Stopping Brownian Motion without Anticipation as Close as Possible to its Ultimate Maximum

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Abstract

Let $B = (B_t)_{0 \le t \le 1}$ be standard Brownian motion started at zero, and let $S_t = \max_{0 \le r \le t} B_r$ for $0 \le t \le 1$. Consider the optimal stopping problem

$$V_* = \inf_{\tau} E\big(B_\tau - S_1\big)^2$$

where the infimum is taken over all stopping times of B satisfying $0 \le \tau \le 1$. We show that the infimum is attained at the stopping time

$$\tau_* = \inf \left\{ \, 0 \le t \le 1 \mid S_t - B_t \ge z_* \sqrt{1-t} \, \right\}$$

where $z_* = 1.12\ldots$ is the unique root of the equation

$$4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0$$

with $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy$. The value V_* equals $2\Phi(z_*) - 1$. The method of proof relies upon the Itô-Clark representation theorem, time-change arguments, and the solution of a free-boundary problem.

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