On Integral Equations Arising in the First-Passage Problem for Brownian Motion

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Abstract

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g : (0,\infty) \to I\!R$ be a continuous function satisfying $g(0+) \geq 0$, let

$$\tau = \inf \{ t > 0 \mid B_t \geq g(t) \}$$

be the first-passage time of B over g , and let F denote the distribution function of τ . Then the following system of integral equations is satisfied:

$$t^{n/2} H_n\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2} H_n\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) F(ds)$$

for $t > 0$ and $n = -1, 0, 1 \ldots$, where $H_n(x) = \int_x^\infty H_{n-1}(z)\, dz$ for $n \geq 0$ and $H_{-1}(x) = \varphi(x) = (1/\sqrt{2\pi})\, e^{-x^2/2}$ is the standard normal density. These equations are derived from a single 'master equation' which may be viewed as a Chapman-Kolmogorov equation of Volterra type. The initial idea in the derivation of the master equation goes back to Schrödinger [22].

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