Maximal Inequalities for the Ornstein-Uhlenbeck Process

S. E. GRAVERSEN* *and* G. PESKIR*

Abstract

Let $V = (V_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck velocity process solving

$$dV_t = -\beta V_t \, dt + dB_t$$

with $V_0 = 0$, where $\beta > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. Then there exist universal constants $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{C_1}{\sqrt{\beta}} \, E\sqrt{\log(1+\beta\tau)} \leq E\left(\max_{0 \leq t \leq \tau} |V_t| \right) \leq \frac{C_2}{\sqrt{\beta}} \, E\sqrt{\log(1+\beta\tau)}$$

for all stopping times τ of V . In particular, this yields the existence of universal constants $D_1 > 0$ and $D_2 > 0$ such that

$$D_1 \, E\sqrt{\log\bigl(1+\log(1+\tau)\bigr)} \leq E\left(\max_{0 \leq t \leq \tau} \frac{|B_t|}{\sqrt{1+t}} \right) \leq D_2 \, E\sqrt{\log\bigl(1+\log(1+\tau)\bigr)}$$

for all stopping times τ of B . This inequality may be viewed as a stopped law of iterated logarithm. The method of proof relies upon a variant of Lenglart's domination principle [2] and makes use of Itô calculus.

*Centre for Mathematical Physics and Stochastics, supported by the Danish National Research Foundation.

AMS 1980 *subject classifications*. Primary 60J65, 60G40, 60E15. Secondary 60J60, 60G15.

Key words and phrases: The Ornstein-Uhlenbeck velocity process, maximum process, stopping time, maximal inequality, Lenglart's domination principle, Brownian motion, diffusion process, Gaussian process, the Langevin stochastic differential equation.

REFERENCES

[1] BURKHOLDER, D. L. *and* GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* 124 (249-304).

[2] LENGLART, E. (1977). Relation de domination entre deux processus. *Ann. Inst. H. Poincaré Probab. Statist.* 13 (171-179).

[3] NELSON, E. (1967). *Dynamical Theories of Brownian Motion.* Princeton Univ. Press.

[4] PESKIR, G. (1998). Controlling the velocity of Brownian motion by its terminal value. *Research Report* No. 391, *Dept. Theoret. Statist. Aarhus* (11 pp).

[5] REVUZ, D. *and* YOR, M. (1994). *Continuous Martingales and Brownian Motion.* Springer-Verlag.

[6] UHLENBECK, G. E. *and* ORNSTEIN, L. S. (1930). On the theory of Brownian motion. *Physical Review* 36 (823-841).