Martingale Representation, Chaos Expansion and Clark-Ocone Formulas

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Abstract

Ito's representation theorem gives the existence of a martingale representation of stochastic variables with respect to Brownian motion. Similar results exist for instance for compensated Poisson processes and Azema's martingale. We give sufficient conditions for predictable representation (in a weak sense), i.e. there exists predictable processes ϕ^α such that every $F \in L^2(\mathcal{F}_\infty, P)$ can be represented $F = E[F] + \sum_\alpha \phi^\alpha \cdot M^\alpha$ for some given collection of martingales $\{M^\alpha\}_{\alpha\in\mathcal{I}}$. Thereafter we show how one can obtain explicit expressions for the representation using Malliavin calculus methods. The theory is then applied to Lévy processes.

Keywords. chaos expansion ; Clark-Ocone formula ; Lévy process ; property of predictable representation ; stable subspace

1 Introduction

Clark-Ocone formulas have proved to be useful tools to obtain explicit expressions for the martingale representation of stochastic variables. In the Brownian motion case the Clark-Ocone formula has for instance been used frequently to obtain hedging portfolios for derivatives. The object of this paper is to generalize the formula to the case of an infinite collection of stochastic processes, each admitting a chaos decomposition. The Clark-Ocone formula we obtain is in some sense an infinitely dimensional extension of the Clark-Ocone formula appearing in [8].

The first step in order to derive an explicit expression for the martingale representation of stochastic variables is to show that there exists a martingale representation of a given form. In section 2 we first extend some of the results in [11] and introduce the notion of a basis for predictable representation. A basis for predictable representation can be viewed as an analogue of a basis of orthogonal functions in a classical Hilbert space. It is basically a collection of orthogonal square integrable martingales such that every square integrable martingale can be represented as a sum of integrals with respect to these martingales. This property is shown to be equal to the predictable representation property (in a weak sense), i.e. every square integrable functional which is measurable with respect to $\mathcal{F}_\infty$ can be expressed by its expectation plus a sum of integrals with respect to the martingales constituting the basis of predictable representation. The main result in the section concerning martingale representation is a result which gives sufficient conditions for a collections of martingales to have this property.

Section 3 deals with the possibility of chaos expansions. In order to use Malliavin calculus methods we want to have a chaos expansion property. We assume that each of the square integrable martingales constituting the basis for predictable representation have the chaos representation property. By results in [5] this is the same as assuming that each of the martingales have the predictable representation property. The chaos decomposition is obtained by using a monotone class argument, the density of the Doléans-Dade exponential and iterative use of the representation property. Thereafter we obtain a chaos expansion consisting of sums of integrals with respect to product stochastic measures, similar to the Wiener chaos expansion.

In section 4 we define a Gross derivative of stochastic variables via their chaos expansion. The definition is similar to the definition in [8], and the results in this section extend some of the results in that paper. We show a correspondence between this derivative and the directional Gross derivative and the directional Gross derivative (difference) for Poisson processes. This enables us to actually compute the derivative in these cases. The Clark-Ocone formula is obtained by similar techniques as its counterpart in [8]. Our formula is in a sense an infinitely dimensional generalisation of the Clark-Ocone formula in [8]. As in this paper we only assume that the martingales constituting the basis for predictable representation satisfis certain properties, and do not specify these martingales to be for instance Brownian motions.

A basis for predictable representation which satisfies the sufficient conditions for a chaos expansion can often be seen directly from a known representation of a given process. In Section 5 we apply the theory to Lévy processes which is one such example. Such processes can be expressed in terms of a Brownian motion and Poisson processes. The Brownian motion and the compensated Poisson processes will then constitute a basis for predictable representation which also allows a chaos expansion. The Clark-Ocone formula then provides a practical tool to obtain expressions for stochastic variables.

In order to illustrate the theory we give two examples in the Lévy process case.

2 Martingale representation

We assume as given a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. As usual we assume that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and that the filtration $\{\mathcal{F}_t\}$ is right-continuous and contains all P-null sets. Following the notation and ideas in [11],

Definition 1. Denote by $\mathbf{M}^2$ the space of all square integrable martingales M such that $\sup_{t\geq 0} E\big[M_t^2\big] < \infty$, and $M_0 = 0$ a.s.

Notice that if $M \in \mathbf{M}^2$ then $\lim_{t\to\infty} E\big[M_t\big] = E\big[M_\infty\big] < \infty$. Such martingales are identified by their terminal value. We endow $\mathbf{M}^2$ with the inner-product $(M, N) = E\big[M_\infty N_\infty\big]$, and notice that $\mathbf{M}^2$ is a Hilbert space. (see [11] for more details)

A closed subspace $F \subset \mathbf{M}^2$ is called *stable* if $M \in F$ implies $M^\tau \in F$ for every stopping time τ. (where $M_t^\tau = M_{t\wedge\tau}$). Let $\mathcal{A}$ be a subset of $\mathbf{M}^2$. We define *the stable subspace generated by* $\mathcal{A}$, denoted $\mathcal{S}(\mathcal{A})$, as the intersection of all closed, stable subspaces containing $\mathcal{A}$.

Definition 2. We will call a collection $\{M^\alpha\}_{\alpha\in\mathcal{I}}$ of martingales in $\mathbf{M}^2$ a *Basis for Predictable Representation* (BPR) if M^α and M^β are orthogonal for all $\alpha \neq \beta$ and $\mathcal{S}(\{M^\alpha\}_{\alpha\in\mathcal{I}}) = \mathbf{M}^2$.

Note that since $\mathbf{M}^2$ is a Hilbert space a BPR for $\mathbf{M}^2$ always exists. The next two results, Lemma 3 and Proposition 4 are almost identical to [2, remark, pp. 360].

Lemma 3. *Assume that* $\{M^\alpha\}$ *is a BPR for* $\mathbf{M}^2$. *Then*

$$\mathbf{M}^2 = \bigoplus_{\alpha\in\mathcal{I}} \mathcal{S}(M^\alpha)$$

Proof. First note that $\mathcal{S}(\mathcal{A})$ is a closed subspace of a Hilbert space and hence complete for every $\mathcal{A} \subset \mathbf{M}^2$. Therefore $X := \bigoplus_{\alpha\in\mathcal{I}} \mathcal{S}(M^\alpha)$ is complete and therefore also closed. Now choose $N \in \mathbf{M}^2$ and assume that N is orthogonal to X. By [11, Th.35, pp.149] we have $\mathcal{S}(M^\alpha, \ldots, M^\gamma) = \mathcal{S}(M^\alpha) \oplus \cdots \oplus \mathcal{S}(M^\gamma)$ for every finite collection $\alpha, \ldots, \gamma$. Since $\mathbf{M}^2 = \mathcal{S}(\{M^\alpha\})$ we see that N must be zero. Hence, X is dense in $\mathbf{M}^2$. Since X is closed, $X = \mathbf{M}^2$. $\square$

The direct sum $\bigoplus_\alpha \mathcal{S}(M^\alpha)$ is interpreted in the usual way, i.e. if $\mathcal{I}$ is a countable set we denote by $\bigoplus_\alpha \mathcal{S}(M^\alpha)$ the set

$$\left\{ x = (x_\alpha, x_\beta, \dots) \in \mathcal{S}(M^\alpha) \times \mathcal{S}(M^\beta) \times \cdots ; \sum_{\alpha \in \mathcal{I}} \|x_\alpha\|_\alpha^2 < \infty \right\} \tag{1}$$

where $\|\cdot\|_\alpha$ denotes the norm in $\mathcal{S}(M^\alpha)$. The concept is extended to general infinite index sets $\mathcal{I}$ by simply noting that since all the Hilbert-spaces $\mathcal{S}(M^\alpha)$ are mutually orthogonal, at most a countable number of the x_α's are nonzero. Note that by Lemma 3 we have the isometry $\|M\|^2_{L^2(\Omega)} = \sum_{\alpha \in \mathcal{I}} \|Y^\alpha\|^2_{L^2(\Omega)}$ for every $M \in \mathbf{M}^2$, where $M = (Y^\alpha, Y^\beta, \dots)$ is in $\bigoplus_\alpha \mathcal{S}(M^\alpha)$.

Proposition 4. *Assume $\{M^\alpha\}$ is a BPR for $\mathbf{M}^2$. Let $F \in L^2(\mathcal{F}_\infty, P)$. Then F has a representation*

$$F = E\big[F\big] + \sum_{\alpha \in \mathcal{I}} H^\alpha \cdot M^\alpha$$

where H^α is predictable and $\sum_{\alpha \in \mathcal{I}} E\big[(H^\alpha)^2 \cdot [M^\alpha, M^\alpha]\big] < \infty$.

Proof. $F = E\big[F\big] + E\big[F - E[F] | \mathcal{F}_\infty\big]$. The process $M_t = E\big[F - E[F] | \mathcal{F}_t\big]$ is a square integrable martingale with $M_0 = 0$, hence $M \in \mathbf{M}^2$. Therefore every $F \in L^2(\mathcal{F}_\infty, P)$ has a decomposition $F = E\big[F\big] + M_\infty$ where $M \in \mathbf{M}^2$. By Lemma 3 every $M \in \mathbf{M}^2$ can be represented as

$$M = (Y^\alpha, Y^\beta, \dots), \qquad Y^\alpha \in \mathcal{S}(M^\alpha) \tag{2}$$

The result now follows since $\bigoplus_{\alpha \in \mathcal{I}} \mathcal{S}(M^\alpha)$ and $\sum_{\alpha \in \mathcal{I}} \mathcal{S}(M^\alpha)$ are unitarily equivalent and by [11, Th.35, pp. 149],

$$\mathcal{S}(M^\alpha) = \big\{ H \cdot M^\alpha ; E\big[(H)^2 \cdot [M^\alpha, M^\alpha]\big] < \infty \big\} \tag{3}$$

□

Given a collection $\{M^\alpha\}_{\alpha \in \mathcal{I}}$ of martingales in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, we want to determine sufficient conditions for $\{M^\alpha\}$ to be a BPR. Denote by $\mathcal{N}$ the P-null sets of $\mathcal{F}$. Let $\{M^\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of mutually independent stochastic processes, each square integrable martingales with respect to their own completed filtration, $\mathcal{F}^\alpha_t = \sigma(M^\alpha_s; 0 \le s \le t) \vee \mathcal{N}$. Denote by $\mathbf{M}^2_\alpha$ the set of martingales $\mathbf{M}^2$ in $(\Omega, \mathcal{F}, \{\mathcal{F}^\alpha\}, P)$. We assume that all the filtrations $\{\mathcal{F}^\alpha_t\}$ are right-continuous. Let $\mathcal{F}_t \subset \sigma(\mathcal{F}^\alpha_t, \mathcal{F}^\beta_t, \dots)$ be a right-continuous filtration such that all the processes M^α are martingales with respect to $\{\mathcal{F}_t\}$. Notice that the filtration $\{\mathcal{F}_t\}$ given by $\mathcal{F}_t = \sigma(\mathcal{F}^\alpha_t, \mathcal{F}^\beta_t, \dots)$ satisfies these requirements, so such filtrations exist.

Theorem 5. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space and $\{M^\alpha\}_{\alpha \in \mathcal{I}}$ a collection of martingales, where M^α and $\{\mathcal{F}_t\}$ are as specified above. Assume that $[M^\alpha, M^\beta] = 0$ for all $\alpha \neq \beta$. If $\mathcal{S}(M^\alpha) = \mathbf{M}^2_\alpha$ (i.e. M^α has the predictable representation property). Then the collection $\{M^\alpha\}$ is a BPR for $\mathbf{M}^2$.*

Proof. Define a multiplicative class $\mathcal{M}$ by

$$\mathcal{M} := \left\{ \prod_{\substack{i=1 \\ \alpha_i \in \mathcal{I}}}^{n} F^{\alpha_i};\ \big| \prod_{\substack{i=1 \\ \alpha_i \in \mathcal{I}}}^{n} F^{\alpha_i} \big| < \infty, n < \infty, n \in \mathbb{N},\ F^{\alpha_i} \text{ is } \mathcal{F}^{\alpha_i}_\infty\text{-measurable} \right\} \tag{4}$$

Define a space $\mathcal{H}$ by

$$\mathcal{H} := \Big\{ F \in \mathbb{R} + \mathbf{M}^2;\ \text{there exist } \{H^\alpha\}, E\Big[\int_0^\infty (H^\alpha_t)^2 d[M^\alpha, M^\alpha]_t\Big] < \infty$$

$$H^\alpha \text{ is predictable, and such that } F = E\big[F\big] + \sum_{\alpha \in \mathcal{I}} \int_0^\infty H^\alpha_t dM^\alpha_t \Big\} \tag{5}$$

In order to apply the monotone class theorem we first want to show that $\mathcal{H}$ is a monotone vector space. Let F_n be a monotone increasing sequence in $\mathcal{H}$. For each n there exists a collection of predictable processes $\{H^\alpha_n\}$ such that $F_n = E\big[F_n\big] + \sum_{\alpha \in \mathcal{I}} \int_0^\infty H^\alpha_n(t) dM^\alpha_t$. Since F_n converges to F and $\mathcal{S}(M^\alpha)$ which is equal to $\big\{H \cdot M^\alpha; E\big[H^2 \cdot [M^\alpha, M^\alpha]\big] < \infty, H \in \mathcal{P}\big\}$ and $\mathcal{S}(M^\alpha)$ is a closed subset of a Hilbert space, F must be in $\mathcal{H}$.

In order to show that $\mathcal{M}$ is contained in $\mathcal{H}$ consider first for simplicity one element $F^1 F^2 \in \mathcal{M}$. Since F^1 is $\mathcal{F}^1_\infty$-measurable and $\mathcal{S}(M^1) = \mathbf{M}^2_1$, there exists a predictable process H^1 with $E\big[\int_0^\infty (H^1_t)^2 d[M^1, M^1]\big] < \infty$ such that $F^1 = E\big[F^1\big] + \int_0^\infty H_t^1 dM_t^1$. Similarly $F^2 = E\big[F^2\big] + \int_0^\infty H_t^2 dM_t^2$. Now,

$$\begin{aligned}
F^1F^2 &= E\big[F^1\big]E\big[F^2\big] + \int_0^\infty E\big[F^1\big]H_t^2 dM_t^2 + \int_0^\infty E\big[F^2\big]H_t^1 dM_t^1 \\
&\quad + \Big(\int_0^\infty H_t^1 dM_t^1\Big)\Big(\int_0^\infty H_t^2 dM_t^2\Big) \\
&= E\big[F^1\big]E\big[F^2\big] + \int_0^\infty E\big[F^1\big]H_t^2 dM_t^2 + \int_0^\infty E\big[F^2\big]H_t^1 dM_t^1 \\
&\quad + \int_0^\infty \Big(\int_0^{t-} H_s^1 dM_s^1\Big) H_t^2 dM_t^2 + \int_0^\infty \Big(\int_0^{t-} H_s^2 dM_s^2\Big) H_t^1 dM_t^1 \\
&\quad + \int_0^\infty H_t^1 H_t^2 d[M^1, M^2]_t \\
&= E\big[F^1\big]E\big[F^2\big] + \int_0^\infty \Big(E\big[F^1\big] + \int_0^{t-} H_s^1 dM_s^1\Big) H_t^2 dM_t^2 \\
&\quad + \int_0^\infty \Big(E\big[F^2\big] + \int_0^{t-} H_s^2 dM_s^2\Big) H_t^1 dM_t^1
\end{aligned}$$

where we have used that $[M^1, M^2] = 0$. This shows that $F^1F^2 \in \mathcal{H}$. The proof that $F^1 \cdots F^n \in \mathcal{M}$ is in $\mathcal{H}$ is similar. Hence, $\mathcal{M}$ is contained in $\mathcal{H}$. Since $|M^\alpha| < \infty$ almost surely, $\mathcal{M}$ contains all the martingales M^α. Obviously, $\sigma(\mathcal{M}_t) = \sigma(\{\mathcal{F}_t^\alpha\})$. By the monotone class theorem $\mathcal{H} = \mathbf{M}^2$ and the result follows. □

Standard examples of processes satisfying the conditions in Theorem 5 are independent Brownian motions and independent Poisson processes. Both possess the predictable representation property which is the same as $\mathcal{S}(M^\alpha) = \mathbf{M}_\alpha^2$. If B^1 and B^2 are two independent Brownian motions then $[B^1, B^2] = 0$. This follows from the fact that $[B, B]_t = \langle B, B\rangle_t = t$ and the polarization identity. If N^1 and N^2 are two independent compensated Poisson processes then $[N^1, N^2] = 0$. This is because two independent Poisson processes jump at different times almost surely, and if X is a pure jump semimartingale then $[X, Y]_t = \sum_{0<s\leq t} \triangle X_s \triangle Y_s$ for any semimartingale Y ([11, Th. 28. pp.68]). This also shows that $[B, N] = 0$ when B is a Brownian motion and N is a compensated Poisson process. Hence any mutually independent collection of Brownian motions and compensated Poisson processes satisfy the conditions of Theorem 5.

A not so standard example is the case when the filtration $\mathcal{F}_t = \sigma(\mathcal{M}_t, \mathcal{G}_t)$ where $\mathcal{M}_t$ is the completed filtration of a Brownian motion W and $\mathcal{G}_t$ is the completed filtration of an Azema martingale A independent of W. Since Azema's martingale is quadratic pure jump it follows from [11, Th. 28, pp.68] that $[W, A] = 0$. Both the Brownian motion and Azema's martingale possess the predictable representation property with respect to their own filtration. Hence by Theorem 5, $\{W, A\}$ is a BPR for $\mathbf{M}^2$ in this case. (for properties of Azema's martingale see [11, pp. 180]).

3 Chaos expansion

We will now specialize to the case where there exists a BPR $\{M^\alpha\}_{\alpha\in\mathcal{I}}$ with some desired properties. These properties are that $[M^\alpha, M^\beta] = 0$ and M^α and M^β are independent for $\alpha \neq \beta$. I.e. the conditions of Theorem 5 are satisfied. In addition we will assume that $\langle M^\alpha \rangle$ is deterministic and absolutely continuous with respect to Lebesgue measure for every $\alpha \in \mathcal{I}$. Let $[0, T]$ be a finite fixed time horizon. Define the integrals

$$J_n(H_n) := \sum_{\alpha_1\in\mathcal{I}} \int_0^T \sum_{\alpha_2\in\mathcal{I}} \int_0^{t_1-} \cdots \\ \cdots \sum_{\alpha_n\in\mathcal{I}} \int_0^{t_{n-1}-} H^{\alpha_1,\dots,\alpha_n}(t_1, \dots, t_n) dM_{t_n}^{\alpha_n} \cdots dM_{t_n}^{\alpha_1} \quad (6)$$

$$H_n(t_1, \dots, t_n) \in \bigoplus_{(\alpha_1,.,\alpha_n)\in\mathcal{I}^n} L^2\big(d\langle M^{\alpha_1}\rangle \otimes \cdot \cdot \otimes d\langle M^{\alpha_n}\rangle \otimes dP\big) \quad (7)$$

Where, $H_n(t_1, \dots, t_n) = \big(H_n^{\alpha_1,\dots,\alpha_n}(t_1, \dots, t_n), H_n^{\beta_1,\dots,\beta_n}(t_1, \dots, t_n), \dots\big)$, and $H_n^{\alpha_1,\dots,\alpha_n}(t_1, \dots, t_n)$ is predictable. Notice that we have the Ito isometry,

$$E\big[J_1(H)J_1(G)\big] = \sum_\alpha E\big[\int_0^T H_t^\alpha G_t^\alpha d\langle M^\alpha\rangle_t\big] \quad (8)$$

Define spaces $\mathcal{D}_n$ by

$$\mathcal{D}_n := \Big\{ H_n \in \bigoplus_{\alpha\in\mathcal{I}^n} L^2\big([0, T]^n, d\langle M^{\alpha_1}\rangle \otimes \cdots \otimes d\langle M^{\alpha_n}\rangle\big)\Big\} \quad (9)$$

and the corresponding n'th homogeneous chaos $\mathcal{H}_n$ as the intersection of all closed spaces containing

$$\Big\{ \sum_{\alpha\in\mathcal{I}} H^\alpha \cdot M^\alpha; H = (H^\alpha, H^\beta, \dots) \in \mathcal{D}_n \Big\} \quad (10)$$

From the Ito isometry we see that $\mathcal{H}_n$ and $\mathcal{H}_m$ are orthogonal whenever $n \neq m$. For notational convenience we make the convention, $\mathcal{H}_0 := \mathbb{R}$.

Theorem 6. *Assume $\{M^\alpha\}$ is a BPR for $\mathbf{M}^2$, where the M^α's are mutually independent and every M^α has the chaos representation property (with respect to its own filtration). If $[M^\alpha, M^\beta] = 0$ for every $\alpha \neq \beta$ and $\langle M^\alpha \rangle \ll dt$ is deterministic, then*

$$L^2(\mathcal{F}_T, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

Proof. Choose one $F \in L^2(\mathcal{F}_T, P)$. By Theorem 6,

$$F = E\big[F\big] + J_1(H_1), \qquad H_1 = H^{\alpha_1}(t_1) \tag{11}$$

Now we use the same procedure again to obtain a representation of H_1 to get

$$\begin{aligned} F &= E\big[F\big] + J_1\big(E\big[H_1\big] + J_1(H_2)\big) \\ &= E\big[F\big] + J_1\big(E\big[H_1\big]\big) + J_2(H_2) \end{aligned}$$

we proceed in this way, and after n steps we have

$$F = E\big[F\big] + J_1(F_1) + \cdots + J_{n-1}(F_{n-1}) + J_n(H_n) \tag{12}$$

where $F_i \in \mathcal{D}_i$. By the triangle inequality we see that the norm of $R_n := J_n(H_n)$ is bounded by the norm of F. Therefore $R := \lim_{n\to\infty} R_n$ exists as a limit in $L^2(P)$. We want to show that $R = 0$.

Since every M^α has the chaos expansion property the linear span of the Doléans-Dade exponentials $\mathcal{E}(\theta \cdot M^\alpha)$ is a dense subset of $\mathbf{M}^2_\alpha$ where $\theta = \theta_t$ is a deterministic process. This is due to the fact that it holds in the Brownian motion case. Therefore there exist for every $f_n \in L^2([0,T]^n, dt)$ a sequence $\{\theta_i\}_{i=1}^\infty$ of deterministic functions in $L^2([0,T])$ and constants $\{c_i\}_{i=1}^\infty$ such that $\lim_{N\to\infty} \sum_{i=1}^N c_i(\theta_i)^{\otimes n} = f_n$ in $L^2([0,T]^n, dt)$ (where $\theta^{\otimes n} = \theta_{t_1}\theta_{t_2}\cdots\theta_{t_n}$). Since we have assumed that $\langle M^\alpha \rangle$ is absolutely continuous with respect to Lebesgue measure, this implies that for every $f_n \in L^2([0,T]^n, d\langle M^\alpha\rangle)$ one can find sequences as above such that $\lim_{N\to\infty} \sum_{i=1}^N c_i(\theta_i)^{\otimes n} = f_n$. The density of the linear span of Doléans-Dade exponentials then follows from their chaos expansion.

By a monotone class argument similar to that in the proof of Theorem 5 one can then show that the linear span of $\big\{\prod_\alpha \mathcal{E}\big(\theta^\alpha \cdot M^\alpha\big)\ ;\ \theta^\alpha \text{ deterministic } \big\}$ is dense in $L^2(\mathcal{F}_T, P)$. Since $[M^\alpha, M^\beta] = 0$ for $\alpha \neq \beta$ it follows from [11, Th. 37, pp.79] that $\prod_\alpha \mathcal{E}\big(\theta^\alpha \cdot M^\alpha\big) = \mathcal{E}\big(\sum_\alpha \theta^\alpha M^\alpha\big)$ which shows that the linear span of $\big\{\mathcal{E}\big(\sum_{\alpha\in\mathcal{I}} \theta^\alpha \cdot M^\alpha\big)\ ;\ \theta^\alpha \text{ deterministic } \big\}$ is dense in $L^2(\mathcal{F}_T, P)$.

Consider a Doléans-Dade martingale $Z = \mathcal{E}\big(\sum_\alpha \theta^\alpha \cdot M^\alpha\big)$. That is Z solves the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-} \sum_\alpha \theta^\alpha dM^\alpha$. Hence

$$E\big[Z_t^2\big] = 1 + E\big[\big(\int_0^t Z_{s-} \sum_\alpha \theta^\alpha dM^\alpha\big)^2\big] = 1 + \int_0^t \sum_\alpha E\big[Z_{s-}^2\big] d\langle M^\alpha\rangle_s \quad (13)$$

We therefore have that $E\big[Z_t^2\big] = \exp\big\{\sum_\alpha \langle M^\alpha\rangle_t\big\}$. Now,

$$\exp\big\{\sum_\alpha \langle M^\alpha\rangle_t\big\} = 1 + \sum_\alpha \langle M^\alpha\rangle_t + \frac{(\sum_\alpha \langle M^\alpha\rangle_t)^2}{2} + \cdots \quad (14)$$

by comparing the norm of the different terms in the chaos expansion of Z given by (12) with the expansion (14), we deduce that all exponential martingales $\mathcal{E}\big(\sum_\alpha \theta^\alpha \cdot M^\alpha\big)$ are contained in $\bigoplus_{n=0}^\infty \mathcal{H}_n$. Hence R is orthogonal to a dense subset of $L^2(\mathcal{F}_T, P)$ and must therefore be zero. Since $\bigoplus_{n=0}^\infty \mathcal{H}_n$ is closed it must be equal to $L^2(\mathcal{F}_T, P)$. $\square$

$$\begin{array}{ccc}
\mathbb{R} \oplus \mathbb{M}^2 & \longleftrightarrow & \mathbb{R} \oplus \bigoplus_{\alpha \in \mathcal{I}} \mathcal{S}(M^\alpha) \\
\updownarrow & & \updownarrow \\
L^2(\mathcal{F}_T, P) & \longleftrightarrow & \bigoplus_{n=0}^\infty \mathcal{H}_n
\end{array}$$

It is clear from proceeding discussions that if we fix the timehorizon to $[0, T]$ then there exists bijective isometries between all the spaces in the diagram if there exists a BPR $\{M^\alpha\}$ for $\mathbf{M}^2$ which fulfills the conditions in Theorem 6. These conditions are satisfied if for instance $\{M^\alpha\}$ consists of Brownian motions and/or compensated Poisson processes.

4 Clark-Ocone formula

So far the attention has been on the existence of predictable processes such that a given stochastic variable can be represented as integrals of these predictable processes with respect to some given martingales $\{M^\alpha\}_{\alpha \in \mathcal{I}}$. We will now try to obtain explicit expressions for the predictable processes appearing in this representation. This can be achieved by Clark-Ocone formulas which express these processes in terms of a Gross derivative. We will start by defining a Gross derivative via the chaos expansion. The Clark-Ocone formula can then be proved in a way similar to that in [8]. Then we show how the Gross derivative is related to the Gross derivative in the Brownian motion case and the Poisson process case. This has the advantage that we can use existing theory of the properties of the Gross derivative in these cases.

For the rest of the paper we assume that $\{M^\alpha\}_{\alpha\in\mathcal{I}}$ is a BPR for $\mathbf{M}^2$ which satisfies the conditions in Theorem 6. That is $\{M^\alpha\}$ consists of mutually independent martingales with $\langle M^\alpha\rangle \ll dt$ deterministic and the quadratic covariation process $[M^\alpha, M^\beta] = 0$ for $\alpha \neq \beta$. When $\langle M^\alpha\rangle \ll dt$ the diagonals $\triangle_n = \big\{(t_1,\ldots,t_n); t_i = t_j \text{ for some } i,j \text{ and } i \neq j\big\}$ are null sets of the measures $d\langle M^{\alpha_1}\rangle \otimes \cdots \otimes d\langle M^{\alpha_n}\rangle$.

Define the spaces $\mathbf{X}^n := \bigoplus_{\alpha\in\mathcal{I}^n} L^2([0,T]^n \times \Omega)$, and denote by

$$\|H\|_n^2 = \sum_{(\alpha_1,.,\alpha_n)\in\mathcal{I}^n} E\Big[\int_{[0,T]^n} \big(H^{(\alpha_1,.,\alpha_n)}(t_1,.,t_n)\big)^2 d\langle M^{\alpha_1}\rangle_{t_1} \cdot\cdot \langle M^{\alpha_n}\rangle_{t_n}\Big] \quad (15)$$

the norm on $\mathbf{X}^n$. We are now extending the integral J_n defined previously on the increasing simplex to its symmetrization $[0,T]^n$. As pointed out by several authors, e.g. [8], the symmetrization does not cover the diagonal. Since the diagonals are null sets, every function on $[0,T]^n$ is equal in $L^2(dt)$ to a function on $[0,T]^n$ which are zero on the diagonal in $\mathbf{X}^n$. We can therefore assume that every function is zero on the diagonal. Denote by $\mathbf{X}_s^n := \bigoplus_{\alpha\in\mathcal{I}^n} L_s^2([0,T]^n \times \Omega)$ the space of all functions $f^\alpha(t_1,\ldots,t_n)$ which are symmetric in the variables $(t_1,\ldots,t_n)$ and where $f \in \mathbf{X}^n$. For constant functions $f \in \mathbb{R} := \mathbf{X}_s^0$ we define $I_0(f) = f$, and for functions $f \in \mathbf{X}_s^n$ we define the integrals

$$I_n(f) := \sum_{(\alpha_1,.,\alpha_n)\in\mathcal{I}^n} \int_{[0,T]^n} f^{(\alpha_1,\ldots,\alpha_n)}(t_1,\ldots,t_n) dM_{t_1}^{\alpha_1} \otimes \cdots \otimes dM_{t_n}^{\alpha_n} \quad (16)$$

We now restate Theorem 6 to the case of symmetric functions and integrals I_n:

Theorem 7. *Assume $\{M^\alpha\}$ is a BPR for $\mathbf{M}^2$, that $\langle M^\alpha, M^\alpha\rangle \ll dt$ is deterministic. Then for every $F \in L^2(\mathcal{F}_T, P)$ there exists a sequence of functions $\{f_n\}_{n=0}^\infty$ where $f_n \in \mathbf{X}_s^n$ such that*

$$F = \sum_{n=0}^\infty I_n(f_n), \qquad \textit{moreover} \quad \|F\|_{L^2(\Omega)}^2 = \sum_{n=0}^\infty n!\|f_n\|_n^2$$

Proof. The first part follows from Theorem 6, so we only have to prove that $\|F\|_{L^2(\Omega)}^2 = \sum_{n=0}^\infty n!\|f_n\|_n^2$. This isometry follows from the Ito isometry and that the increasing simplex $S_n = \big\{(t_1,\ldots,t_n); t_1 < t_2 < \cdots < t_n < T\big\}$ is $\frac{1}{n!}$ of $[0,T]^n$. $\square$

Define a subset $\mathbb{D}_{1,2}$ of $L^2(\Omega)$ which will be the domain of our derivative operator,

$$\mathbb{D}_{1,2} := \left\{ F = \sum_{n=0}^{\infty} I_n(f_n) \,\Big|\, \sum_{n=1}^{\infty} n \cdot n! \|f_n\|_n^2 < \infty \right\} \tag{17}$$

Since every $F \in L^2(\Omega)$ of the form $F = \sum_{n=0}^{N} I_n(f_n)$ is in $\mathbb{D}_{1,2}$ we see that $\mathbb{D}_{1,2}$ is dense in $L^2(\Omega)$.

Definition 8. The operator $D : \mathbb{D}_{1,2} \mapsto \bigoplus_{\alpha\in\mathcal{I}} L^2([0,T] \times \Omega, d\langle M^\alpha\rangle \otimes dP)$, defined by

$$D_{t,\alpha}F := \sum_{n=1}^{\infty} n I_{n-1}(f_n^\alpha(\cdot, t)) \tag{18}$$

will be referred to as the derivative operator.

Similar to the Brownian motion case one can define a Skorohod integral δ as the adjoint of the derivative operator. One can show that the integral δ is an extension of the Ito integral, i.e. the Ito integral and the integral δ coincide if the Ito integral exists. Since lots of the results concerning the Skorohod integral only relies on the chaos expansion and the martingale property of Brownian motion, much of these results can be extended to the δ integral case.

The next observation is that if a variable $F \in L^2(\mathcal{F}_T, P)$ is in $\mathbb{D}_{1,2}$ then $DF \in \bigoplus_{\alpha\in\mathcal{I}} L^2\big([0,T] \times \Omega, d\langle M^\alpha\rangle \otimes dP\big)$ as claimed in the definition of D. Assume $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{D}_{1,2}$ then

$$\begin{aligned} E\big[(\sum_\alpha \int_0^T D_{t,\alpha}F dM_t^\alpha)^2\big] &= E\big[\sum_\alpha \int_0^T \big(D_{t,\alpha}F\big)^2 d[M^\alpha, M^\alpha]_t\big] \\ &= \sum_\alpha \int_0^T \| \sum_{n=1}^{\infty} n I_n(f_n^\alpha(\cdot, t) \|_{L^2(\Omega)}^2 d\langle M^\alpha\rangle_t \\ &= \sum_\alpha \int_0^T \sum_{n=1}^{\infty} n^2 (n-1)! \|f_n^\alpha(\cdot, t)\|_{n-1}^2 d\langle M^\alpha\rangle_t \\ &= \sum_{n=1}^{\infty} n \cdot n! \|f_n\|_n^2 < \infty \end{aligned} \tag{19}$$

Proposition 9. *Assume $\{F_n\}_{n=1}^{\infty}$ is a sequence of stochastic variables in $\mathbb{D}_{1,2}$ which converges in $L^2(P)$ to $F \in \mathbb{D}_{1,2}$. Then $DF = \lim_{n\to\infty} DF_n$.*

Proof. By (19), $D_{t,z}F_n$ and $D_{t,z}F$ is in $\bigoplus_{\alpha\in\mathcal{I}} L^2([0,T]\times\Omega)$. The result therefore follows by dominated convergence. $\square$

Given a stochastic variable $F \in L^2(\mathcal{F}_T, P)$, we are now able to describe its martingale representation by the following formula:

Theorem 10 (Clark-Ocone formula). *Let $F \in L^2(\mathcal{F}_T, P)$. If $F \in \mathbb{D}_{1,2}$ then*

$$F = E\big[F\big] + \sum_{\alpha\in\mathcal{I}} \int_0^T E\big[D_{t,\alpha}F|\mathcal{F}_{t-}\big] dM_t^\alpha$$

where by $E\big[D_{t,\alpha}F|\mathcal{F}_{t-}\big]$ is meant the predictable projection of $D_{t,\alpha}F$.

Proof. The proof follows the ideas in [8]. Note that for $f_n \in \mathbf{X}_s^n$ we have that $I_n(f_n) = n!J_n(f_n)$. By Theorem 7, we can for every $F \in L^2(\mathcal{F}_T, P)$ write,

$$F = E\big[F\big] + \sum_{n=1}^\infty I_n(f_n) = E\big[F\big] + \sum_{\alpha\in\mathcal{I}} \int_0^T u_t^\alpha dM_t^\alpha \tag{20}$$

where u_t^α is a predictable process given by

$$u_t^\alpha = \sum_{n=1}^\infty n!\Big(\sum_{(\alpha_1,..,\alpha_{n-1})\in\mathcal{I}^{n-1}} \int_{t_1<\cdot\cdot<t_{n-1}<t-} f_n^{\alpha_1,..,\alpha_{n-1},\alpha}(t_1,\ldots,t_{n-1},t) \times dM_{t_1}^{\alpha_1}\cdots dM_{t_{n-1}}^{\alpha_{n-1}}\Big) \tag{21}$$

Hence we have to show that $u_\tau^\alpha = E\big[D_{\tau,\alpha}F|\mathcal{F}_{\tau-}\big]$ for any predictable stopping time $\tau \in [0,T]$, where $D_{\tau,\alpha}F = D_{t,\alpha}F|_{t=\tau}$. By definition of D we have

$$\begin{aligned} D_{t,\alpha}F &= \sum_{n=1}^\infty nI_n(f_n^\alpha(\cdot,t)) \\ &= \sum_{n=1}^\infty n!\Big(\sum_\alpha \int_0^T h_n^{\alpha_{n-1},\alpha}(t_{n-1},t) dM_{t_{n-1}}^{\alpha_{n-1}}\Big) \end{aligned} \tag{22}$$

where

$$h_n^{\beta,\alpha}(s,t) = \tag{23}$$

$$\sum_{(\alpha_1,..,\alpha_{n-2})\in\mathcal{I}^{n-2}} \int_{t_1<\cdot\cdot<t_{n-2}<s-} f_n^{\alpha_1,..,\alpha_{n-2},\beta,\alpha}(t_1,..,t_{n-2},s,t) dM_{t_1}^{\alpha_1}\cdot\cdot dM_{t_{n-2}}^{\alpha_{n-2}} \tag{24}$$

$h_n^{\beta,\alpha}(s,\tau)$ is predictable for every predictable stopping time τ since clearly $h_n^{\beta,\alpha}(s,t)$ is predictable for fixed t,α. Define

$$M_n^\alpha(r,t) := \sum_{\beta\in\mathcal{I}} \int_0^r h_n^{\beta,\alpha}(s,t)dM_s^\beta, \qquad r\in[0,T] \tag{25}$$

Since $E\big[\int_0^T \big(h^{\beta,\alpha}(s,\tau)\big)^2 d[M^\beta,M^\beta]_s\big] < \infty$ for every $\beta,\alpha\in\mathcal{I}$ and predictable stopping time τ and M^β is a square integrable martingale, it follows from [11, Lem, pp. 142] that $M_n^\alpha(\cdot,\tau)$ is a martingale for every predictable stopping time τ and α. In particular,

$$E\big[M_n^\alpha(T,\tau) - M_n^\alpha(\tau,\tau)|\mathcal{F}_\tau\big] = 0 \tag{26}$$

Hence,

$$E\big[M_n^\alpha(T,\tau) - M_n^\alpha(\tau,\tau)|\mathcal{F}_{\tau-}\big] = E\big[E\big[M_n^\alpha(T,\tau) - M_n^\alpha(\tau,\tau)|\mathcal{F}_\tau\big]|\mathcal{F}_{\tau-}\big] = 0 \tag{27}$$

By (21) we then have that $\sum_{n=1}^\infty n!M_n^\alpha(t,t) = u_t^\alpha$ for all t a.s. Consequently we see that for any predictable stopping time $\tau\in[0,T]$,

$$E\big[D_{\tau,\alpha}F|\mathcal{F}_{\tau-}\big] = \sum_{n=1}^\infty n!E\big[M_n^\alpha(T,\tau)|\mathcal{F}_{\tau-}\big] = \sum_{n=1}^\infty n!E\big[M_n^\alpha(\tau,\tau)|\mathcal{F}_{\tau-}\big] = u_t^\alpha \tag{28}$$

where the last equality is due to the predictability of u. □

Theorem 10 might provide an expression for the martingale representation of a general stochastic variable F, but since the chaos expansion of F generally is not known it might seem hard to find an expression for DF. The next result provides a link between D and the directional Gross derivative for Brownian motion and Poisson processes. Properties of the derivative in these cases are known, and thus provide a key to derive an expression for the derivative DF.

Proposition 11. *Assume that M^β is a Brownian motion and that M^η is a compensated Poisson process. Then $D_{t,\beta}$ coincides with the directional Gross derivative, and $D_{t,\eta}$ coincides with the directional Gross derivative (difference) for compensated Poisson processes.*

Proof. Let D_t^β denote the directional Gross derivative with respect to the Brownian motion M^β. Then

$$\begin{aligned} D_t^\beta I_1(f_1) &= D_t^\beta\big(\int_{[0,T]} f_1^\beta(s)dM_s^\beta + \sum_{\alpha\neq\beta}\int_{[0,T]} f_1^\alpha(s)dM_s^\alpha\big) \\ &= f_1^\beta(t) = D_{t,\beta}I_1(f_1) \end{aligned} \tag{29}$$

We will now show by induction on n that $D_t^\beta\big(I_n(f_n)\big) = nI_{n-1}(f_n^\beta(\cdot,t))$. We see from the calculations above that it holds for $n=1$. Assume it holds for $n=N$. $I_n(f_n)$ fulfills the conditions for the commutativity relation between Skorohod integration and Gross differentiation (see [9, pp. 38]). Hence,

$$
\begin{aligned}
D_t^\beta I_{N+1}(f_{N+1}) &= D_t^\beta\bigg(\sum_{\alpha\in\mathcal{I}}\int_0^T I_N\big(f_{N+1}^\alpha(\cdot,s)\big)dM_s^\alpha\bigg)\\
&= D_t^\beta\int_0^T I_N\big(f_{N+1}^\beta(\cdot,s)\big)dM_s^\beta\\
&\quad + D_t^\beta\sum_{\alpha\neq\beta}\int_0^T I_N\big(f_{N+1}^\alpha(\cdot,s)\big)dM_s^\alpha\\
&= \int_0^T D_t^\beta I_N\big(f_{N+1}^\beta(\cdot,s)\big)dM_s^\beta + I_N\big(f_{N+1}^\beta(\cdot,t)\big)\\
&\quad + \sum_{\alpha\neq\beta}\int_0^T D_t^\beta I_n\big(f_{N+1}^\alpha(\cdot,s)\big)dM_s^\alpha\\
&= \int_0^T NI_{N-1}\big(f_{N+1}^{\beta,\beta}(\cdot,t,s)\big)dM_s^\beta + I_N\big(f_{N+1}^\beta(\cdot,t)\big)\\
&\quad + \sum_{\alpha\neq\beta}\int_0^T NI_{N-1}\big(f_{N+1}^{\beta,\alpha}(\cdot,t,s)\big)dM_s^\alpha\\
&= I_N\big(f_{N+1}^\beta(\cdot,t)\big) + NI_N(f_{N+1}^\beta(\cdot,t)\big)\\
&= (N+1)I_N\big(f_{N+1}^\beta(\cdot,t)\big)
\end{aligned}
\tag{30}
$$

Hence $D_t^\beta I_n(f_n) = nI_{n-1}(f_n^\beta(\cdot,t)) = D_{t,\beta}I_n(f_n)$. By linearity it follows that $D_t^\beta\sum_{n=0}^N I_n(f_n) = D_{t,\beta}\sum_{n=0}^N I_n(f_n)$. So the directional Gross derivative D_t^β and $D_{t,\beta}$ are equal on a dense subset of $L^2(\mathcal{F}_T,P)$, and must therefore coincide. The proof for the compensated Poisson case is the same. $\square$

When M^β is a Brownian motion and M^η is a compensated Poisson process we now deduce from the properties of the Gross derivative and the properties of the Poisson Gross derivative that D obeys the following rules:

$$D_{t,\beta}f(G) = f'(G)D_{t,\beta}G \tag{31}$$

if f is differentiable and $f(G)\in\mathbb{D}_{1,2}$,

$$D_{t,\beta}\big(FG\big) = D_{t,\beta}F\cdot G + F\cdot D_{t,\beta}G \tag{32}$$

if $F,G,FG\in\mathbb{D}_{1,2}$. See for instance [9]. And,

$$D_{t,\eta}f(G) = f(G+D_{t,\eta}G) - f(G) \tag{33}$$

if $f(G), G \in \mathbb{D}_{1,2}$, and

$$D_{t,\eta}\big(FG\big) = D_{t,\eta}F \cdot G + F \cdot D_{t,\eta}G + D_{t,\eta}F \cdot D_{t,\eta}G \tag{34}$$

if $F, G, FG \in \mathbb{D}_{1,2}$ (see for instance [10]).

5 Lévy processes

In order to illustrate the theory we will give some examples using Lévy processes. Lévy processes are in this sense very suitable since they in an easy way can be expressed via a Brownian motion and Poisson processes. If L is a square integrable Lévy process then L can be represented

$$L_t = \gamma t + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z(\mu - \nu)(t, dz) \tag{35}$$

for some constants γ, σ and some Poisson random measure μ with compensator ν being the Lévy measure, W being the Brownian motion. See for instance [1], [3] and [4] for results on this topic. If two sets Λ_1 and Λ_2 are disjoint we know that $\mu(\cdot, \Lambda_1)$ and $\mu(\cdot, \Lambda_2)$ are independent Poisson processes.

Lemma 12. *There exists a disjoint countable partition $\{\Lambda_n\}_{n=1}^{\infty}$ of $\mathbb{R} \setminus \{0\}$ and constants $z_n \in \Lambda_n$ such that*

$$\int_{\mathbb{R}\setminus\{0\}} z(\mu - \nu)(t, dz) = \sum_{n=1}^{\infty} z_n (\mu - \nu)(t, \Lambda_n)$$

in $L^2(P)$.

Proof. There exists a sequence of simple functions ϕ_i converging to z pointwise and in $L^2(\nu)$ where $\phi_i(z) = \sum_{j=1}^{N_i} z_{i,j} \mathbf{1}_{A_j^i}(z)$. $A^i = \{A_j^i\}_{j=1}^{N_i}$ being a partition of $\mathbb{R} \setminus \{0\}$. Define

$$\mathcal{P} := \sigma(A_1^1, \dots, A_{N_1}^1, A_1^2, \dots, A_{N_2}^2, A_1^3, \dots) \tag{36}$$

There exists a countable number of disjoint sets $\Lambda_1, \Lambda_2, \Lambda_3, \dots$ such that

$$\mathcal{P} = \sigma(\Lambda_1, \Lambda_2, \Lambda_3, \dots) \tag{37}$$

Since $\phi := \lim_{i\to\infty} \phi_i$ is of the form $\phi(z) = \sum_{n=1}^{\infty} z_n \mathbf{1}_{\Lambda_n}(z)$ and

$$(\mu - \nu)(\cdot, \Lambda_1) + (\mu - \nu)(\cdot, \Lambda_2) = (\mu - \nu)(\cdot, \Lambda_1 \cup \Lambda_2) \tag{38}$$

when Λ_1 and Λ_2 are disjoint, the result follows. $\square$

For each $\Lambda_n \subset \mathbb{R} \setminus \{0\}$ the processes $(\mu - \nu)(t, \Lambda_n)$ are compensated Poisson processes with intensity $\nu(\Lambda_n)$. By Lemma 12 and Theorem 6 we see that $\big\{W, \{(\mu-\nu)(\cdot, \Lambda_n)\}_{n=1}^{\infty}\big\}$ constitutes a BPR for $\mathbf{M}^2$ and admits a chaos expansion. From now on we simply write $\big\{W, \{(\mu-\nu)(\cdot, dz)\}\big\}$ for the BPR $\big\{W, \{(\mu-\nu)(\cdot, \Lambda_n)\}_{n=1}^{\infty}\big\}$. Instead of using α as the parameter, we use z and w. Hence $f^{\alpha}(t)$ is now either $f(t, z)$ or $f^{w}(t)$. To further clarify the notation,

$$\sum_{\alpha \in \mathcal{I}} \int_0^T f^{\alpha}(t) dM_t^{\alpha} = \int_0^T f^{w}(t) dW_t + \int_0^T \int_{\mathbb{R}\setminus\{0\}} f(t,z)(\mu - \nu)(dt, dz) \quad (39)$$

Let now $F \in L^2(\mathcal{F}_T, P)$ where $\{\mathcal{F}_t\}$ is the completed, right-continuous σ-algebra generated by a square integrable Lévy process L up to time t. By Theorem 6 we have that there exists predictable processes ϕ and ψ such that

$$F = E\big[F\big] + \int_0^T \phi(t) dW_t + \int_0^T \psi(t,z)(\mu - \nu)(dt, dz) \quad (40)$$

where $E\big[\int_0^T \phi^2(t)dt\big] < \infty$ and $E\big[\int_0^T \int_{\mathbb{R}\setminus\{0\}} \psi^2(t,z)\nu(dz)dt\big] < \infty$. See also [6] for results on this topic.

Instead of $D_{t,\alpha}$ we will now write $D_{t,w}$ for the directional derivative with respect to the Brownian motion, and $D_{t,z}$ for the directional difference (derivative) with respect to the different compensated Poisson processes. We can restate Theorem 10 for the case of Lévy processes.

Theorem 13 (Clark-Ocone formula for Lévy processes). *Assume L is a square integrable Lévy process and $\{\mathcal{F}_t\}$ its completed filtration. If a stochastic variable $F \in L^2(\mathcal{F}_T, P) \cap \mathbb{D}_{1,2}$ then*

$$F = E\big[F\big] + \int_0^T E\big[D_{t,w}F|\mathcal{F}_{t-}\big] dW_t + \int_0^T \int_{\mathbb{R}\setminus\{0\}} E\big[D_{t,z}F|\mathcal{F}_{t-}\big](\mu - \nu)(dt, dz)$$

where by $E\big[\cdot|\mathcal{F}_{t-}\big]$ is meant the predictable projection.

Example 1. Let L be a square integrable Lévy process and assume L has a Lévy-Ito representation of the form (35). Let $M_{t,T} = \max_{t \le s \le T} L_s$ and let $F = M_{0,T}$. Let τ be the time where the maximum is achieved, i.e. $L_\tau = M_{0,T}$. From [7, pp. 367], $D_{t,w}F = \sigma \mathbf{1}_{\tau > t}$. In order to compute $D_{t,z}F$ we approximate F by

$$F_n = f_n(L_{t_1}, \ldots, L_{t_n}) = \max_{1 \le j \le n} \big\{L_{t_j}\big\} \quad (41)$$

Assume that this maximum is achieved for $j = i$. Then by the relation (33),

$$\begin{aligned} D_{t,z}F_n &= f_n\big(L_{t_1} + D_{t,z}L_{t_1}, \ldots, L_{t_n} + D_{t,z}L_{t_n}\big) - f_n\big(L_{t_1}, \ldots, L_{t_n}\big) \\ &= \max_{1 \le j \le n}\big\{L_{t_j} + z\mathbf{1}_{t_j > t}\big\} - \max_{1 \le j \le n}\big\{L_{t_j}\big\} \\ &= z\mathbf{1}_{t_i > t} \end{aligned} \tag{42}$$

It is easy to see that $D_{t,z}F_n$ converges to $\mathbf{1}_{\tau > t}$. Lévy processes renew themselves at stopping times. If we use this we obtain

$$\begin{aligned} \phi_t := &E\big[\mathbf{1}_{\tau > t}|\mathcal{F}_{t-}\big] \\ &= P(\tau > t|\mathcal{F}_{t-}) \\ &= P(M_{t,T} > M_{0,t}|\mathcal{F}_{t-}) \\ &= P(M_{0,T-t} > b)|_{b = M_{0,t-} - L_{t-}} \end{aligned}$$

Hence we have derived an integral representation for the stochastic variable $F = \max_{0 \le s \le T} L_s$ given by

$$\begin{aligned} \max_{0 \le s \le T} L_s &= E\big[\max_{0 \le s \le T} L_s\big] + \int_0^T \sigma\phi_t dW_t + \int_0^T \int_{\mathbb{R}\setminus\{0\}} \phi_t z(\mu - \nu)(dt, dz) \\ &= E\big[\max_{0 \le s \le T} L_s\big] + \int_0^T \phi_t dL_t \end{aligned} \tag{43}$$

Example 2. Consider the function $f(x) = (x - K)_+$ where $K \in \mathbb{R}_+$ is a constant. It can be noted that f is the payoff function of a European call option. The function is not differentiable so we approximate f by a sequence of functions f_n where $f_n \in C^1$ and $f_n(x) = f(x)$, $|x| \ge \frac{1}{n}$ and $0 \le f_n' \le 1$. Let $F = f(L_T)$, and $F_n = f_n(L_T)$. Since $0 \le f_n' \le 1$ and $|f_n(x+y) - f(x)| \le |y|$, we have that $F_n \in \mathbb{D}_{1,2}$. If we now use the properties of the directional Gross derivative with respect to Brownian motion and Poisson processes we obtain that $D_{t,w}F_n = f_n'(L_T)D_{t,w}L_T = f_n'(L_T)\sigma$ and

$$D_{t,z}F_n = f_n(L_T + D_{t,z}L_T) - f_n(L_T) = f_n(L_T + z) - f_n(L_T) \tag{44}$$

By the Clark-Ocone formula we obtain that

$$\begin{aligned} F_n = E\big[F_n\big] &+ \int_0^T E\big[f_n'(L_T)\sigma|\mathcal{F}_{t-}\big]dW_t \\ &+ \int_0^T \int_{\mathbb{R}\setminus\{0\}} E\big[f_n(L_T + z) - f_n(L_T)|\mathcal{F}_{t-}\big](\mu - \nu)(dt, dz) \end{aligned} \tag{45}$$

As n tends to infinity, $D_{t,w}F_n$ converges to $\mathbf{1}_{[K,\infty)}(L_T)\sigma$ and $D_{t,z}F_n$ converges to $(L_T - K + z)_+ - (L_T - K)_+$. Hence

$$\begin{aligned}F = E\big[F\big] &+ \int_0^T E\big[\mathbf{1}_{[K,\infty)}(L_T)|\mathcal{F}_{t-}\big]\sigma dW_t \\ &+ \int_0^T \int_{\mathbb{R}\setminus\{0\}} E\big(L_T - K + z)_+ - (L_T - K)_+|\mathcal{F}_{t-}\big](\mu - \nu)(dt, dz)\end{aligned}$$

This example also shows that contrary to the stochastic variable given in Example 1, a stochastic variable F cannot in general be expressed as an integral with respect to L itself. I.e. L does not have the predictable representation property, or in our notation, $\{L\}$ is not a BPR for $\mathbf{M}^2$.

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