = \{*\}$ and $r_i(*,a) = x_{i,a}$ for all $i \in I$. Hence $P = \prod_I A_i$, all f_i as well as f are the unique functions and (7.3) follows from the equality of (7.4) and (7.5).

Given $r : X \nrightarrow Y$ and a subset inclusion $m : Y \longrightarrow Z$, we have

$$m \cdot r(x,z) = \bigvee_{y \in m^{-1}\{z\}} r(x,y) = \begin{cases} r(x,z) & \text{if } z \in Y, \\ \bot & \text{else.} \end{cases}$$

Suppose every f_i in (7.1) is a monomorphism, then

$$\bigwedge_I f_i \cdot r_i(x,a) = \begin{cases} \bigwedge_I r_i(x,a) & \text{if } a \in A_i \text{ for all } i \in I, \\ \bot & \text{else,} \end{cases}$$

and

$$f \cdot \bigwedge_I p_i^* \cdot r_i(x,a) = \begin{cases} \bigwedge_I r_i(x, p_i(a)) & \text{if } a \in P, \\ \bot & \text{else,} \end{cases}$$

hold. Hence (7.2) holds in this special situation without any further conditions on V. ■

Lemma 7.2.10 *Let* $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ *be a scene. If* $\mathcal{S}$ *is* $\mathcal{M}$*-pullback modular, so is* $\mathrm{Dist}(\mathcal{S}) = (\mathbf{S}, \mathrm{Cat}(\mathbf{S}, \mathbf{D}), \mathrm{Dist}(\mathbf{S}, \mathbf{D}))$.

Proof Observe that $\mathrm{Dist}(\mathbf{S}, \mathbf{D})$ is closed under composition and local infima in $\mathbf{D}$. Moreover for a morphism $X \xrightarrow{f} Y$ between discrete $(\mathbf{S}, \mathbf{D})$-categories the corresponding distributors f_* and f^* are just the $\mathbf{D}$-morphisms $f_\circ$ and f°. Thus $\mathcal{M}$-pullback modularity of $\mathrm{Dist}(\mathbf{V})$ follows trivially. ■

Since OrdMat($\mathbf{V}$) is a 1-full and 2-full subcategory of Dist($\mathbf{V}$), we obtain:

Corollary 7.2.11 *For every* $\mathbf{V}$, $\mathcal{O}(\mathbf{V})$ *is Mono-pullback modular. If* V *is completely distributive, then* $\mathcal{O}(\mathbf{V})$ *is pullback-modular.* ∎

Thus, we immediately see that $(\mathbf{Set}, \mathbf{Rel})$, $\mathcal{D}$, and $(\mathbf{Set}, \mathrm{Mat}\,\overline{\mathbb{R}}_+)$ are pullback-modular. This implies:

Corollary 7.2.12 *The following topologies are stable:*

- $\{$*proper morphisms*$\}$ *on* **Top***;*
- $\{$*proper contractions*$\}$ *on* **Ap***;*
- $\{$*closed morphisms*$\}$ *on* **Clos**. ∎

7.3 Factorization systems and closure operators

Fix a topological and finitely complete scene $\mathcal{S} = (\mathbf{S}, \mathbf{C}, \mathbf{D})$ together with a splitable factorization system $(\mathcal{E}, \mathcal{M})$ such that intersections (= wide pullbacks) of arbitrary families of $\mathcal{M}$-morphisms exist in $\mathbf{S}$. Moreover assume that $\mathcal{S}$ has the (BC)-property and that $\mathcal{S}$ is $\mathcal{M}$-pullback modular (thus $(\mathbf{D}, \mathcal{M})$ satisfies the Frobenius law).

For instance, one could choose $\mathcal{S} = (\mathbf{Set}, \mathrm{Mat}\,\mathbf{V})$ or $\mathcal{S} = \mathcal{O}(\mathbf{V})$ equipped with the (Epi,Mono)-factorization system for *any* quantale $\mathbf{V}$.

Fix any (monadic) actor T on $\mathcal{S}$. We will state the following results only for $\mathbf{Alg}\,T$, but they are also valid in $\mathbf{Alg}_r\mathsf{T}$, $\mathbf{Alg}_u\mathsf{T}$, and $\mathbf{Alg}_t\mathsf{T}$, since the latter categories are closed under limits in $\mathbf{Alg}\,T$. From the results obtained so far, we can deduce easily the following properties of the class $\mathcal{M}_{\mathrm{S}} \subset \mathrm{mor}\,\mathbf{Alg}\,T$:

(1) $\mathcal{M}_{\mathrm{S}}$ is closed under composition and contains all isomorphisms;

(2) $\mathcal{M}_{\mathrm{S}}$ is stable under pullback;

(3) $\mathcal{M}_{\mathrm{S}}$ is closed under arbitrary intersections.

This immediately leads to the following result.

Proposition 7.3.1 *There is a conglomerate* $\mathbb{E}$ *of sinks in* $\mathbf{Alg}\,T$ *such that* $(\mathbb{E}, \mathcal{M}_{\mathrm{S}})$ *is a factorization structure for sinks in* $\mathbf{Alg}\,T$.

Proof Use [AHS04, Proposition 15.14]. ∎

For the next result, recall the notion of (categorical) closure operator from [DT95].

Proposition 7.3.2 *For* $(X,a) \in \mathbf{Alg}\,T$ *define* $c_{(X,a)} : \mathrm{Sub}_{\mathcal{M}_\mathrm{I}}(X,a) \longrightarrow \mathrm{Sub}_{\mathcal{M}_\mathrm{I}}(X,a)$ *by* $c_{(X,a)}(m) = \bigwedge\{\, s \in \mathrm{Sub}_{\mathcal{M}_\mathrm{S}}(X,a) \mid m \leq s \,\}$*, where* $\bigwedge$ *is intersection in the (complete) lattice of* $\mathcal{M}_\mathrm{I}$*-subobjects of* (X,a)*. Then* $(c_{(X,a)})$ *defines a weakly hereditary and idempotent (categorical) closure operator on* $\mathbf{Alg}\,T$*. Moreover, the class of closed embeddings with respect to this closure operator is precisely* $\mathcal{M}_\mathrm{S}$*.*

Proof Use [DT95, Exercise 2.D]. ∎

Let us call the closure operator defined above the *strict closure operator* of $\mathbf{Alg}\,T$.

Examples The strict closure operator on

- $\mathbf{Ord} \cong \mathbf{Alg}_t\mathbf{1}$ is the up-closure operator of **Ord**, see [DT95].
- $\mathbf{Top} \cong \mathbf{Alg}_t\mathbf{U}$ is the Kuratowski-closure operator.

Groundedness Assume $\mathbf{S}$ to have an initial object 0 such that for every X in $\mathbf{S}$, the unique morphism $0 \xrightarrow{\mathrm{i}_X} X$ is in $\mathcal{M}$. Under this conditions, we can prove:

Proposition 7.3.3 *The strict closure operator of* $\mathbf{Alg}\,T$ *is grounded provided* T *sends* 0 *to an initial object of* $\mathbf{D}$*.*

Proof It suffices to show that the least element of $\mathcal{M}_\mathrm{I}(X,a)$ is strict. Our general assumptions imply that this least element is just $(0, ¡^\diamond_X a) \xrightarrow{\mathrm{i}_X} (X,a)$. Clearly, this morphism is strict provided $T0$ is initial in $\mathbf{D}$. ∎

Additivity Using the results from Section 6.1, we can give conditions under which the strict closure is additive. Let us assume that $\mathbf{D}$ locally has (finite) suprema, that these are preserved by composition, and that $\mathbf{S}$ has (finite) coproducts, which are biproducts in $\mathbf{D}$ (cf. Section 6.1).

By [DT95, Proposition 2.6], the strict closure operator of $\mathbf{Alg}\,T$ will be additive if and only if $\mathcal{M}_\mathrm{S}$ is closed under binary suprema in the complete lattice $\mathcal{M}_\mathrm{I}$, and fully additive if and only if $\mathcal{M}_\mathrm{S}$ is closed under non-empty suprema in $\mathcal{M}_\mathrm{I}$. Using these facts, we can prove:

Proposition 7.3.4 *Suppose* $T[\mathcal{E}] \subset \mathrm{Cov}\,\mathbf{D}$ *holds. The strict closure operator of* $\mathbf{Alg}\,T$ *is fully additive (additive, respectively) provided* T *preserves non-empty coproducts (preserves binary coproducts, respectively).*

Proof Recall the construction of suprema in $\mathcal{M}_\mathrm{I}$ already used in Section 6.1.2: Given a set-indexed family $(A_i \xrightarrow{m_i} X)_I$ subobjects, its supremum can be formed by first forming the induced morphism $[m_i] : \coprod_I A_i \longrightarrow X$ and then $(\mathcal{E}, \mathcal{M}_\mathrm{I})$ factoring $[m_i]$ as, say, $[m_i] = m \cdot e$; m will be the supremum of the family (m_i).

Now, in case each m_i is in $\mathcal{M}_{\mathrm{S}}$, we obtain $[m_i] \in \mathcal{M}_{\mathrm{S}}$ by Proposition 6.1.9. Since for any $(\mathcal{E}, \mathcal{M}_{\mathrm{I}})$-factorization $[m_i] = m \cdot e$ as above, Te will be a cover, Proposition 3.5.7(2a) implies that m will be strict. Thus $\mathcal{M}_{\mathrm{S}}$ is closed under non-empty set-indexed suprema in $\mathcal{M}_{\mathrm{I}}$.

The binary case is proved in the same manner. ■

Over Set We consider now the case $\mathbf{S} = \mathbf{Set}$, with (Epi, Mono)-factorization system. Since for any lax algebra (X, a), $\mathrm{Sub}_{\mathcal{M}_{\mathrm{I}}}(X, a)$ is isomorphic to the powerset PX, the strict closure on $\mathbf{Alg}\, T$ gives rise to a closure space $\mathcal{C}(X, a) = (X, c_a)$ for each lax algebra (X, a), where c_a is defined by

$$c_a A = \bigcap \{\, B \subset X \mid i_B : (B, i_B^{\diamond} a) \longhookrightarrow (X, a) \text{ is strict} \,\}.$$

Since the strict closure on $\mathbf{Alg}\, T$ is a categorical closure operator, we have defined a concrete functor $\mathbf{Alg}\, T \longrightarrow \mathbf{Clos}$.

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