# Bounding the Maximal Height of a Diffusion by the Time Elapsed

By Goran Peskir*


### Abstract

Let $X = (X_t)_{t \geq 0}$ be a one-dimensional time-homogeneous diffusion process associated with the infinitesimal generator

$$\mathbb{L}_X = \mu(x) \frac{\partial}{\partial x} + \frac{\sigma^2(x)}{2} \frac{\partial^2}{\partial x^2}$$

where $x \mapsto \mu(x)$ and $x \mapsto \sigma(x) > 0$ are continuous. We show how the question of finding a function $x \mapsto H(x)$ such that

$$c_1 E\big(H(\tau)\big) \leq E\Big(\max_{0 \leq t \leq \tau} |X_t|\Big) \leq c_2 E\big(H(\tau)\big)$$

holds for all stopping times $\tau$ of $X$ relates to solutions of the equation:

$$\mathbb{L}_X(F) = 1 \ .$$

Explicit expressions for $H$ are derived in terms of $\mu$ and $\sigma$ . The method of proof relies upon a domination principle established by Lenglart and Itô calculus.


* Centre for Mathematical Physics and Stochastics, supported by the Danish National Research Foundation.

*MR 1991 Mathematics Subject Classification*. Primary 60J60, 60J65, 60E15. Secondary 60G40, 34B05, 60G44.

*Key words and phrases:* Diffusion process, stopping time, maximal inequality, Lenglart's domination principle, Brownian motion, stochastic differential equation, scale function, speed measure.

## REFERENCES


[1] BURKHOLDER, D. L. *and* GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* 124 (249-304).

[2] GRAVERSEN, S. E. *and* PESKIR, G. (1998). Maximal inequalities for the Ornstein-Uhlenbeck process. *Research Report* No. 393, *Dept. Theoret. Statist. Aarhus* (8 pp). To appear in *Proc. Amer. Math. Soc.*

[3] KARATZAS, I. *and* SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer-Verlag.

[4] LENGLART, E. (1977). Relation de domination entre deux processus. *Ann. Inst. H. Poincaré Probab. Statist.* 13 (171-179).

[5] PESKIR, G. *and* SHIRYAEV, A. (1999). Maximal inequalities for reflected Brownian motion with drift. *Research Report* No. 409, *Dept. Theoret. Statist. Aarhus*. To appear.

[6] REVUZ, D. *and* YOR, M. (1994). *Continuous Martingales and Brownian Motion.* Springer-Verlag.