Temporary bubbles in an economy with under-accumulation[1]

Philippe Michel[2]   Bertrand Wigniolle[3]

[1]The authors thank E. Minelli for helpfull comments and suggestions.

[2]Université de la Méditerranée II and I.U.F, GREQAM, Centre de la Vieille Charité, 2 rue de la Charité, 13002 Marseille, France.

[3]EUREQua Université de Paris I and L.I.B.R.E. Université de Franche-Comté. Address : EUREQua, Université de Paris-I, Maison des Sciences de l'Economie, 106-112, boulevard de l'hopital, 75647 Paris, Cedex 13, France. Email : wignioll@asterix.univ-paris1.fr. Tel : 33+ (0)1 44 07 81 98, Fax : 33+ (0)1 44 07 82 02

Abstract

This paper studies the equilibrium dynamics of an overlapping generations model with capital, money and cash-in-advance constraints. At each date the economy can experience two different regimes. In the first one the cash-in-advance constraint is binding and money is a dominated asset. In the second one, the constraint is strictly satisfied and money has the same return as capital. When the second regime holds on some finite interval, we say that the economy experiences a temporary bubble. We prove that temporary bubbles can exist in an economy which would experience under accumulation without money. We also show, in an example, that cyclical bubbles and sunspot equilibria may occur. Finally, we prove that money creation has the power to eliminate bubbles. JEL numbers : D9, E4 and G1. Key words : overlapping generations model, bubbles, cash-in-advance constraint.

1 Introduction

From Samuelson's article in 1958, the overlapping generations model is become a reference framework for monetary analysis. The first works on this subject pointed out the store-of-value role of money, which allows for exchanges between agents of different generations. It gives some foundations to the idea of confidence in money : money is hold only if agents expects that it will have a positive value in the future. A general presentation of these works can be found in Sargent [1987], Blanchard and Fisher [1989], Azariadis [1993] or Champ and Freeman [1994].

However, this first approach focused on a very specific aspect of money holding. In the presence of another asset, money is valued only if its rate of return was equal to the rate of return on capital. Money is in fact equivalent in such a framework to a rational bubble. Tirole [1985] analyzes the global dynamics of an overlapping generations economy with capital and money. He shows that real money balances hold by the agents can only be valued in an economy which is inefficient without money. In this case, there exists an infinity of equilibria: one of these equilibria leads to a constant and positive value in the long run for the bubble held by each agent, and the economy converges toward the golden rule; the other trajectories converge on the stationary state of Diamond's model without bubbles.

In order to rule out this multiplicity, Tirole [1985] proposes to introduce a liquidity constraint which forces agents to hold some fixed share of their total savings in money. He shows that this type of constraint can eliminate all trajectories converging toward a bubbleless equilibrium, because such an equilibrium cannot satisfy the liquidity constraint. Then, the bubbly equilibrium is the only existing one of the economy with a liquidity constraint. The unicity of the equilibrium is now assured for an economy which is inefficient without bubble.

Recent studies in the overlapping generations framework focused on the introduction of cash-in-advance or liquidity constraints. These works allow to take into account the medium-of-exchange role of money. For example, Hahn and Solow [1995] assume that agents have to hold money between their two periods of life, the amount of money being proportional to the amount of consumption in their second period of life. Money is held, even if it brings a rate of return smaller than the rate of return on capital, because it is necessary for consumption. Champ and Freeman [1990] assume that a constant amount of real monetary balances is required. A general study of various types of liquidity constraints is maide by Crettez, Michel and Wigniolle [1999]. This work shows notably that the Hahn and Solow constraint is the more tractable one for studying the dynamical properties of the economy.

But, Hahn and Solow [1995] assume that the liquidity constraint is binding at each period, which is verified only if the rate of return for money is smaller that the one for capital. We wish to study the intertemporal equilibrium of the overlapping generations economy with a liquidity constraint in the general case. So, we do not exclude *a priori* the existence of a bubble on money, which may appear when money and capital have the same return. At each period, two regimes may occur: the Hahn and Solow's regime, where money is a dominated asset and where the liquidity constraint is binding; the Tirole's regime where money and capital have the same return and where the liquidity constraint is verified with inequality. The economy may jump from one regime to the other, and then temporary bubbles appear.

We define the intertemporal equilibrium of the economy with liquidity constraint, and we first make a dynamical study. In assuming that a stationary equilibrium with under-accumulation exists in the economy without money (the Diamond [1965] economy), we prove that there exists in the neighborhood of this equilibrium a monetary equilibrium in the Hahn and Solow's regime for a sufficiently low liquidity constraint. The study of the local dynamics in a neighborhood of this equilibrium shows that the dynamics of the monetary economy can be as close as one wish to the equilibrium of the non-monetary economy, when the liquidity constraint tends to zero.

But, this monetary equilibrium is not unique. Indeed, we prove that it is possible to jump in the Tirole's regime, and that the economy can experience sometimes periods with bubbles on money. So there is a multiplicity of equilibria, with the existence of temporary bubbles.

In the case where the utility and production functions are Cobb-Douglas, we are able to explicitly study the global dynamics of the economy. In this framework, a unique stationary equilibrium exists in the Hahn and Solow's regime. Two cases may occur, as this stationary state is in over or under accumulation. In the first case, we prove that the Tirole's regime with a permanent bubble is the only possible equilibrium and that the economy converges toward the golden rule. We find again Tirole [1985]'s result: all bubbleless trajectories in the long run are ruled out by the existence of the liquidity constraint. But, in the second case (if the equilibrium with money in the Hahn and Solow's regime is in under-accumulation), different situations may appear. If the liquidity constraint is sufficiently hard, the Hahn and Solow's regime is the only possible one, and the intertemporal equilibrium is unique. But, if the liquidity constraint is relatively low, it is possible that the economy oscillates between the two regimes. Money can periodically support bubbles, but these bubbles are temporary. There is multiplicity of dynamical equilibria, the number of intertemporal equilibria decreasing with the weight of the liquidity constraint.

The multiplicity of equilibria leads to the question of the coordination of anticipations. This coordination could be achieved if all agents believed in the same theory on the appearance of the two regimes. We assume that this theory consists in setting that the state of the economy is related to the occurrence of an exogenous phenomenon called sunspots. We show the existence of such sunspot equilibria, where the dynamics of the economy become stochastic.

The appearance of a temporary bubble reduces capital accumulation, because it absorbs some part of savings. As the economy is in under-accumulation, bubbles tend to push the economy far from the golden rule. So we consider that an objective for monetary policy could be to fight bubbles. Indeed, monetary creation diminishes the return on money and can forbid the jump to the Tirole's regime where money and capital have the same return. We show that a sufficiently high rate of monetary creation can eliminate temporary bubbles. But this policy also has another negative impact on agents welfare: it increases the distortion related to money detention.

The model is presented in the second section. The study of the intertemporal equilibrium is achieved in section three. The fourth section characterizes the global dynamics of a Cobb-Douglas economy and studies the influence on bubbles of monetary creation.

2 The model

We consider an overlapping generations model with capital accumulation *à la* Allais [1947]-Diamond [1965]. Agents live two periods. They supply one unit of labor in the first period (when young) and they are retired and consume the return of their savings in the second period (when old). The number of young agents at a date t, N_t, grows at the constant rate $n : N_t = (1+n)N_{t-1}$.

2.1 The liquidity constraint

In the model with a infinite-lived representative consumer, the cash-in-advance constraint assumes that the agent has to save at each period the quantity of money necessary to finance his consumption at the following period. In the overlapping generations model with two periods of life, if all the second period consumption value $P_{t+1}d_{t+1}$ was financed by the amount of money saved in the first period M_t, all savings would be in money, and it would be impossible to accumulate capital.

So we have to modify the cash-in-advance constraint. Following Hahn et Solow [1995], we assume the following constraint:

$$M_t \geq \mu P_{t+1} d_{t+1} \tag{1}$$

A share μ, $0 < \mu < 1$ of consumption expenses in the second period must be financed by the amount of money saved in the first period.

2.2 Savings choice

2.2.1 The utility function

Agents born in t are endowed with an intertemporal utility function:

$$U_t = U(c_t, d_{t+1}) \tag{2}$$

c_t is the first period consumption, and d_{t+1} is the second period consumption.
Assumption 1 : U is strictly quasi-concave, twice continuesly differentiable, and such that:

$$\forall c \text{ and } d \text{ positive, } U'_c(0, d) = U'_d(c, 0) = +\infty$$

From this assumption, one deduces the existence of a C^1 function $\sigma(w, R)$ defined on $\mathbb{R}^2_{++}$ by :

$$\sigma(w, R) = \arg\max_{\sigma} U(w - \sigma, R\sigma)$$

2.2.2 Budgetary constraints

Agents can save under two different forms: they can invest in firms capital, as in the Diamond model, but they also have to hold money, which has a nominal rate of return equal to zero.

In real terms, the two budget constraints of a generation t agent are:

$$c_t + s_t + \frac{M_t}{P_t} = w_t \tag{3}$$

$$d_{t+1} = R_{t+1} s_t + \frac{M_t}{P_{t+1}} \tag{4}$$

where w_t is the real wage in period t and s_t is the amount of savings invested in capital. R_{t+1} is the real return factor expected for period $t + 1$. M_t is the money amount hold in period t and P_t is the price of the good in money.

Eliminating s_t between the two constraints (3) and (4), we obtain the intertemporal budget constraint:

$$c_t + \frac{1}{R_{t+1}} d_{t+1} + \frac{M_t}{R_{t+1} P_t} \left(R_{t+1} - \frac{P_t}{P_{t+1}} \right) = w_t \tag{5}$$

In equation (5) P_t/P_{t+1} is the real return on money. The third term is the expected loss resulting from savings hold in money. This loss is the difference between returns on capital and on money. In order to make an optimal choice, agents have to compare the returns on these two assets. Two cases must be distinguished:

- Either the liquidity constraint is binding (we call this case the Hahn and Solow's case),

$$\frac{M_t}{P_{t+1}} = \mu d_{t+1} \tag{6}$$

  and then the expected return on money is no larger than the return on financial savings

$$P_t/P_{t+1} \leq R_{t+1}$$

- Or the expected liquidity constraint si not binding (we call this case the Tirole's case),

$$\frac{M_t}{P_{t+1}} > \mu d_{t+1} \tag{7}$$

  and then the expected return on money must be equal to the return on financial savings,

$$P_t/P_{t+1} = R_{t+1} \tag{8}$$

2.3 Agents behavior

2.3.1 Young consumers decisions in period t

Each young agent maximizes his utility given by (2) under the budgetary constraints (3) and (4) and the liquidity constraint (1). Two cases may appear:

The Hahn and Solow's case (H.S.) : the liquidity constraint is binding. Then, using (6) and (5) to eliminate M_t, we obtain the constraint:

$$c_t + \frac{d_{t+1}}{\rho_{t+1}} = w_t$$

with:

$$\frac{1}{\rho_{t+1}} = \frac{1-\mu}{R_{t+1}} + \mu \frac{P_{t+1}}{P_t} \tag{9}$$

ρ_{t+1} is the real expected return of total savings, when the liquidity constraint is binding. $1/\rho_{t+1}$ is the mean of the inverse return of money weighted by μ and the inverse return of capital weighted by $1-\mu$. The assumption on expected returns $P_t/P_{t+1} \leq R_{t+1}$ can be written equivalently as:

$$\rho_{t+1} \leq R_{t+1}$$

The resolution of the consumer program leads to the expression of total savings:

$$\sigma_t = \sigma(w_t, \rho_{t+1}) = s_t + m_t \tag{10}$$

where m_t is the real money holding:

$$m_t = \frac{M_t}{P_t}$$

Using (4) and (6), we obtain:

$$(1-\mu)m_t = \mu \frac{P_{t+1}}{P_t} R_{t+1} s_t \tag{11}$$

Equations (10) and (11) give s_t et m_t. Finally, using (11), one can replace the condition (9) by :

$$\rho_{t+1} = \frac{R_{t+1} s_t}{(1-\mu)(m_t + s_t)} \tag{12}$$

The three conditions (10), (11) and (12) characterize the behavior of a generation t agent expecting a binding liquidity constraint.

The Tirole's case (T.) : the liquidity constraint is not binding:

$$M_t/P_{t+1} > \mu d_{t+1}$$

In this case, total savings of the consumer is the same as in the Diamond's model:

$$\sigma_t = \sigma(w_t, R_{t+1}) = s_t + m_t \tag{13}$$

Savings of the consumer can be shared by any proportion of money or capital. The only constraint is the cash-in-advance constraint (7) which is equivalent to:

$$(1-\mu)m_t > \mu s_t \tag{14}$$

The sharing of savings between m_t and s_t is indeterminate, but there is a unique solution for consumptions:

$$\begin{aligned} c_t &= w_t - \sigma(w_t, R_{t+1}) \\ d_{t+1} &= R_{t+1}\sigma(w_t, R_{t+1}) \end{aligned}$$

The condition $P_t/P_{t+1} = R_{t+1}$ being equivalent to $\rho_{t+1} = R_{t+1}$, we see that relations (13) and (10) are identical, for $\rho_{t+1} = R_{t+1}$.

2.3.2 Old agents in period $t = 0$

In the initial period, old agents in 0 hold a quantity of money M_{-1}, and they inherit from the last period capital savings s_{-1} which are remunerated by the interest factor R_0. Their consumption is :

$$d_0 = \frac{M_{-1}}{P_0} + R_0 s_{-1} \tag{15}$$

But, they could only reach this consumption level if their cash in hand is sufficient:

$$M_{-1} \geq \mu P_0 d_0$$

We deduce that they should to hold a quantity of money M_{-1} such that:

$$(1 - \mu) M_{-1}/P_0 \geq \mu R_0 s_{-1} \tag{16}$$

Then, d_0 is given by (15).[1]

2.3.3 Firms

We assume that at each period t exists one competitive firm using a neo-classical technology with constant returns to scale $F(K_t, L_t)$. F is increasing in its two arguments, concave, twice continuesly differentiable. The capital stock K_t is determined by savings of generation $t-1$

$$K_t = N_{t-1} s_{t-1}$$

L_t is the quantity of labor used in production, paid by the real wage w_t. The profit maximization of the firm gives labor demand:

$$w_t = F_L(K_t, L_t) \tag{17}$$

The return on capital investments made by agents at the precedent period is:

$$R_t = \frac{F(K_t, L_t) - w_t L_t}{K_t} = F_K(K_t, L_t)$$

using the homogeneity of degree 1 of the function F.

[1] In the case where $(1-\mu)M_{-1}/P_0 < \mu R_0 s_{-1}$, d_0 would be given by the cash-in-advance constraint: $d_0 = M_{-1}/(\mu P_0)$. In this case, the first old agents would be rationed. This desequilibrium case is excluded in the sequel.

3 Intertemporal equilibrium

3.1 Equilibrium characterization

The equilibrium on the labor market leads to: $L_t = N_t$. We define the variable k_t as the level of capital by worker:

$$k_t = \frac{K_t}{N_t}$$

We deduce from that the equilibrium wage:

$$w_t = F_{L_t}(k_t, 1) \tag{18}$$

The factor of return for productive investments is:

$$R_t = F_{K_t}(k_t, 1) \tag{19}$$

The total money stock $\overline{M}$ is constant. In the beginning of period t, it is held with equal shares by old agents: $M_{t-1} = \overline{M}/N_{t-1}$. Equilibrium on the money market gives:

$$\overline{M} = N_t M_t$$

or :

$$\frac{\overline{M}}{P_t} = N_t m_t \tag{20}$$

Finally, we express that capital in period $t+1$ results from savings of generation t agents:

$$K_{t+1} = N_t s_t \Leftrightarrow (1+n)k_{t+1} = s_t \tag{21}$$

Using (20), money gross return is given by:

$$P_t/P_{t+1} = \frac{\overline{M}/P_{t+1}}{\overline{M}/P_t} = (1+n)\frac{m_{t+1}}{m_t} \tag{22}$$

This return cannot be larger than the return of physical capital, or, with $k_{t+1} > 0$:

$$m_{t+1} \leq \frac{R_{t+1}}{1+n} m_t \tag{23}$$

with :

$$R_{t+1} = F_K(k_{t+1}, 1)$$

The two preceding cases will be studied separately.

The (H. S.) case : the liquidity constraint is binding between t and $t+1$, and $P_t/P_{t+1} \leq R_{t+1}$. Using the expression of the cash-in-advance constraint (11), and equations (22) and (21), one obtains :

$$(1-\mu)m_{t+1} = \mu R_{t+1} k_{t+1} \tag{24}$$

The dynamics of capital is given in replacing savings s_t in (21) by its expression (13), and in using ρ_{t+1} which is given by (12):

$$\begin{aligned} (1+n)k_{t+1} &= \sigma(w_t, \rho_{t+1}) - m_t \\ \text{with } \rho_{t+1} &= \frac{(1+n)R_{t+1}k_{t+1}}{(1-\mu)(m_t + (1+n)k_{t+1})} \end{aligned} \tag{25}$$

Finally, the condition which ensures that money is a dominated asset corresponds to equation (23). With (24), one can write:

$$\mu(1+n)k_{t+1} \leq (1-\mu)m_t \tag{26}$$

The (T.) case: Money is not dominated between t and $t+1$: $P_t/P_{t+1} = R_{t+1}$. In this case, (23) is verified with an equality and we have:

$$m_{t+1} = \frac{R_{t+1}}{1+n} m_t \tag{27}$$

The dynamics of capital is always given by (25), but with $\rho_{t+1} = R_{t+1}$:

$$(1+n)k_{t+1} = \sigma(w_t, R_{t+1}) - m_t \tag{28}$$

Finally, one has to write that the cash-in-advance constraint of generation t agents is not binding, or (14). Using (21), we find:

$$\mu(1+n)k_{t+1} < (1-\mu)m_t \tag{29}$$

One recognizes the same relation as (26) written with a strict inequality. Using (27), we can express this last inequality as:

$$(1-\mu)m_{t+1} > \mu R_{t+1} k_{t+1} \tag{30}$$

This last form is the same as in the (H.S.) case (24), but with a strict inequality. For an intertemporal equilibrium, $\forall t \geq 0$, (24) or (30) is verified.

Finally, we express that the first old agents satisfy their cash-in-advance constraint. We saw that this property could be written by (16):

$$(1-\mu)M_{-1}/P_0 \geq \mu R_0 s_{-1}$$

Aggregating this relation, one obtains:

$$(1-\mu)\overline{M}/P_0 \geq \mu R_0 K_0$$

or:

$$(1-\mu)m_0 \geq \mu R_0 k_0 \tag{31}$$

It is possible to give the following definition:

Definition 1 *: All sequence* $(k_t, m_t)_{t\geq 0}$ *with* $k_t > 0$ *and* $m_t > 0$ *which verifies equation (31) and for all* $t \geq 0$,

- *either equilibrium conditions (H. S.): (24), (25) and (26)*
- *or equilibrium conditions (T.) : (27), (28) and (29)*

defines an intertemporal equilibrium with perfect foresight.

3.2 Steady states

We are studying constant trajectories (k^*, m^*) with $k^* > 0$ and $m^* > 0$. We begin by characterizing existence of stationary states in the (T.) case. After that, we shall analyze the existence of a stationary state in the (H.S.) regime.
The T. case : From (27), k^* must be such that $R^* = 1+n$. k^* is the golden rule capital stock. The real stock of money at the stationary state m^* is given by (28) :

$$m^* = \sigma(w^*, 1+n) - (1+n)k^* \tag{32}$$

Such a stationary equilibrium exists if and only if m^* given by (32) strictly satisfies the cash-in-advance constraint (29), or:

$$(1+n)k^* < (1-\mu)\sigma(w^*, 1+n) \tag{33}$$

A necessary condition for such an equilibrium to exist is:

$$(1+n)k^* < \sigma(w^*, 1+n)$$

For the values of wage and interest rate in the golden rule, savings is higher than the level needed for financing the investment corresponding to the golden rule. Consequently, there exists a stationary equilibrium in the Diamond's economy (without money) associated with over-accumulation. Indeed, the function

$$\phi(k) = \sigma(w(k), R(k)) - (1+n)k$$

is positive in k^* and goes to $-\infty$ when $k \to +\infty$.[2] So, there exists $k^D > k^*$ such that $\phi(k^D) = 0$.

k^* is the stationary state of the Diamond's economy with asymptotic bubbles studied by Tirole [1985]. This last contribution establishes the existence of a unique trajectory associated with an asymptotic bubble. If from a date t_0, all the points of this trajectory, following the dynamical equations (27) and (28), also verify the liquidity constraint (29), then this trajectory is an equilibrium of the economy with a liquidity constraint.

The (H.S.) case : From (24) and (26), (k^*, m^*) are such that money is a dominated asset if: $R^* \geq 1+n$. Then, k^* is associated with under-accumulation. Eliminating m^* with (24) in (25), one obtains the following equation defining k^* :

$$(1+n)k^* + \frac{\mu}{1-\mu} R^* k^* = \sigma(w^*, \rho^*)$$

with

$$\rho^* = \frac{(1+n)R^*k^*}{\mu R^* k^* + (1-\mu)(1+n)k^*} = \frac{(1+n)}{\mu + (1-\mu)(1+n)/R^*}$$

3.3 Local study of the dynamics in the (H.S.) case

We make a local study of convergent trajectories in the (H.S.) case. So, we consider a neighborhood of a stationary equilibrium of the Diamond's economy that we assume stable and in under-accumulation. We consider the economy with the liquidity constraint for a small value of μ.

Assumption 1 : k^D is a solution of $(1+n)k^D = \sigma(w(k^D), R(k^D))$ which verifies:

$$R(k^D) > 1+n \text{ and } \sigma'_w w'(k^D) + \sigma'_R R'(k^D) < 1+n$$

The last condition is equivalent to:

$$\frac{dk_{t+1}}{dk_t} = \frac{\sigma'_w w'(k^D)}{1+n-\sigma'_R R'(k^D)}$$

strictly between 0 and 1.

---

[2] This property comes from:

$$\phi(k) < k\left(\frac{w(k)}{k} - (1+n)\right)$$

and

$$\lim \frac{w(k)}{k} = 0$$

Proposition 1 *: There exists a neighborhood* $I = (\underline{k}, \overline{k})$ *of* k^D *and* $\varepsilon > 0$ *such that, for all* $\mu \in (0, \varepsilon)$ *and all* $k_0 \in I$*, there exists a unique intertemporal equilibrium* $(k_t, m_t)_{t \geq 0}$ *of the economy with a liquidity constraint, with initial conditions* k_0 *and* $m_0 = \frac{\mu}{1-\mu} R_0 k_0$*, this equilibrium being located in I (i.e.* $k_t \in I \ \forall t$*), and such that at each date the liquidity constraint is binding (case (H.S.)). This trajectory converges toward a stationary equilibrium* $k^*(\mu)$ *of the (H.S.) regime. And the sequence* $k_t(\mu)$ *converges uniformly toward the Diamond trajectory* $k_t(0)$ *starting from* k_0 *when* μ *tends to 0.*

Proof: see appendix 1.

This proposition shows that in a neighborhood of a stationary state in under-accumulation, it is possible to define an intertemporal equilibrium with money and with binding liquidity constraints at each period, when μ is suficiently small. The trajectory of the monetary economy converges uniformly toward the non-monetary one as μ tends to 0. But we are going to see that this intertemporal equilibrium in the (H.S.) regime is not unique, and that other equilibria with temporary bubbles exist.

3.4 Trajectories with bubbles in under-accumulation

We are studying now the following question: is it possible that a trajectory which converges in the (H.S.) regime in the long run includes bubbles at some dates ? Let us consider an intertemporal equilibrium entirely in the (H.S.) regime, as defined in proposition 1. Is it possible to modify in one point this trajectory in order to obtain a bubble (and to be during one period in the (T.) regime) and then to go back to the (H.S.) regime ?

When the economy experiences two consecutive periods in the (H.S.) regime (periods t and $t+1$), the dynamics can be written:

$$g(k_{t+1}, k_t, \mu) = 0, \ t \geq 0$$

with

$$g(k_{t+1}, k_t, \mu) \equiv (1+n)k_{t+1} - \sigma\left(w(k_t), \rho(k_{t+1}, k_t, \mu)\right) + \frac{\mu}{1-\mu} R_t k_t$$

and

$$\rho(k_{t+1}, k_t, \mu) \equiv \frac{(1+n)R(k_{t+1})k_{t+1}}{\mu R(k_t)k_t + (1-\mu)(1+n)k_{t+1}}$$

A trajectory with one period in the (T.) regime would be characterized by the following conditions:

1. In $t = 0$, with k_0 and $m_0 = \frac{\mu}{1-\mu} R_0 k_0$, there is no expected bubble, the regime (H.S.) happens. So we have:

$$g(k_1, k_0, \mu) = 0$$

$$m_1 = \frac{\mu}{1-\mu} R_1 k_1$$

2. In $t = 1$, there is an expected bubble, and one jumps to the (T.) regime.

$$(1+n)k_2 = \sigma(w_1, R_2) - m_1$$

$$m_2 = \frac{R_2}{1+n} m_1$$

The liquidity constraint can be written:

$$(1-\mu)m_2 > \mu R_2 k_2$$

or equivalently:

$$R_1 k_1 > (1+n)k_2$$

3. In $t = 2$, there is no expected bubble: one goes back to the (H.S.) regime.

$$\begin{aligned} (1+n)k_3 &= \sigma(w_2, \rho_3) - m_2 \\ \text{with } \rho_3 &= \frac{(1+n)R_3 k_3}{(1-\mu)(m_2 + (1+n)k_3)} \end{aligned}$$

$$m_3 = \frac{\mu}{1-\mu} R_3 k_3$$

4. Then, the dynamics of the (H.S.) regime applies: for $t \geq 3$,

$$\begin{aligned} g(k_{t+1}, k_t, \mu) &= 0 \\ m_{t+1} &= \frac{\mu}{1-\mu} R_{t+1} k_{t+1} \end{aligned}$$

Proposition 2 : *Under assumption 1, one can modify a trajectory of the (H.S.) regime sufficiently close to k^D and for μ small enough in introducing a bubble during one period.*

Proof: In a stable Diamond's equilibrium associated with under-accumulation and in a neighborhood of this equilibrium, all these conditions are satisfied with $\mu = 0$ and the inequality $R_1 k_1 > (1+n)k_2$. The implicit function theorem allows to determine k_1, m_1, k_2, m_2, k_3 in function of μ, for μ small enough and k_0 sufficiently close to k^D, which verify conditions 1, 2 and 3. By continuity, it is possible to use neighborhoods such that $k_3 \in I$. Proposition 1 can be applied from k_3.

Following the same argument, one also find that bubbles can be introduced during a finite number of periods. One can even introduce bubbles at an infinite number of periods, which prevent the trajectory from converging toward a stationary state. We need only to introduce a bubble every time k is close enough to the stationary state of the dynamics without bubbles.

4 Global dynamics in the Cobb-Douglas case

4.1 Expression of the dynamics

We consider Cobb-Douglas specifications for utility and production functions. We show that it is possible to explicitly characterize the global dynamics in this case.

Agents born in t are endowed with the utility function:

$$U_t = U(c_t, d_{t+1}) = (1-a)\ln c_t + a \ln d_{t+1}$$

The total savings function takes the following expression:

$$\sigma(w, R) = aw$$

The representative firm produces with a Cobb-Douglas technology:

$$F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$$

Under these assumptions, appendix 2 establishes that the perfect foresight dynamics take a simple form. Introducing the variable $x_t = k_{t+1}k_t^{-\alpha}$, the dynamics verifies :

(H.S.) regime between t and $t+1$:

$$x_{t+1} = \widetilde{x} \text{ with } \widetilde{x} = \frac{A}{1+n}\left(a(1-\alpha) - \frac{\mu\alpha}{1-\mu}\right) \tag{34}$$

$$\text{and } x_t \leq \overline{x} \text{ with } \overline{x} = \frac{A}{1+n}a(1-\alpha)(1-\mu) \tag{35}$$

(T.) regime between t and $t+1$:

$$x_{t+1} = x_d - \widehat{x}\left(\frac{x_d}{x_t} - 1\right) \quad (36)$$

$$\text{and } x_t < \overline{x} \quad (37)$$

with the following notations:

$$x_d = \frac{aA(1-\alpha)}{1+n} \text{ and } \widehat{x} = \frac{A\alpha}{1+n}$$

x_d corresponds to the value of the variable x at the Diamond's stationary state , when $\widehat{x}$ corresponds to the golden rule.

Initial condition for the first old people:

$$x_0 \leq \widetilde{x}$$

From the sequence of (x_t), it is possible to obtain the sequences of (k_t) and (m_t). From k_0 which is given, k_t is defined, from the definition of x_t, by the equation :

$$k_{t+1} = x_t k_t^{\alpha}$$

and then, m_t is such that (cf. appendix 2) :

$$m_t = a(1-\alpha)Ak_t^{\alpha} - (1+n)k_{t+1} \quad (38)$$

The following properties will be useful in the sequel:

$$\overline{x} = (1-\mu)x_d \text{ and } \widetilde{x} = x_d - \frac{\mu}{1-\mu}\widehat{x}$$

Finally, one stresses that x_t is a forward variable, for which no initial condition exists.

The dynamical equation of the (T.) regime admits two stationary states x_d and $\widehat{x}$, the greater being stable and the smaller unstable.

Two cases have to be distinguished, corresponding to the comparison of the stationary state of the (H.S.) regime with the golden rule.

4.2 The first case : $\widetilde{x} > \widehat{x} \Leftrightarrow (1-\mu)x_d > \widehat{x}$

In this case, the stationary state in the (H.S.) regime is associated with over-accumulation. The following inequalities are verified:

$$x_d > \widetilde{x} > \overline{x} > \widehat{x}$$

These inequalities show notably that the Diamond's stationary state is in over-accumulation: $x_d > \widehat{x}$. From that one deduces that in the dynamics of the (T.) regime, x_d is the stable stationary state, when $\widehat{x}$ is the unstable one.

As $\widetilde{x} > \overline{x}$, the (H.S.) regime is impossible. The economy follows necessarily equations of the (T.) regime, and the dynamics is given by (36). $x_t < \widehat{x}$ is impossible, because x_t would converge toward negative values. $x_t > \widehat{x}$ is impossible because x_t would converge in the long run toward $x_d > \overline{x}$. Consequently $x_t = \widehat{x}$, $\forall t$, as one has $\widehat{x} < \overline{x}$. The economy converges in the long run toward the golden rule. The equilibrium is determined.

The interpretation of this result is simple: all trajectories of the model of Tirole [1985] converging toward a long run stationary state without bubbles are eliminated by the existence of the liquidity constraint. Agents are constrained to hold some cash-in-hand. The only existing trajectory is the one converging toward the golden rule. We find again Tirole [1985]'s result: all bubbleless trajectories in the long run are ruled out by the existence of the liquidity constraint. In the Tirole's economy, this constraint was that a minimum amout of savings had to be in gold. Here, the constraint is a minimum requirement of money in order to finance second period consumption. But these two constraints have the same effect in ruling out the possibility of overaccumulation in the long run.

4.3 The second case : $\widetilde{x} \leq \widehat{x} \Leftrightarrow (1-\mu)x_d \leq \widehat{x}$

In this case, the stationnary state in the (H.S.) regime is associated with under-accumulation. The following inequalities are verified:

$$\begin{aligned} \widetilde{x} &\leq \overline{x} \leq \widehat{x} \\ x_d &> \overline{x} \end{aligned}$$

The (H.S.) regime at any date is an equilibrium:

$$\begin{aligned} x_t &= \widetilde{x} \ \forall t \geq 1 \\ x_0 &\leq \widetilde{x} \end{aligned}$$

Two cases can happen: either this equilibrium is the only possible one, or the (T.) regime is possible simultaneously. We consider successively these two cases.

4.3.1 The (H.S.) regime is the only possible one

Let us assume that at a date t the economy is in the (T.) regime. Then, one has :

$$\begin{aligned} x_{t+1} &= x_d - \widehat{x}\left(\frac{x_d}{x_t} - 1\right) \text{ with } x_t < \overline{x} \\ x_0 &\leq \widetilde{x} \end{aligned}$$

The inequality $x_t < \overline{x} \leq \min(x_d, \widehat{x})$ ensures that the dynamics in the (T.) regime leads to negative values for the variable x. Such dynamics cannot be followed in the long run, and there exists a date where the economy jumps to the (H.S.) regime. Considering the function

$$f(x) = x_d - \widehat{x}\left(\frac{x_d}{x} - 1\right)$$

f is an increasing function. As $x_0 \leq \widetilde{x}$, if $f(\widetilde{x}) \leq 0$, one concludes that the economy cannot be in the (T.) regime at the initial period, and that it remains at all future dates in the (H.S.) regime. The condition $f(\widetilde{x}) \leq 0$ can be written on the form:

$$\mu \geq \frac{x_d^2}{x_d^2 + x_d\widehat{x} + \widehat{x}^2}$$

In this case, the dynamics is determined from period $t = 1$ and one has:

$$\begin{aligned} x_t &= \widetilde{x} \ \forall t \geq 1 \\ x_0 &\leq \widetilde{x} \end{aligned}$$

The dynamics is determined when the liquidity constraint is sufficiently strong.

4.3.2 Transitions between the two regimes

We consider the case where the precedent inequality is not satisfied[3]:

$$\mu < \frac{x_d^2}{x_d^2 + x_d\widehat{x} + \widehat{x}^2} \text{ and } \mu \geq 1 - \frac{\widehat{x}}{x_d}$$

[3]It is easy to verify that these two conditions are compatible, as the property

$$\frac{x_d^2}{x_d^2 + x_d\widehat{x} + \widehat{x}^2} > 1 - \frac{\widehat{x}}{x_d}$$

is equivalent to:

$$\widehat{x}^3 > 0.$$

We show that multiple equilibria exist.

(H.S.) Equilibrium

A first equilibrium consists in staying at each date in the (H.S.) regime:

$$x_t = \widetilde{x} \ \forall t$$

Cyclical equilibria of period 2 (H.S.)-(T.)

It is possible to construct equilibria with oscillations between the two regimes. Let us assume that at each date one jumps in the other regime. One observes a cyclical dynamics of the variable x_t :

$$\widetilde{x}, f(\widetilde{x}), \widetilde{x}, f(\widetilde{x}), \widetilde{x}, f(\widetilde{x}), \widetilde{x}, f(\widetilde{x}).....$$

Such dynamics for x_t implies a particular dynamics for the variables k_t and m_t which converges toward a cycle of period 2 denoted by:

$$\begin{pmatrix} k_1^* \\ m_1^* \end{pmatrix}, \begin{pmatrix} k_2^* \\ m_2^* \end{pmatrix}$$

We assume that one belongs to the (T.) regime when one jumps from 1 to 2, and to the (H.S.) regime when one jumps from 2 to 1. So:

$$\begin{aligned} \widetilde{x} &= k_2 k_1^{-\alpha} \\ f(\widetilde{x}) &= k_1 k_2^{-\alpha} \end{aligned}$$

or:

$$\begin{aligned} k_1^* &= [\widetilde{x}^{\alpha} f(\widetilde{x})]^{\frac{1}{1-\alpha^2}} \\ k_2^* &= [\widetilde{x} f(\widetilde{x})^{\alpha}]^{\frac{1}{1-\alpha^2}} \end{aligned}$$

Finally, real money balances are given by:

$$\begin{aligned} m_1^* &= \frac{\mu}{1-\mu} R(k_1^*) k_1^* = \frac{\mu A \alpha}{1-\mu} [\widetilde{x}^{\alpha} f(\widetilde{x})]^{\frac{\alpha}{1-\alpha^2}} \\ m_2^* &= \frac{R(k_2^*)}{1+n} m_1^* = \frac{\mu (A\alpha)^2}{(1-\mu)(1+n)} [\widetilde{x} f(\widetilde{x})]^{\frac{\alpha^2+\alpha-1}{1-\alpha^2}} \end{aligned}$$

Cyclical equilibria with a period greater that 2

Let us assume that the n first iterates of $\widetilde{x}$ by f are positive:

$$f^k(\widetilde{x}) > 0 \ \forall k \leq n$$

We consider a cyclical dynamics for the variable x_t with the following form:

$$\left(\widetilde{x}, f(\widetilde{x}), f^2(\widetilde{x}), ...f^n(\widetilde{x}), \underbrace{\widetilde{x}, ..., \widetilde{x}}_{p-1 \text{ times}}\right)$$

which is periodic of order $n+p$. The economy remains n consecutive periods in the (T.) regime, and p periods in the (H.S.) regime.

More generally, the multiplicity of possible equilibrium dynamics increases with the number of positive iterates of $\widetilde{x}$ by f. The lower is μ, the closer is $\widetilde{x}$ from $\min\left(x_d, \widehat{x}\right)$, and the greater is the multiplicity of equilibria. So, the importance of the liquidity constraint plays in favor of the determination of the equilibrium.

Sunspot equilibria

Considering the multiplicity of possible equilibria, one may assume that agents coordinate their expectations on a particular equilibrium, if they believe in the same theory. This theory consists in setting that the state of the economy is bound with an exogenous phenomenon, by example the appearance of sunspots. In our framework, agents could link this exogenous phenomenon to the occurrence of each regime (H.S.) and (T.).

More precisely, one assumes that the n first iterates of $\widetilde{x}$ by f are positive:

$$f^k(\widetilde{x}) > 0 \ \forall k \leq n$$

We assume that agents observes an exogenous markovian phenomenon at each period t which can take $n+1$ different states. Transition probabilities from a state to another one are given by the matrix:

	State in $t+1$					
State in t	0	1	2	3	4	n
0	π_0	$1-\pi_0$	0	0	0	0
1	π_1	0	$1-\pi_1$	0	0	0
2	π_2	0	0	$1-\pi_2$	0	0
$n-1$	π_{n-1}	0	0	0		$1-\pi_{n-1}$
n	1	0	0	0	0	0

At each period t, agents observe the exogenous phenomenon before they make their choice, and they deduce from this observation if the economy will be between t and $t+1$ in the (H.S.) or in the (T.) regime. If agents observe the state 0, they deduce that the economy will be in the regime (H.S.). If agents observe the realization of the state k, it means that the economy is going to experience its k^{th} consecutive period in the (T.) regime. Finally, the economy cannot be more than n consecutive periods in the (T.) regime.

Under these assumptions, one obtains a sunspot equilibrium of a particular type. The observation of the phenomenon gives to agents informations on the regime which is expected to be, (H.S.) or (T.), and then on the value of the variable x_t. So, the observation of the state 0 leads to the realization of $x_t = \widetilde{x}$. The observation of the state k leads to the realization of $x_t = f(\widetilde{x})^k$. But, the corresponding dynamics of the economy (for the variables k_t et m_t) is complex. It depends on the risky realizations of the exogenous phenomenon. Then, we obtain a non-stationary sunspot equilibrium.

These results can be compared with Tirole (1985). In his paper, Tirole consider the Cobb-Douglas case for the utility function with a liquidity constraint of the form $m_t \geq \mu s_t$: some minimal amout of savings has to be hold in money. In this framework, it is easy to show that no temporary bubbles can exist, because the dynamics when the constraint is binding loses its forward dimension[4].

4.4 The counter-bubbles monetary policy

We consider the following question: is it possible to rule out temporary bubbles with an appropriate monetary policy ? The intuition suggests that monetary creation induces inflation which causes a fall in the return of money detention. Then, the return on money could no more be the same as the return on capital. When a bubble appears, it absorbs a share of savings, which can no more finance productive investment. The bubble diminishes capital accumulation. In a situation of under-accumulation, to fight the possible appearance of bubbles could be an objective of the monetary policy.

In order to study this question, we always consider Cobb-Douglas functions, what allows to have an explicit form of the global dynamics followed by the economy.

4.4.1 Monetary creation

We assume that the government creates money, and that it gives this money as a lump sum transfer to young people. We note $\overline{M}_t$ the total supply of

[4]For a general discussion on different formulations of the liquidity constraint and on their dynamical properties, cf. Crettez, Michel and Wigniolle [1999].

money in period t, and λ_t the rate of money creation. So we have:

$$\overline{M}_t = (1+\lambda_t)\overline{M}_{t-1} \tag{39}$$

T_t is the lump sum transfer received by each of the N_t young agents living during period t. This transfer is financed by money creation:

$$\lambda_t \overline{M}_{t-1} = N_t T_t \tag{40}$$

Remark 1 *The assumption that money creation is redistributed to young people has two justifications. The first one is that it allows to keep for the dynamics of x_t a dimension equal to one. The second one is that it is the most unfavourable assumption for ruling out temporary bubbles. Indeed, the distribution of the created money in the first period of life increases the productive savings of the agents and decreases the interest rate. This indirect effect of money creation plays in the opposit direction as the direct effect which lowers the return on money. Consequently, we are going to prove that money can rule out bubbles in the most unfavourable case for this concern.*

4.4.2 The dynamics with a constant rate of monetary creation

Appendix 3 presents the equilibrium dynamics with money creation. We assume now that the rate of money creation is constant: $\lambda_t = \lambda, \forall t$. We study the dynamics in introducing the variable $x_t = k_{t+1}k_t^{-\alpha}$. As it is shown in appendix 3, the dynamics can be written under the following form:
The (H.S.) regime:

$$\begin{aligned} x_{t+1} &= \widetilde{x}(\lambda) \\ \text{with } \widetilde{x}(\lambda) &= \frac{A}{1+n}\left(a(1-\alpha) - \frac{\mu\alpha(1+\lambda(1-a))}{1-\mu}\right) \end{aligned} \tag{41}$$

Money is a dominated asset if:

$$\begin{aligned} x_t &\leq \overline{x}(\lambda) \\ \text{with } \overline{x}(\lambda) &= \frac{Aa(1-\alpha)(1-\mu)}{1+n}\frac{1+\lambda}{1+\lambda(1-\mu a)} \end{aligned} \tag{42}$$

The (T.) regime:

$$x_{t+1} = x_d - (1+\lambda)\widehat{x}\left(\frac{x_d}{x_t} - 1\right) \equiv f_\lambda(x_t) \tag{43}$$

The stationary states of this dynamics are x_d and $(1+\lambda)\widehat{x}$.

The liquidity constraint is given by:

$$x_t < \overline{x}(\lambda) \tag{44}$$

Finally, the liquidity constraint for the first old agents writes:

$$x_0 \leq \widetilde{x}(\lambda) \tag{45}$$

4.4.3 The counter-bubbles monetary policy

We assume that the economy without monetary creation ($\lambda = 0$) is such that:

$$\widetilde{x}(0) < \widehat{x}$$

We know that there exists in this case an equilibrium at all date in the (H.S.) regime, and such that $\forall t, x_t = \widetilde{x}(0)$. But, we assume that others equilibria with temporary bubble may exist, what is verified under the condition:

$$f_0(\widetilde{x}(0)) > 0$$

The economy may also oscillate between the two regimes (H.S.) and (T.). We wonder if a policy of monetary creation can eliminate these equilibria with bubbles, in ensuring the uniqueness of the equilibrium in the (H.S.) regime such that $\forall t, x_t = \widetilde{x}(\lambda)$.

We show this result in establishing successively the three following points:

- If $\lambda > 0$, $\forall t, x_t = \widetilde{x}(\lambda)$ is a possible equilibrium.
- If $\lambda > 0$, an equilibrium in the (T.) regime cannot exist.
- In increasing λ, it is possible to have $f_\lambda(\widetilde{x}(\lambda)) < 0$. Then, the transition from the (H.S.) regime toward the (T.) regime becomes impossible.

The monetary policy does no more allow the existence of a temporary bubble, and ensures the uniqueness of the equilibrium. Let us show successively this three points.

1. By assumption, we know that $\widetilde{x}(0) < \overline{x}(0)$. As $\widetilde{x}(\lambda)$ is a decreasing function of λ and $\overline{x}(\lambda)$ an increasing function, we deduce that for $\lambda > 0$, $\widetilde{x}(\lambda) < \overline{x}(\lambda)$. So $x_t = \widetilde{x}(\lambda)$ $\forall t$ is a possible equilibrium.

2. By assumption, we know that:

$$\begin{aligned} \widehat{x} &> \overline{x}(0) \\ x_d &> \overline{x}(0) \end{aligned}$$

We need to verify:

$$\begin{aligned} \widehat{x}(1+\lambda) &> \overline{x}(\lambda) \\ x_d &> \overline{x}(\lambda) \end{aligned}$$

Consequently, the (T.) regime will be impossible in the long run. The first inequality is evident, the second is obtain in calculating:

$$x_d - \overline{x}(\lambda) = \frac{x_d \mu \left[1 + \lambda(1-a)\right]}{1 + \lambda(1 - a\mu)}$$

3. Let us show that the monetary creation do no more allow the transition from the (H.S.) regime to the (T.) regime. We define $x_l(\lambda)$ as the solution of the equation: $f_\lambda(x_l(\lambda)) = 0$. One has :

$$x_l(\lambda) = \frac{(1+\lambda)\widehat{x}x_d}{x_d + (1+\lambda)\widehat{x}}$$

It is sufficient to prove that, for λ sufficiently high, one has:

$$\widetilde{x}(\lambda) < x_l(\lambda)$$

Taking into account expressions of $\widetilde{x}(\lambda)$ and of $x_l(\lambda)$, this property is verified.

Finally, we have shown that monetary policy allows to obtain a unique equilibrium in the (H.S.) regime. Temporary bubbles can no more occur. The monetary policy diminishes the return of money, and then suppresses the possibility for money and capital to have the same return. This result has been obtained in the most unfavorable case: the case where the created money is distributed to young agents. This way of distribution plays in favor of savings, and tends to diminish the return of capital. Our study shows that this indirect effect is dominated by the direct effect of the monetary policy.

We have considered in this section that the objective of the government was to eliminate bubbles. Such a government objective is partly *ad-hoc*, because it is not possible to prove that a counter-bubbles policy is Pareto improving. We know that the existence of a bubble at some period is beneficial for the generation living during this period, because it increases savings return. But it is detrimental for the following generations, because it reduces capital accumulation in an economy with under-accumulation. Then, suppressing bubbles cannot be Pareto improving. The counter-bubbles monetary policy also has a negative impact on agents welfare, because it increases the difference between returns on the two assets, money and capital, and then it increases the distortion related to money holding.

5 Conclusion

In this paper, we have studied the dynamical properties of an overlapping generations model with capital and money. The medium-of-exchange role of money is taken into account in assuming that agents are subject to a cash-in-advance constraint. We study the intertemporal equilibrium of this economy in the general case, without excluding a priori the existence of bubbles (temporary or permanent).

We first have made a dynamical study in the general case. In assuming that a stationary equilibrium with under-accumulation exists in the economy without money, we have shown that there exists in the neighborhood of this equilibrium a monetary equilibrium with a binding liquidity constraint for a sufficiently low liquidity constraint. But, this monetary equilibrium is not unique: we have shown that temporary bubbles on money may appear at some periods. So there is a multiplicity of equilibria, with the existence of temporary bubbles.

In the case where the utility and production functions are Cobb-Douglas, the global dynamics of the economy has been analysed. A unique stationary equilibrium exists in the Hahn and Solow's regime. Two cases may occur, as this stationary state is in over or under accumulation. In the first case, we have found that the Tirole's regime with a permanent bubble is the only possible equilibrium and that the economy converges toward the golden rule[5]. But, in the second case (if the equilibrium with money in the Hahn and Solow's regime is in under-accumulation), different situations may appear. If the liquidity constraint is sufficiently hard, the Hahn and Solow's regime is the only possible one, and the dynamical equilibrium is unique. But, if the liquidity constraint is relatively low, it is possible that the economy oscillates between the two regimes. Money can periodically support bubbles, but these bubbles are temporary. There is multiplicity of dynamical equilibria, the degree of indetermination being inversely related to the weight of the liquidity constraint.

[5]This result was obtained by Tirole [1985] for another form of liquidity constraint.

In this last case, it is possible to assume that agents expectations are based on the observation of an exogenous phenomenon called sunspot. This phenomenon is supposed to govern the appearance of bubbles. We have shown that such a theory can be self-fulfilling, and that sunspot equilibria may exist.

Finally, we have assumed that the government objective was to fight bubbles. Its weapon is the monetary policy. We have proved that a sufficiently high rate of monetary creation could eliminate temporary bubbles.

Appendix 1:

Proof of proposition 1:

A trajectory of the (H.S.) regime is characterized by the dynamical equation:

$$g(k_{t+1}, k_t, \mu) = 0, \ t \geq 0$$

with

$$g(k_{t+1}, k_t, \mu) = (1+n)k_{t+1} - \sigma\left(w(k_t), \rho(k_{t+1}, k_t, \mu)\right) + \frac{\mu}{1-\mu} R_t k_t$$

and

$$\rho(k_{t+1}, k_t, \mu) = \frac{(1+n)R(k_{t+1})k_{t+1}}{\mu R(k_t)k_t + (1-\mu)(1+n)k_{t+1}}$$

The condition which assures that money is a dominated asset (26), with (24), is equivalent to:

$$(1+n)k_{t+1} < R(k_t)k_t$$

By assumption, one has $g(k^D, k^D, 0) = 0$. g being continuously differentiable with the first partial derivative $g'_1(k^D, k^D, 0) \neq 0$, one can apply the implicit functions theorem. There exists $\varepsilon > 0$ and I neighborhood of k^D such that, for all $k \in I$ and all μ, $|\mu| < \varepsilon$, the equation $g(x, k, \mu) = 0$ admits a unique solution $x = h(k, \mu)$ in a neighborhood of k^D. One has : $k^D = h(k^D, 0)$ and $(1+n)k^D < R(k^D)k^D$. It is possible to restrict the neighborhoods in order that the solution verifies : $(1+n)x < R(k)k$, for all $k \in I$ and all μ, $|\mu| < \varepsilon$.

The function h is differentiable and verifies :

$$h'_1(k^D, 0) = -g'_2/g'_1 = \frac{\sigma'_w w'(k^D)}{1 + n - \sigma'_R R'(k^D)}$$

This derivative is strictly between 0 and 1. One can again restrict the neighborhoods in order that the derivative $h'_1(k, \mu)$ also is between 0 et 1. Applying the implicit function theorem to the two variables function $g(k, k, \mu)$, we obtain a stationary equilibrium $k^*(\mu) \in I$ for all $\mu < \varepsilon$.

Let us show that for all $k_0 \in I$, the trajectory $k_t(\mu)$ such that:

$$k_{t+1}(\mu) = h\left(k_t(\mu), \mu\right), \ k_0(\mu) = k_0$$

is defined, belongs to I and converges toward $k^*(\mu)$.

By recurrence, if $k_t(\mu) \in I$,

$$\begin{aligned} k_{t+1}(\mu) - k^*(\mu) &= h\left(k_t(\mu), \mu\right) - h\left(k^*(\mu), \mu\right) \\ &= h'_1\left(x_t(\mu), \mu\right)\left(k_t(\mu) - k^*(\mu)\right) \end{aligned}$$

with $x_t(\mu)$ between $k^*(\mu)$ and $k_t(\mu)$ and belonging to I. As $h_1' \in (0,1)$, the distance between $k_{t+1}(\mu)$ and $k^*(\mu)$ is strictly smaller than the one between $k_t(\mu)$ and $k^*(\mu)$. So, $k_{t+1}(\mu)$ is defined and belongs to I, and the sequence $k_t(\mu) - k^*(\mu)$ tends toward 0.

The sequence $k_t(\mu)$ belongs to the (H.S.) regime as it verifies:

$$(1+n)k_{t+1}(\mu) < R(k_t(\mu))k_t(\mu)$$

from the precedent restrictions taken on I.

The uniform convergence of this sequence when $\mu \to 0$ results from its convergence and from the convergence of $k^*(\mu)$ toward $k^*(0) = k^D$. Indeed, one can write:

$$|k_t(\mu) - k_t(0)| \leq |k_t(\mu) - k^*(\mu)| + \left|k^*(\mu) - k^D\right| + \left|k_t(0) - k^D\right|$$

From the precedent recurrence, the first and the third terms can be bounded above by a sequence $z_t \geq 0$, which has a null limit and does not depend on μ. So, $\forall \varepsilon > 0$, for t high enough ($t \geq T$), these two terms can be made smaller than ε. One also has: $\lim_{\mu \to 0} \left|k^*(\mu) - k^D\right| = 0$. Finally, for $t < T$, the continuity in relation with the variable μ allows to obtain the convergence: $\sup_{t<T} |k_t(\mu) - k_t(0)| \to 0$ when $\mu \to 0$.

Appendix 2:

First we remark that equations (25) and (28) have the same form, as σ does no more depend on R or ρ :

$$(1+n)k_{t+1} = a(1-\alpha)Ak_t^\alpha - m_t \tag{46}$$

The (H.S.) regime: the liquidity constraint is binding between t and $t+1$. Then, (24) can be written:

$$(1-\mu)m_{t+1} = \mu\alpha Ak_{t+1}^\alpha \tag{47}$$

Condition (26) which assures that the return on money is smaller than the return on capital can be written:

$$\mu(1+n)k_{t+1} \leq (1-\mu)m_t \tag{48}$$

In using (46) written in $t+1$, (47) becomes a dynamical equation of order one in the variable k_t :

$$\frac{\mu}{1-\mu}A\alpha k_{t+1}^\alpha = aA(1-\alpha)k_{t+1}^\alpha - (1+n)k_{t+2}$$

We introduce the variable $x_t = k_{t+1}k_t^{-\alpha}$. The last equation can be written:

$$x_{t+1} = \widetilde{x} \text{ with } \widetilde{x} = \frac{A}{1+n}\left(a(1-\alpha) - \frac{\mu\alpha}{1-\mu}\right)$$

To be in the regime with a binding liquidity constraint implies that the value of x_{t+1} is determined.

In the same way, in using (46), (48) becomes:

$$\frac{1-\mu}{\mu}\left[aA(1-\alpha)k_t^{\alpha} - (1+n)k_{t+1}\right] \geq (1+n)k_{t+1}$$

This last inequality can be expressed with the variable x_t:

$$x_t \leq \overline{x} \text{ with } \overline{x} = \frac{A}{1+n}a(1-\alpha)(1-\mu)$$

The (T.) regime : The liquidity constraint is not binding between t and $t+1$.

In using (46) written in t and $t+1$, (27) becomes a dynamical equation of the second order in k_t :

$$\left[aA(1-\alpha)k_{t+1}^{\alpha} - (1+n)k_{t+2}\right](1+n) = \left[aA(1-\alpha)k_t^{\alpha} - (1+n)k_{t+1}\right]A\alpha k_{t+1}^{\alpha-1}$$

We introduce the following notations:

$$x_d = \frac{aA(1-\alpha)}{1+n} \text{ and } \widehat{x} = \frac{A\alpha}{1+n}$$

x_d corresponds to the value of the variable x at the Diamond's stationary state , when $\widehat{x}$ corresponds to the golden rule. These two notations being introduced, it is possible to give from the precedent equation the following expression:

$$x_{t+1} = x_d - \widehat{x}\left(\frac{x_d}{x_t} - 1\right)$$

The liquidity constraint of generation t agents remains to be expressed, either (29) or (30). We find again:

$$x_t < \overline{x}$$

This condition is the same as (35), but with a strict inequality.

Finally, for the first old people in $t = 0$, their liquidity constraint is verified if (31) holds, or:

$$(1-\mu)m_0 \geq \mu R_0 k_0$$

In using (46) in $t = 0$, this last condition leads to:

$$x_0 \leq \frac{A}{1+n}\left(a(1-\alpha) - \frac{\mu\alpha}{1-\mu}\right) = \widetilde{x}$$

All dynamical equations in the two regimes have been expressed in function of the variable x_t.

Appendix 3:

- **Agents behavior**

Budgetary constraints of generation t agents now depend from monetary transfers:

$$\begin{aligned} c_t + s_t + m_t &= w_t + \theta_t & (49) \\ d_{t+1} &= R_{t+1}s_t + \frac{M_t}{P_{t+1}} & (50) \end{aligned}$$

with:

$$\theta_t = \frac{T_t}{P_t}$$

as the value of the real transfer.
The (H.S.) case: In this case, the utility maximization under the budgetary constraints gives consumptions and total savings:

$$\begin{aligned} c_t &= (1-a)(w_t + \theta_t) & (51) \\ d_{t+1} &= \rho_{t+1}a(w_t + \theta_t) \\ \sigma_t &= a(w_t + \theta_t) = s_t + m_t \end{aligned}$$

with:

$$\rho_{t+1} = \frac{(1+n)R_{t+1}s_t}{(1-\mu)(m_t + (1+n)s_t)}$$

Real monetary balances of agents are:

$$(1-\mu)m_t = \mu\frac{P_{t+1}}{P_t}R_{t+1}s_t \quad (52)$$

Finally, money has a smaller return than capital under the condition:

$$P_{t+1}/P_t \leq R_{t+1} \Leftrightarrow \rho_{t+1} \leq R_{t+1}$$

The (T.) case : In this case, consumptions and savings are:

$$\begin{aligned} c_t &= (1-a)(w_t+\theta_t) \\ d_{t+1} &= R_{t+1}a(w_t+\theta_t) \\ \sigma_t &= a(w_t+\theta_t) = s_t + m_t \end{aligned} \tag{53}$$

m_t is determined by the equilibrium, but an agent has to satisfy his liquidity constraint:

$$(1-\mu)m_t > \mu s_t \tag{54}$$

- **The intertemporal equilibrium**

The equilibrium on the money market gives:

$$\overline{M}_t = N_t P_t m_t \tag{55}$$

From that one deduces the real value of the lump sum monetary transfer:

$$\theta_t = \frac{\lambda_t \overline{M}_{t-1}}{N_t P_t} = \frac{\lambda_t}{1+\lambda_t} m_t$$

The dynamics of capital is obtained in expressing that capital results from past savings:

$$(1+n)k_{t+1} = s_t$$

As total savings σ_t is the same in the two regimes, it is possible to express this equation in using (51) and (53) by:

$$(1+n)k_{t+1} = aw_t - \left(1 - \frac{a\lambda_t}{1+\lambda_t}\right) m_t \tag{56}$$

Finally, one writes in each regime the specific conditions.

(H.S.) Regime :

With (39) and (55), the real return on money can be written:

$$\frac{P_{t+1}}{P_t} = \frac{m_{t+1}(1+n)}{m_t(1+\lambda_{t+1})}$$

Real balances of the agents , given by (52), become now:

$$(1-\mu)m_{t+1} = (1+\lambda_{t+1})\mu\alpha A k_{t+1}^{\alpha} \tag{57}$$

In using (56) written in $t+1$, (57) becomes a dynamical equation of the first order in the variable k_t:

$$(1+\lambda_{t+1}(1-a))\frac{\mu}{1-\mu} A\alpha k_{t+1}^{\alpha} = aA(1-\alpha)k_{t+1}^{\alpha} - (1+n)k_{t+2} \tag{58}$$

The condition ensuring that the return on money do not exceed the one on capital remains identical:

$$\mu(1+n)k_{t+1} \leq (1-\mu)m_t$$

In using (56), this condition becomes:

$$(1+n)k_{t+1}\left(\frac{1+\lambda_t(1-a\mu)}{1+\lambda_t}\right) \leq (1-\mu)aA(1-\alpha)k_t^{\alpha} \tag{59}$$

From (58), one obtains:

$$\begin{aligned} x_{t+1} &= \widetilde{x}(\lambda) \\ \text{with } \widetilde{x}(\lambda) &= \frac{A}{1+n}\left(a(1-\alpha) - \frac{\mu\alpha(1+\lambda(1-a))}{1-\mu}\right) \end{aligned}$$

From (59), money is a dominated asset if:

$$\begin{aligned} x_t &\leq \overline{x}(\lambda) \\ \text{with } \overline{x}(\lambda) &= \frac{Aa(1-\alpha)(1-\mu)}{1+n}\frac{1+\lambda}{1+\lambda(1-\mu a)} \end{aligned}$$

The (T.) regime: money is not a dominated asset between t and $t+1$:

$$P_t/P_{t+1} = \frac{m_{t+1}(1+n)}{m_t(1+\lambda_{t+1})} = R_{t+1}$$

In using (56) written between t and $t+1$, the variable m_t is eliminated in order to obtain a dynamical equation in k :

$$\frac{\left[aA(1-\alpha)k_{t+1}^{\alpha} - (1+n)k_{t+2}\right]\left(\frac{1+\lambda_t(1-a)}{1+\lambda_t}\right)}{\left[aA(1-\alpha)k_t^{\alpha} - (1+n)k_{t+1}\right]\left(\frac{1+\lambda_{t+1}(1-a)}{1+\lambda_{t+1}}\right)} = \frac{(1+\lambda_{t+1})(A\alpha k_{t+1}^{\alpha-1})}{1+n} \tag{60}$$

The condition ensuring that the liquidity constraint is satisfied can be written:

$$(1+n)k_{t+1}\left(\frac{1+\lambda_t(1-a\mu)}{1+\lambda_t}\right) < (1-\mu)aA(1-\alpha)k_t^{\alpha} \tag{61}$$

From (60), the dynamics of x_t can be written:

$$x_{t+1} = x_d - (1+\lambda)\widehat{x}\left(\frac{x_d}{x_t} - 1\right) \equiv f_\lambda(x_t)$$

Its stationary states are x_d and $(1+\lambda)\widehat{x}$.

From (61), the liquidity constraint is given by:

$$x_t < \overline{x}(\lambda)$$

Finally, it remains to express that the liquidity constraint for the first old agents is satisfied, or:

$$(1-\mu)M_{-1}/P_0 \geq \mu R_0 s_{-1}$$

This condition can be written:

$$(1-\mu)\frac{m_0}{1+\lambda_0} \geq \mu R_0 k_0$$

In suppressing m_0 by (56), one has:

$$(1+\lambda_0(1-a))\,\frac{\mu}{1-\mu}A\alpha k_0^{\alpha} = aA(1-\alpha)k_0^{\alpha} - (1+n)k_1 \tag{62}$$

From (62), one obtains:

$$x_0 \leq \widetilde{x}(\lambda)$$

References

Abel, A. B. [1987], "Optimal Money Growth", *Journal of Monetary Economics*, 19, 437-450.

Allais, M. [1947], *Economie et Intérêt*, Paris, Imprimerie Nationale.

Azariadis, C. [1993], *Intertemporal Macroeconomics*, Basil Blackwell.

Blanchard, O.J. et S. Fischer [1989], *Lectures on macroeconomics*, Cambridge, U.S.A. M.I.T. Press.

Champ, B. et S. Freeman, [1990], "Money, output, and the nominal national debt", *American Economic Review*, june, 390-397.

Champ, B. and S. Freeman, [1994], *Modeling monetary economies*, John Wiley&Sons.

Crettez, B., P. Michel et B. Wigniolle [1999], "Cash-in-advances constraints in the Diamond overlapping generations model : neutrality and optimality of monetary policy", *Oxford Economic Papers,* 51, 431-452.

Diamond, P [1965], "National debt in a neo-classical growth model", *American economic review*, 55, 1126-1150.

Gahvari, F. [1988], Lump-sum taxation and the superneutrality and optimum quantity of money in life-cycle growth models, *Journal of public economics*, 338-366.

Gale, D. [1983], *Money : in desequilibrium*, Cambridge, Cambridge University Press (The Cambridge Economic Handbooks).

Hahn, F. et R. Solow [1995], *A critical essay on modern macroeconomic theory*, Basil Backwell.

Sargent, T.J. [1987], *Dynamic Macroeconomic Theory*, Cambridge, U.S.A., Harvard University Press.

Tirole, J. [1985], "Asset bubbles and overlapping generations", *Econometrica*, 53, 1499-1528.

Villieu, P. [1993], "Les modèles à encaisses préalables : un renouveau des fondements microéconomiques de la macroéconomie monétaire ?", *Revue d'Economie Politique*, 5, 613-692.