LARGE CORES AND EXACTNESS.

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Abstract. In this paper we will answer the question if every exact game has a large core. For games with 5 or more players, the answer will be negative. For TU-games with 3 or 4 players every exact game has a large core. For totally balanced *symmetric* games a large core, a stable core and exactness will turn out to be equivalent.

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1. Preliminaries.

To start with we recall some definitions and introduce the notation we will use.

A *TU-game with player set* N is a map $v: 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$. A TU-game is*symmetric* if the value of a coalition S depends on the size of the coalition only. In that case we introduce a map $f: [0\,1\,\ldots\,n] \to \mathbf{R}$ with $f(0) = 0$. Then $v(S) = f(s)$ whenever $s = |S|$, the size of S.

The *core of a TU-game* (N, v) consists of the (payoff)vectors $x \in \mathbf{R}^N$ satisfying the conditions $x(S) := \sum_{i \in S} x_i \geq v(S)\,(S \subset N)$ and $x(N) = v(N)$.

A TU-game (N, v) is a *balanced game* if for every nonnegative solution of the vector equations $\sum_{S \subset N} y_S\, e_S$ the inequality $\sum_{S \subset N} y_S\, v(S) \leq v(N)$ holds.
According to a well known theorem (Bondareva (1963), Shapley (1967)) the core of a TU-game is nonempty if and only if the game is balanced.

A TU-game is *totally balanced* if every subgame (S, v_S) is balanced.

A TU-game (N, v) is said to *have a large core* (Sharkey (1982)), if for every vector $y \in \mathbf{R}^N$ with $y(S) \geq v(S)$ for all $S \subseteq N$, there is a core element $x \in \text{Core}\,(N, v)$ with $x \leq y$.
A vector y satisfying the conditions $y(S) \geq v(S)$ for $S \subset N$ will be called an *upper vector* of the game (N, v). The set of all upper vectors is denoted by $U(v)$. It is a polyhedral set and **not** dependent on the value of the grand coalition $v(N)$ (as we only take the inequalities for $S \subset N$). The recession cone of the polyhedral set $U(v)$, i.e the set

$$\{u \in \mathbf{R}^N : z + t\,u \in U(v) \text{ for all } z \in U(v),\, t \geq 0\}$$

is the cone $\mathbf{R}^N_+$, as can be seen easily and by the structure theorem for polyhedral sets

$$U(v) = \underline{\text{ch}}\,(\underline{\text{extr}}\,U(v)) + \mathbf{R}^N_+.$$

($\underline{\text{ch}}$ means 'convex hull of' and $\underline{\text{extr}}$ means 'extreme points of').
From this description one can see that the core is large if and only if $v(N) \geq z(N)$ for all extreme points z of $U(v)$.

A TU-game (N, v) is called *exact* (Schmeidler (1972)) if for every coalition $T \subset N$ there is a core allocation $x \in \text{Core}\,(N, v)$ with $x(T) = v(T)$.
Note that an exact game is balanced and that every subgame (S, v_S) is also balanced: (N, v)

is *totally balanced.*

The core of a TU-game is *stable* if for every imputation $y \notin \text{Core}\,(N,v)$ there exists a core allocation $x \in \text{Core}\,(N,v)$ and a coalition $S \subset N$ such that $x(S) = v(S)$ and $x_i > y_i$ for $i \in S$. If the last conditions hold, we say that x *dominates* y by S.

Games with large core have been studied, mainly because of the following property:

Proposition 1. [Sharkey (1982)] *If a TU-game has a large core, the core is the unique stable set.*

Proof: For completeness we repeat the proof. If $y \in I(v)$ (the imputation set of (N,v)) and y is not a core allocation, there is a coalition S with $y(S) < v(S)$ and $y(T) \geq v(T)$ for $T \subset S$. Take $z_S > y_S$ and $z_S(S) = v(S)$. It is possible to extend z_S with payoffs for players outside S such that $z = (z_S, z_{N\setminus S}) \in U(v)$ and $z(N) \geq v(N)$. There is a core allocation x with $x \leq z$ (largeness of the core). Then $v(S) \leq x(S) \leq z(S) = v(S)$ and $x_S = z_S$. Then x dominates y by S. Every imputation outside the core is dominated by a core allocation. So, the core is a stable set and there cannot be an other one, as every stable set contains the core and no imputation can be added without violating internal stability. $\triangleleft$

If (N,v) is a balanced game we define the *totally balanced cover* $(N,\bar{v})$ by

$$\bar{v}(S) := \max\left\{\textstyle\sum_{T\subseteq S} y_T\, v(T) : y_T \geq 0 \text{ and } \sum_{T\subseteq S} y_T\, e_T = e_S\right\}.$$

As $U(v) = U(\bar{v})$ for every TU-game and $\text{Core}\,(N,v) = \text{Core}\,(N,\bar{v})$ for every balanced game, the game has a large core if and only if the totally balanced cover has a large core. So, we can restrict our attention to totally balanced games. For totally balanced games we have

Proposition 2. [Sharkey (1982)] *If a totally balanced game has a large core, it is an exact game.*

Proof: Take any $T \subset N$ and a core element $z_T \in \text{Core}\,(T, v_T)$. As before we can extend z_T to an element $z = (z_T, z_{N\setminus T})$ in $U(v)$ and $z(N) \geq v(N)$. There exists a core allocation $x \in \text{Core}\,(N,v)$ with $x \leq z$. Then $v(T) \leq x(T) \leq z(T) = v(T)$. So, x is a core allocation with $x(T) = v(T)$. $\triangleleft$

Remark. What is really needed in the proof of the propositions 1 and 2 is that every core element of a subgame (S, v_S) can be extended to a core element of the whole game. In an unpublished manuscript Shapley and Kikuta (1986) proved:

If in a balanced game (N, v) every core element of a subgame can be extended to a core element of the whole game, then the core is a stable set.

They conjectured that this condition is also needed for core stability. On page 15, remark (i) they bring forward the comment of Sharkey on the Lucas' game (see below) and conclude that this conjecture is false. Unfortunately Sharkey's counterexample is not a counterexample. At the end of the paper we will give a real counterexample i.e. a TU-game with stable core not satisfying the Shapley/Kikuta conditions.

The question we will answer in this paper is the converse of Proposition 2:

Does every exact game have a large core?

The history of this problem is curious. In Sharkey (1982) the author claims to have a counterexample in Lucas' famous 10-person game without stable set. This counterexample, however, is false.

Example: [Lucas (1969), cf. Sharkey (1982)]

Consider the TU-game (N, v) with $n = 10$ and coalition values $v(S) = 0$ except for the following coalitions:

$v(12) = v(34) = v(56) = v(78) = v(90) = 1,$

$v(137) = v(139) = v(157) = v(159) = v(357) = v(359) = 2,$

$v(1479) = v(2579) = v(3679) = 2,$

$v(1379) = v(1579) = v(3579) = 3,$

$v(13579) = 4.$

Sharkey states that the totally balanced cover of the game (N, v) is exact if $v(N) \geq 5$, that the game has a stable core for $v(N) \geq 7$ and a large core if $v(N) \geq 8$. None of these statements is true. For exactness $v(N)$ must be at least 11 and, by Proposition 2, the same must be true for a large core. Finally, the core is not stable for $v(N) < 11$. We show the statement about exactness first.

Take the coalition (1679). This coalition and each of its subcoalitions has value 0. To have exactness there must be a core allocation with $x_i = 0$ for $i = 1, 6, 7, 9$. Then $x_2 \geq 1$ (by coalition (12)), $x_3 \geq 3$ (by coalition (1379)), $x_4 \geq 2$ and $x_5 \geq 3$ (by the coalitions (1479) and (1579)). Finally $x_8 \geq 1$ and $x_0 \geq 1$ (by the coalitions (78) and (90)). Then

$x(N) \geq 11$. The allocation obtained by taking equalities in the before mentioned coalitions is $z = (0, 1, 3, 2, 3, 0, 0, 1, 0, 1)$. It is an extreme point of $U(v)$. It can be proved that the allocation obtained by diminishing the coordinates of z on the places 3, 4 and 5 with the *same* amount $t \in (0, 1]$ gives an imputation in the game with $v(N) = 11 - 3t$ that cannot be dominated by any core allocation of that game. The vector x is also the smallest extension in $U(v)$ of the core allocation $(0, -, -, -, -, 0, 0, -, 0, -)$ of the subgame (S, v_S) with $S = (1679)$. So, core stability, largeness of the core and the Kikuta/ Shapley conditions start at the same level of $v(N) = 11$.

More recently Biswas et al. (1996) proved that the converse of Proposition 2 holds for *symmetric* TU-games (coalition values depend only on the size of coalitions).

2. Non-symmetric games.

In this note we prove that the converse of Proposition 2 holds if $|N| =: n \leq 4$ and we give a counterexample for every number $n \geq 5$.

Theorem 3 *If (N, v) is an exact game and $n \leq 4$, then (N, v) has a large core (and the core is a stable set). If $n \geq 5$, there are exact games (N, v) not having a large core.*

Remark. Kulakovskaja (1980) gives, for $|N| = 4$, the following necessary and sufficient conditions for a zero-normalized balanced game to have a stable core. For every ordering (i, j, k, l) of the players in N is

$$v(ij) + v(ik) + v(il) \leq v(N),$$

$$v(ijk) + v(il) \leq v(N),$$

$$v(ijk) + v(ijl) - v(ij) \leq v(N).$$

In fact these conditions check the exactness of the (totally balanced cover of) the game. The first two relations check the exactness condition for coalition (i) and the last condition is the exactness condition for (ij). For 3-person coalitions the exactness conditions follow from totally balancedness.

Proof: The proof of the theorem and the distinction between $n \leq 4$ and $n \geq 5$ mainly rest on the following observation: only if $n \geq 5$, there exists a collection $\mathcal{T}$ of proper coalitions in N, $T_1, \ldots, T_n$, with the following properties:

(a) $\{e_T : T \in \mathcal{T}\}$ is a basis of $\mathbf{R}^N$,

(b) the vector equation $\sum_{T\in\mathcal{T}} y_T\, e_T = e_N + a\, e_S$ **does not have** a nonnegative solution (y_T, a) for any coalition S.

Condition (b) says that $\mathcal{T}$ does not contain a balanced collection and also that it is not possible to combine the vectors $e_T, T \in \mathcal{T}$ in such a way that every player get either weight 1 or $1+a$. For $n \geq 5$, the collection $\mathcal{T}_n := \{(1),(1,2),(1,3),(1,2,4),(1,2,5),\ldots(1,2,n)\}$ has the properties (a) and (b). For $n \leq 4$ such a collection $\mathcal{T}$ does not exist (see Lemma 4).

If one has a collection $\mathcal{T}$ with the properties (a) and (b), one can proceed as follows: the vector e_N can be written as $\sum_{i=1}^{n} y_{T_i}\, e_{T_i}$. Let the vectors $\{w_i\}$ be defined by $w_i(T_j) = 0$ if $i \neq j$ and $w_i(T_i) > 0$. If we compute $w_i(N)$, we find that $w_i(N) < 0$ if and only if $y_{T_i} < 0$. We normalize the vectors w_j by $w_j(N) = -1$ if $y_{T_j} < 0$ and denote the set of indices j with $y_{T_j} < 0$ by J. By property (b) J contains at least *two* elements. Next we define a game (N, v) by $\quad v(S) := \min_{j\in J} w_j(S)$ for $S \subseteq N$.

The game (N, v) is the minimum of $|J|$ additive games. Note that (N, v) has the following properties:

(c) for every coalition $S \subset N$ there is an index $j \in J$ with $w_j(S) = v(S)$,

(d) for every index $j \in J$ and every coalition S we have $w_j(S) \geq v(S)$.

From (c) and (d) we find that (N, v) is an exact game.

(e) $v(N) = -1$ and $v(T) = 0$ for all $T \in \mathcal{T}$ (here we use $|J| \geq 2$).

If the game (N, v) *happens to be nonpositive,* the vector $z = 0$ is an element of $U(v)$ and even an extreme point of $U(v)$ (as the collection of 'tight' coalitions T i.e. coalitions with $z(T) = v(T)$, contains the basis $\mathcal{T}$ of $\mathbf{R}^N$). Then $z \in U(v)$ cannot be decreased to become a core allocation and the core of (N, v) is not large.

For the collections $\mathcal{T}_n$ we find $\quad e_N = -e_{(1)} - (n-4)\, e_{(12)} + e_{(13)} + \cdots + e_{(12n)}$.

Then $J = \{1, 2\}$ (two coefficients are negative) and

$$w_1 := (1, -1, -1, 0, \ldots, 0) \text{ and } w_2 := \frac{1}{n-4}\,(0, 1, 0, -1, \ldots, -1).$$

We check that $v(S) = \min\{w_1(S), w_2(S)\} \leq 0$. $w_1(S) > 0$ iff $1 \in S$ and $S \cap \{2,3\} = \emptyset$; $w_2(S) > 0$ iff $2 \in S$ and $S \cap \{4, 5, \ldots, n\} = \emptyset$.

So, the game (N, v) is exact and non-positive, $v(N) = -1$ and $z = 0$ is an extreme point of $U(v)$. The game (N, v) does not have a large core.

It is an open question whether the game (N, v) defined according to the previous rules is always a non-positive game. If this is true, every collection $\mathcal{T}$ with the properties (a) and (b) gives a counterexample for the converse of Proposition 2.

To prove the first part of the Theorem we assume that $n \leq 4$ and that (N, v) is an exact game. We assume that no collection $\mathcal{T}$ with properties (a) and (b) exists. Let z be any extreme point of $U(v)$ and $\mathcal{T}(z) := \{T \subset N : z(T) = v(T)\}$. The collection $\mathcal{T}(z)$ contains a basis $\mathcal{T}$ of $\mathbf{R}^N$ (as z is an extreme point of $U(v)$). By the absence of property (b) there is, for at least one coalition S, a nonnegative solution of the vector equation

$$\sum_{T \in \mathcal{T}} y_T \, e_T = e_N + a \, e_S. \tag{1}$$

There exists a core element x of (N, v) with $x(S) = v(S)$ (exactness of (N, v)). From equation (1) we find

$$\sum_{T \in \mathcal{T}} y_T \, z(T) = z(N) + a \, z(S) \text{ and therefore, } \sum_{T \in \mathcal{T}} y_T \, v(T) \geq z(N) + a \, v(S)$$

But also

$$\sum_{T \in \mathcal{T}} y_T \, x(T) = x(N) + a \, x(S) \text{ implying } \sum_{T \in \mathcal{T}} y_T \, v(T) \leq v(N) + a \, v(S)$$

So, $v(N) \geq z(N)$ for every extreme point z of $U(v)$.

If $y \in U(v)$ is an upper vector and $y(N) > v(N)$, there is a vector $w \in \underline{\mathrm{ch}}\,(\underline{\mathrm{extr}}\,(U(v))$ and a vector $u \geq 0$ with $y = w + u$. As $y(N) > v(N)$ and $w(N) \leq v(N)$, there is a number $t \in [0, 1)$ with $w(N) + t\,u(N) = v(N)$. Then $w + t\,u$ is a core element $\leq y$. So, (N, v) has a large core $\triangleleft$

We are left with the following combinatorial lemma:

Lemma 4 *If $n \leq 4$, there is no collection $\mathcal{T}$ satisfying (a) and (b).*

Proof: Suppose that $\mathcal{T}$ is a basis of $\mathbf{R}^N$ with $n = 4$. If $\mathcal{T}$ contains a 3-coalition, say w.l.o.g. $(1, 2, 3)$, there is also a coalition $T \ni 4$ in $\mathcal{T}$. Then $e_{(123)} + e_T = e_N + e_{T \cap (123)}$. If $\mathcal{T}$ contains only 1-coalitions, it is the balanced collection $\{(1), (2), (3), (4)\}$. So, to satisfy (b), $\mathcal{T}$ must contain at least one 2-coalition and no 3-coalitions. Say, $(1, 2) \in \mathcal{T}$. There is also a coalition T_3 containing 3 in $\mathcal{T}$ and a coalition T_4 containing player 4. These coalitions are not the same. Otherwise $(3, 4) \in \mathcal{T}$ and $\mathcal{T}$ contains the balanced collection $\{(12), (34)\}$. At least one of the coalitions T_3 or T_4 is a 2-coalition; otherwise $\mathcal{T} \supset \{(1, 2), (3), (4)\}$, a balanced collection. W.l.o.g we may assume $T_3 = (1, 3)$. Then $e_{(12)} + e_{(13)} + e_{T_4} = (2, 1, 1, 1)$, $(2, 1, 2, 1)$, $(3, 1, 1, 1)$ or $(2, 2, 1, 1)$. There exists no collection $\mathcal{T}$ with the properties (a) and

(b) for $n = 4$. For $n = 3$ the proof is even easier. $\triangleleft$

2. Symmetric games.

Let $f: [0, 1, 2, \ldots, n] \to \mathbf{R}_+$ be a map with the property $f(0) = f(1) = 0$. Such a map defines a *symmetric game* by $v(S) := f(s)$ whenever $|S| = s$.

For symmetric games we also will investigate the relation between exactness, having a stable core and having a large core.

For any totally balanced TU-game we have:

If the game has a large core, it is exact and it has a stable core.

(see Proposition 1 and 2). For symmetric games we prove

Theorem 5: *For totally balanced, symmetric TU-games (N, v) the statements '(N, v) has a large core', '(N, v) has a stable core' and '(N, v) is exact' are equivalent.*

The proof of the Theorem follows from Lemma 6, Proposition 7 and Proposition 8.

It is well known that symmetric game defined by a function f is balanced iff $\frac{f(s)}{s} \leq \frac{f(n)}{n}$ for all $1 \leq s \leq n-1$. The game is, therefore, totally balanced iff the function $s \to \frac{f(s)}{s}$ is a weakly monotonic function.

Lemma 6. (Shapley (1974)) *A symmetric game defined by a function f is exact if and only if the function $s \to \frac{f(s)}{s}$ is weakly monotonic and, for every pair of numbers $s < t < n$,*

$$\frac{f(n)-f(s)}{n-s} \geq \frac{f(t)-f(s)}{t-s} \text{ (Shapley conditions).}$$

Proof: If f satisfies the conditions in the Lemma and $1 \leq t \leq n-1$, we define $x' := \frac{f(t)}{t}$ and $x'' := \frac{f(n)-f(t)}{n-t}$. If T is a coalition of size t, we give the players in T the payment x' and the players outside T the payment x''. Call the vector $x := x'\, e_T + x''\, e_{N \setminus T}$. Then $x(N) = f(n)$ and $x(T) = f(t)$. We have to prove that x is a core allocation. From $\frac{f(t)}{t} \leq \frac{f(n)}{n}$ we find $x' \leq x''$. For a coalition of size s, we have $x(S) \geq (s \wedge t)\, x' + (s - (s \wedge t))\, x''$. If $s \leq t$ we have $s\, x' \geq f(s)$ (by the totally balancedness condition) and, for $s > t$, we have $t\, x' + (s-t)\, x'' \geq f(t) + (s-t) \frac{f(s)-f(t)}{s-t} = f(s)$ (by the Shapley conditions).

The converse can be proved as follows: if f defines an exact game, for every number $t : 1 \leq t \leq n-1$ an element x of $\mathbf{R}^n$ can be found such that

$$x_1 \leq x_2 \leq \cdots \leq x_n,\ x_1 + \cdots + x_s \geq f(s) \text{ for all } s \text{ and equality for } s = t \text{ and } s = n.$$

If we replace the first t coordinates by their average $x' = \frac{f(t)}{t}$ and the last $n-t$ coordinates also by their average $x'' = \frac{f(n)-f(t)}{n-t}$, we still have a core allocation. So, we find $s\,\frac{f(t)}{t} \geq f(s)$ for $s < t$ and $f(t) + (s-t)\,\frac{f(n)-f(t)}{n-t} \geq f(s)$ for $s > t$. The last inequality gives the Shapley inequalities. $\triangleleft$

Proposition 7. *Every symmetric exact game has a large core.*

Proof: We prove that, if the game has no large core, it is not exact. If the core is not large, there exists an extreme point y of $U(v)$ with $y(N) > f(n)$ and $y_1 \leq y_2 \leq \cdots \leq y_n$.

There exists an index $k : 2 \leq k \leq n-1$ with $y_{k-1} < y_k = y_{k+1} = \cdots = y_n$.
If all coordinate of y are equal (i.e. $k = 1$), $y_i > \frac{f(n)}{n} \geq \frac{f(s)}{s}$ for all indices i and we can decrease y with a positive vector without leaving $U(v)$. Then y is not extreme, as one can always increase y with the same positive vector without leaving $U(v)$. If $k = n$, one can decrease y_n without leaving $U(v)$ and again y is not extremal.

We prove that there exists an index $r : k \leq r \leq n-1$ with $y_1 + \cdots + y_r = f(r)$.
If such an index does not exist and T is a coalition with $y(T) = f(t)$, we have $f(t) \leq y_1 + \cdots + y_t \leq y(T) = f(t)$ and therefore, $T \subseteq [1, \ldots, k-1]$. If we define $u \in \mathbf{R}^n$ by $u_i = 0$ for $i < k$ and $u_k + \cdots + u_n = 0$, we have a vector u with $u(T) = 0$ for all T with $y(T) = f(t)$. We can take $u \neq 0$. Then y is not an extreme point of $U(v)$.

Finally, we prove that there is no core allocation x with $x_1 + \cdots + x_{k-1} = f(k-1)$.
If such a core element x exists, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. Then $x_k + \cdots + x_r \geq f(r) - f(k-1)$ (the index r is as before). For y we have $y_k + \cdots + y_r \leq f(r) - f(k-1)$.

$$x_1 + \cdots + x_n = (x_1 + \cdots + x_{k-1}) + (x_k + \cdots + x_r) + (x_{r+1} + \cdots + x_n) \geq$$
$$f(k-1) + (f(r) - f(k-1)) + (n-r)\,\frac{f(r)-f(k-1)}{r-k+1} \geq f(r) + (n-r)\,y_k = y(N) > f(n). \triangleleft$$

Summarizing, we have the following:

For totally balanced symmetric games (N, v) the following conditions are equivalent:

(a) the Shapley conditions,

(b) exactness,

(c) having a large core.

From each of these properties follows the stability of the core.

We prove, finally, that (d) stability of the core implies (a) the Shapley conditions. In Shapley (1974) the main result is that for a symmetric game (N, v) the core is stable if and only if $(N, -\bar{v}^*)$ is totally balanced. The Shapley conditions are a reformulation of the totally balancedness of $(N, -\bar{v}^*)$. So, the following proposition is, in fact, an (unpublished) result of Shapley (1974). The proof of the proposition we will give is in our opinion simpler.

Proposition 8. (cf. Shapley (1974)) *If the core of a totally balanced symmetric game is a stable set, then it satisfies the Shapley conditions.*

Proof: Suppose that f does not satisfy all the Shapley conditions. Let t^* be an index such that $\frac{f(n)-f(t^*)}{n-t^*} < \frac{f(s)-f(t^*)}{s-t^*}$ for some index $s : t^* < s < n$.

We will give, explicitly, a point $y \in I(v)$ that is not a core allocation and cannot be dominated by a core allocation x.

Take $y_i := \frac{f(t^*)}{t^*}$ for $i \leq t^*$ and $y_j := \frac{f(n)-f(t^*)}{n-t^*}$ for $j > t^*$.

If S is a coalition with $|S| = s \leq t^*$, we have $y(S) \geq s \, \frac{f(t^*)}{t^*} \geq f(s)$ by the totally balancedness condition. If S is a coalition with $|S| = s > t^*$, we find $y(S) \geq f(t^*) + (s - t^*) \, \frac{f(n)-f(t^*)}{n-t^*} \geq f(s)$, if the Shapley condition for $t^* < s < n$ is satisfied. Hence, if $y(S) < v(S)$, then $t^* < s = |S| < n$ and the Shapley condition for $t^* < s < n$ is not satisfied. For $S = [1, \ldots, s]$ we have $y(S) < v(S)$. If y can be dominated by a core allocation x, it must be done by a coalition S of size $s > t^*$, not satisfying the Shapley condition $t^* < s < n$.

Suppose that $x \in U(v)$, $x_S > y_S$ and $x(S) = f(s)$. Let $s' := |S \cap [1, \ldots, t^*]|$. Then,

$$f(s) = x(S) > s' \, \frac{f(t^*)}{t^*} + (s - s') \, \frac{f(n)-f(t^*)}{n-t^*} \geq f(t^*) + (s - t^*) \, \frac{f(n)-f(t^*)}{n-t^*}.$$

Furthermore, $x_j \geq \frac{f(n)-f(t^*)}{n-t^*}$ for $j \notin S$ (otherwise $x(S \backslash i \cup j) < f(s)$, if $x_i > \frac{f(n)-f(t^*)}{n-t^*}$).

$$x(N) > f(t^*) + (f(n) - f(t^*)) = f(n).$$

There is no core allocation dominating y and the core is not stable. ◁

By Lemma 6, the Propositions 7 and 8 we have Theorem 5.

Finally, we prove that *totally exact* TU-games are convex. Totally exact means that the game and all subgames are exact. In Moulin (1990) and Kikuta (1988) the weaker result is proved that "totally large core" (every subgame has a large core) implies convexity.

Proof: Take S and T arbitrarily. As the subgame $(S \cup T, v_{S \cup T})$ is exact, there is a core element x of the subgame with $x(S \cap T) = v(S \cap T)$ (also trivially if $S \cap T = \emptyset$). Then $v(S) + v(T) \leq x(S) + x(T) = x(S \cup T) + x(S \cap T) = v(S \cup T) + v(S \cap T)$. This is the convexity relation. $\triangleleft$

Conclusions: For balanced symmetric games and balanced games with less than 5 players exactness (of the totally balanced cover), stable core and large core are equivalent. For nonsymmetric balanced games with 5 or more players this is no longer true.

The following example, borrowed from van Gellekom et al. (1998), shows that the condition of Shapley/ Kikuta (1986), "any core allocation of any subgame can be extended into a core allocation" is *not* needed for core stability.

Example: Consider the 6-person game $v = u_{12,13} + u_{45,46} + u_N$ where $u_{12,13}$, $u_{45,46}$ and u_N are the simple games with minimal winning coalitions $\{(12),(13)\}$, $\{(45),(46)\}$ and $\{N\}$ respectively.

Then $v(14) = 0$ and $(0,-,-,0,-,-)$ is a core allocation of $((14), v_{(14)})$ that cannot be extended into a core allocation. The smallest extension in $U(v)$ is $(0,1,1,0,1,1)$ with total payoff $4 > v(N) = 3$.

The core of (N, v) is stable, nevertheless. This can be seen as follows. The core of the game (N, v) is given by the relations $x \geq 0$, $x(N) = 3$ and $x(S) \geq 1$ for $S = (12)$, (13), (45) and (46).

Let $y \in \mathbf{R}^6_+$ be an imputation outside the core. Because of the symmetry in the game we may assume that $y_1 + y_2 < 1$. The vector $x = (y_1 + \varepsilon, y_2 + \varepsilon, y_2 + \varepsilon, 1, 0, 0)$ with $\varepsilon = \frac{1}{2}(v(12) - y(12))$ satisfies $x \geq 0$, $x_1 + x_2 = v(12) = 1$ and $x_1 > y_1$ and $x_2 > y_2$. The core conditions $x_1 + x_3 = v(13) = 1$, $x_4 + x_5 = x_4 + x_6 = 1$ are also satisfied. Further, $x(N) = 2 + y_2 + \varepsilon < 3$. If we distribute the difference $1 - y_2 - \varepsilon$ equally among the players $3, 4, 5$ and 6, we obtain a core allocation that dominates y by (12).

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