Two examples of strategic equilibria in approval voting games

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Abstract

In this note we discuss two examples of approval voting games. The first one, with six voters and three candidates, has a unique stable set, where each voter approves only his most preferred candidate. This strategy coincides with the sophisticated one, while other strategy combinations, leading to different outcomes, are selected by the perfect equilibrium concept. Moreover, this example shows that sophisticated voting, as well as strategic stability, does not imply the election of a Condorcet winner, even if it exists. The second example, with four voters and four candidates, shows that strategic stability does not exclude non sincere strategies. Furthermore, the same results hold in complete neighborhoods of the games considered.

Keywords: Approval voting, sophisticated voting, perfect equilibrium, stable set.

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1 Introduction

Approval voting, introduced by Brams and Fishburn (1978), is a system in which a voter can vote for as many candidates as he wants. With approval voting, even if every voter has the same preference order on the various alternatives, voting uniquely for the least preferred candidate is a Nash equilibrium, if there are more than three voters. Given the irrationality of such behavior, it is necessary to use some refinement of the Nash concept that excludes this "implausible" outcome. In the following, we apply the general model of a one stage voting procedure defined by Myerson and Weber (1993) to the approval rule case. In this model, given the set of candidates $K = (1, ..., k)$, each voter submits a ballot, which is a vector of k components. An electoral system is then defined by the set of possible ballots that each voter can submit and by the election rule that, given the ballots cast, selects the winner from the set K. Hence, with approval voting, every voter has the same strategy space, and each pure strategy is a vector with zeros, for non approved candidates, and ones, for approved candidates, while abstention can be represented by the zero vector. With approval, the election rule selects the candidate that receives the largest total number of votes. In case of ties, to preserve the symmetry of the voters, we allow an equal probability lottery among the winners.

The set of candidates, the electoral system, the set of voters and the utility vectors with k components (representing for each voter his payoff for all the possible results of the election) define the associated normal form game. This resulting game is highly non generic, since the same outcome arises from many different pure strategy combinations.

In the next section we show that the solution concept of perfect equilibria is not restrictive enough, in the context of approval games, since there are examples where some outcomes induced by this concept are excluded by an iterative elimination of dominated strategies. Such procedure has a long-standing tradition in voting theory from the pioneering work of Farquharson (1969), who first defined it "sophisticated voting".

However, sophisticated voting, as well as strategic stability, does not imply the election of the Condorcet winner, if it exists.

The second example shows that strategic stability, as well as stronger equilibrium concepts, does not imply the use of sincere strategies[1], not even generically.

Before discussing the two proposed examples, let us introduce some basic notation. Given the set of candidates $K = (1, ..., k)$ and the set of voters $N = (1, ..., n)$, the approval rule determines the strategy space of each player. Since each voter can vote for as many candidates as he wants, everyone has 2^k pure strategies, corresponding to the set of vectors with k components where each entry is either zero or one. The strategy space of each player is

$$\Sigma = \Delta (V)$$

[1] A pure strategy is sincere if and only if whenever a candidate is approved, all the preferred candidates are too.

where V is the set of pure strategies.

In order to determine the winner, we do not need to know the ballots cast by all the voters, it is enough simply to know their sum. Given a pure strategy vector $v \in V^n$, let $\omega = \sum_{i=1}^{n} v^i$. Clearly ω is a k-dimensional vector, and each coordinate represents the total number of votes obtained by the corresponding candidate. Then, denoting by $p(c \mid v)$ the probability that candidate c is elected if v is played, we have:

$$p(c \mid v) = \begin{cases} 0 & if\ \exists m \in K\ s.t\ \omega_c < \omega_m \\ \frac{1}{q} & \begin{array}{l} if\ \omega_c \geq \omega_m\ \forall m \in K\ and \\ \#\ \{d \in K\ s.t.\ \omega_c = \omega_d\} = q. \end{array} \end{cases} \tag{1}$$

Hence, given the utility vectors $\{u^i\}_{i \in N}$, where $u^i = (u_1^i, ..., u_k^i)$ and each u_c^i represents the payoff that player i gets if candidate c is elected, we have a normal form game; for each pure strategy combination v, the payoff of player i is given by:

$$U^i(v) = \sum_{c \in K} p(c \mid v)\, u_c^i. \tag{2}$$

Clearly, we can extend (1) and (2) to mixed strategies. Under a mixed strategy σ we have:

$$p(c \mid \sigma) = \sum \sigma(v)\, p(c \mid v)$$

and

$$U^i(\sigma) = \sum_{c \in K} p(c \mid \sigma)\, u_c^i,$$

where, as usual, $\sigma(v)$ denotes the probability of the (pure) strategy combination v under σ.

Since the election rule depends only upon the sum of the votes cast, the payoff functions and the best reply correspondences also have this property.

Then, to analyze the games, we will often refer to the following set:

$\Omega^{-i} = \left\{ \omega_{-i} \mid \exists v \in V^n\ s.t.\ \sum_{j \neq i} v^j = \omega_{-i} \right\}.$

It is easy to see (cf. Brams and Fishburn (1978)) that an undominated strategy always approves the most preferred candidate(s) and does not approve the least preferred one(s).

With this background in mind, we discuss the two proposed examples.

2 Example I

Let us consider the following example, with six voters and three candidates:

$$
\begin{aligned}
u^1 &= (3,1,0) \\
u^2 &= (3,1,0) \\
u^3 &= (0,3,1) \\
u^4 &= (0,3,1) \\
u^5 &= (0,1,3) \\
u^6 &= (0,1,3)
\end{aligned}
$$

Each voter has only two undominated strategies, namely, approving uniquely his most preferred candidate or approving also the second one. It is easy to see that the strategy combination

$$c = ((1,0,0),(1,0,0),(0,1,1),(0,1,1),(0,0,1),(0,0,1))$$

is an undominated equilibrium, leading to the election of the third candidate. Moreover the equilibrium c is also perfect (Selten (1975)).

Definition 1 *A completely mixed strategy σ^ε is an $\varepsilon-$perfect equilibrium if*

$$
\begin{aligned}
\forall i &\in N,\ \forall v^i, v^{i'} \in V^i \\
if\ U^i\left(v^i,\sigma^\varepsilon\right) &> U^i\left(v^{i'},\sigma^\varepsilon\right)\ then \\
\sigma^\varepsilon\left(v^{i'}\right) &\leq \varepsilon.
\end{aligned}
$$

A strategy combination σ is a perfect equilibrium if there exists a sequence $\{\sigma^\varepsilon\}$ of $\varepsilon-$perfect equilibria converging (for $\varepsilon \to 0$) to σ.

Given the above definition, we can prove that c is a perfect equilibrium. In fact, let us consider the following completely mixed strategy combination σ^ε, where ξ_i denotes the center of the strategy space of player i:
$\sigma_i^\varepsilon = (1-8\varepsilon^2)(1,0,0) + 8\varepsilon^2(\xi_i) \quad i = 1,2$
$\sigma_i^\varepsilon = (1-8\varepsilon^2)(0,1,1) + 8\varepsilon^2(\xi_i) \quad i = 3,4$
$\sigma_i^\varepsilon = (1-\varepsilon-7\varepsilon^2)(0,0,1) + (\varepsilon-\varepsilon^2)(1,0,0) + 8\varepsilon^2(\xi_i) \quad i = 5,6$
It is easy to see that, for ε sufficiently close to zero, this is an $\varepsilon-$perfect equilibrium. In fact, in c, for each voter his two undominated strategies are equivalent. Since for ε going to zero the probability of players 5 and 6 to tremble towards $(1,0,0)$ is infinitely greater than the probability of any other "mistake", it is enough to check that in this event the limiting strategy is preferred to the other undominated strategy. Hence, for player 1, the relevant contingency which allows him to discriminate between his two undominated strategies is when the behavior of the others is summarized by the vector $\omega_{-1} = (2,2,3)$. Since

$$U^1((1,0,0) \mid (2,2,3)) = \frac{3}{2} > \frac{4}{3} = U^1((1,1,0) \mid (2,2,3))$$

we have that approving only the most preferred candidate is, for player 1, the best reply to σ^{ε}. Analogously for player 2.

For player 3, the relevant contingency in order to discriminate between his two undominated strategies is given by $\omega_{-3} = (3, 1, 2)$, with

$$U^3((0,1,1) \mid (3,1,2)) = \frac{1}{2} > 0 = U^3((0,1,0) \mid (3,1,2))$$

hence $(0, 1, 1)$ is the best reply to σ^{ε}, and the same holds for player 4.

For player 5, the relevant event is given by $\omega_{-5} = (3, 2, 2)$ with

$$U^5((0,0,1) \mid (3,2,2)) = \frac{3}{2} > \frac{4}{3} = U^5((0,1,1) \mid (3,2,2))$$

hence $(0, 0, 1)$ is the best reply to σ^{ε}, and the same holds for player 6.

Therefore, $\{\sigma^{\varepsilon}\}$ is a sequence of $\varepsilon-$*perfect* equilibria. Since c is the limit of σ^{ε}, it is perfect.

Nevertheless, we claim that the game has a unique Mertens' stable set[2] (call it e) where each player approves only his most preferred candidate. Then the only stable outcome is an equal probability lottery over the candidates. To prove that e is the only stable set of the game, it is enough to consider the following properties:

i)Stable sets always exist.

ii)Stable sets are connected sets of normal form perfect (hence undominated) equilibria.

iii)A stable set contains a stable set of every game obtained by iterated elimination of dominated strategies.

Once all the dominated strategies have been eliminated, we have a reduced game with the following pure strategy sets:

$$\begin{aligned} V'^i &= \{(1,0,0),(1,1,0)\} \quad i = 1,2 \\ V'^i &= \{(0,1,0),(0,1,1)\} \quad i = 3,4 \\ V'^i &= \{(0,0,1),(0,1,1)\} \quad i = 5,6 \end{aligned}$$

In this reduced game, the last four voters have "approving only the most preferred candidate" as dominant strategy. In fact, let us consider player 3. We do not need to write down all the possible ω_{-3}, to claim that $(0, 1, 0)$ is the dominant strategy for him. In each ω_{-3} the first candidate takes two votes while the second takes at least one and the third at least two. Hence, except for $\omega_{-3} = (2, 1, 2)$, the approval of only the second candidates is either equivalent to the other strategy, since both lead to the election of the same candidate, or it is preferred. Moreover, in $\omega_{-3} = (2, 1, 2)$ we have that $(0, 1, 0)$ is preferred to $(0, 1, 1)$, leading to an utility of $\frac{4}{3}$ instead of 1. The same argument applies to the last three voters. Hence, we can further reduce the game by eliminating the strategy $v^i = (0, 1, 1)$ for $i = 3, 4, 5, 6$. In this game we have that player 1 (resp.

[2]For the definition of this concept see Mertens (1989).

2) can face only two circumstances, namely $\omega_{-1} = (1, 2, 2)$ or $\omega'_{-1} = (1, 3, 2)$. In the latter case his two strategies are equivalent since both lead to the election of the second candidate, while in the former $(1, 0, 0)$ is preferred to $(1, 1, 0)$, giving an utility of $\frac{4}{3}$ instead of 1. Hence $(1, 0, 0)$ is dominant for player 1 (resp. 2). Thus, iterated elimination of dominated strategy isolates the equilibrium e where each voter approves only his most preferred candidate. By property (iii) of stable sets we can conclude that each stable set contains e. Moreover, it is not difficult to see that e is a strict equilibrium (hence isolated); property (ii), therefore, implies that e is the unique stable set of the game.

Furthermore, the second candidate is the Condorcet winner. In the unique stable set e, however, he is elected with the same probability as the other two candidates. In Fishburn and Brams (1981) it is proved that, if a candidate x is a Condorcet winner, then there is a sincere undominated strategy combination that elects x. Our example shows that this result cannot be extended to sophisticated strategies (i.e. strategies that survive to iterated elimination of dominated ones).

Moreover, the exclusion of the "Condorcet outcome" from the solution set does not depend on the definition of stability. In fact, not even a weaker requirement such as perfection guarantees that the set of solutions contains such an outcome. To show this, it is enough to consider the approval game played by 1, 3 and 5. This game has only one perfect equilibrium where each player approves only his most preferred candidates. The second candidate, however, is still the Condorcet winner.[3]

The above results, namely that c is a perfect equilibrium but only e is a stable set, hold for every game with the same preference order and such that, for every voter, the difference in utility between the most preferred candidate and the second preferred one is greater than the difference between the second and the least preferred one.

3 Example II

Let us consider the following example Γ with four voters and four candidates:

$u^1 = (10, \frac{27}{10}, \frac{13}{5}, 0)$
$u^2 = (0, 10, 2, 9)$
$u^3 = (\frac{191}{39}, 10, 1, 0)$
$u^4 = (4, 0, \frac{34}{7}, 10)$

This game has a stable set where player 1 approves the first and the third candidate, hence strategic stability does not imply sincerity. Moreover, this result still holds in a complete neighborhood of the game.

[3] This game has also another undominated equilibrium, the "Condorcet" one, namely $((1, 1, 0), (0, 1, 0), (0, 1, 1))$. However, this is not a perfect equilibrium. We omit a formal proof. Basically it can be shown that $(1, 1, 0)$ is a best reply only if the "Condorcet" player trembles toward approving the third candidate with a large enough probability with respect to other perturbations. A symmetrical argument applies to strategy $(0, 1, 1)$. From these facts it follows that both strategies cannot be simultaneously best replies.

We do not use the explicit definition of stable set, but for our purpose it is enough to use the fact that a strongly stable equilibrium[4] (Kojima *et. al.* (1985)) is a stable set as a singleton[5]. This follows from the fact that every perturbation of the strategies corresponds (continuously) to a perturbation of utilities, and the continuity of the equilibrium function e assures that the projection of the graph of e on the space of perturbed games is homologically non trivial, being a homeomorphism. Cf. also Mertens (1991) (pp.697-699) that shows how the continuity of the map from the space of perturbed games to subsets of equilibria is a stronger requirement than the one included in the definition of stability.

Proposition 2 *The strategy combination*

$$s = ((1,0,1,0),(0,1,0,1),\frac{1}{2}(0,1,0,0)+\frac{1}{2}(1,1,0,0),\frac{9}{10}(0,0,0,1)+\frac{1}{10}(0,0,1,1))$$

is a stable set of Γ. *Moreover, there exists a neighborhood* (Ψ_Γ) *of* Γ, *in the space of approval games with four voters and four candidates, such that every game in* Ψ_Γ *has a stable set with the same support of* s.

Proof. The first step of the proof consists in showing that the strategy combination s is a quasi-strict equilibrium. This is done in the appendix.

The second step requires to prove that the quasi- strict equilibrium s is isolated. To analyze the set of equilibria near s we can limit the analysis by taking the strategy of players 1 and 2 as being fixed, since they are using a strict best reply. Moreover, since s is quasi strict, also players 3 and 4 can use (sufficiently close to s) only the pure strategies in s. Hence, to show that s is isolated it is enough to study the equilibria of the following game between players 3 and 4,

	$(0,0,0,1)$	$(0,0,1,1)$
$(0,1,0,0)$	$5,5$	$\frac{11}{3},\frac{104}{21}$
$(1,1,0,0)$	$\frac{581}{117},\frac{14}{3}$	$\frac{155}{39},\frac{33}{7}$

which has two pure strategies equilibria, i.e. $((0,1,0,0),\ (0,0,0,1))$ and $((1,1,0,0),\ (0,0,1,1))$, and a completely mixed one corresponding to s. Hence, s is isolated.

[4] Remember that if we fix the pure strategy set of each player, we can identify a game with a point $r \in \Re^{nm}$, where m is the number of pure strategy combinations and n the number of players.

Given a game $\tilde{r}$, an equilibrium $\tilde{\sigma}$ is strongly stable if neighborhoods Q of $\tilde{r}$ and W of $\tilde{\sigma}$ exist such that (i) every game in Q has one and only one equilibrium in W and (ii) the mapping $e: Q \to W$ that associates with each game in Q its equilibrium in W is continuous.

[5] For the exact definition of stable set we refer to Mertens (1989), here we simply recall that a set is stable if it is the limit at the true game of a subset of the equilibrium correspondence for perturbed games such that the projection map to perturbed games is homologically non trivial and the subset is locally connected to the true game.

The third step consists in showing that s is a strongly stable equilibrium. Since s is quasi-strict and isolated we can conclude (cf. van Damme (1991) Th. 3.4.4 p.55) that (s_3, s_4) is a strongly stable equilibrium of the reduced game where we take (s_1, s_2) as being fixed. Since the first two players are using their strict best reply, s is a strongly stable equilibrium of the whole game. Hence s is a stable set of Γ.

The second part of the proposition directly follows from corollary 4.1 in Kojima *et. al.* (1985), that assures,given a game and a strongly stable equilibrium, the unique nearby equilibrium of a nearby game is strongly stable too. Since each "approval game" near Γ has a normal form close to that of Γ and since for sufficiently close games and sufficiently close strategies, no other strategy than the ones in s can be a best reply, the claim easily follows. ■

The above proof also shows that not even a more demanding criterion such as strong stability can exclude non sincere strategies. Basically this is due to the fact that a non sincere strategy can be the only best reply to mixed strategy combinations of the opponents and hence, as long as we allow for mixed strategies, there is no reason to exclude non sincere behavior.

Moreover, the second part of the proposition shows that this example is not pathological.

4 Conclusion

In this note we have proposed two examples of approval voting games. The first one allows us to conclude that in the class of approval games, the perfect equilibrium concept is not restrictive enough to capture sophisticated voting, since there are "perfect" outcomes that do not survive to iterated elimination of dominated strategies. Furthermore, some outcomes selected by this concept are not induced by any stable set. This immediately implies that also in this class of games, the stable set is the "right" concept in order to have sensible solutions. Furthermore, we have found that, even if there is a Condorcet winner, strategic stability, as well as sophisticated voting, does not imply his election. Moreover, not even a weaker requirement such as perfection guarantees the existence of a solution in which he is elected. This is not surprising. As a matter of fact, strategic stability concerns individual behavior, while the Condorcet criterion is referred to society as a whole.

The second example shows that strategic stability does not imply sincerity. It is not difficult to see that for every pure strategy of the other players, the set of best replies contains a sincere strategy. As soon as we allow for mixed strategies, not only this is not true, but not even such a strong requirement as strategic stability can justify statements such as "under approval voting, voters are never urged to vote insincerely" (Niemi (1984) p.954). Moreover, our result holds in a complete neighborhood of the game and also for a more demanding criterion such as strong stability. Hence, this phenomenon cannot be considered pathological.

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Appendix

In this appendix we show that

$$s = ((1,0,1,0),(0,1,0,1),\frac{1}{2}(0,1,0,0)+\frac{1}{2}(1,1,0,0),\frac{9}{10}(0,0,0,1)+\frac{1}{10}(0,0,1,1))$$

is a quasi-strict equilibrium of the second proposed example. To this end we calculate the probability, under s, of each contingency that a player can face and, from these probabilities, the expected utility of each undominated strategy. It is easy to see that no dominated strategy is a best reply to s.

Player 1

$\Pr(\omega_{-1} = (0,2,0,2) \mid s_{-1}) = \frac{9}{20}$
$\Pr(\omega_{-1} = (0,2,1,2) \mid s_{-1}) = \frac{1}{20}$
$\Pr(\omega_{-1} = (1,2,0,2) \mid s_{-1}) = \frac{9}{20}$
$\Pr(\omega_{-1} = (1,2,1,2) \mid s_{-1}) = \frac{1}{20}$

From these probabilities it follows that:

$U^1((1,0,1,0),s_{-1}) = \frac{9}{20}\cdot\frac{27}{20}+\frac{1}{20}\cdot\frac{\frac{27}{10}+\frac{13}{5}}{3}+\frac{9}{20}\cdot\frac{10+\frac{27}{10}}{3}+\frac{1}{20}\cdot\frac{10+\frac{27}{10}+\frac{13}{5}}{4} = \frac{6701}{2400}$
$U^1((1,0,0,0),s_{-1}) = \frac{9}{20}\cdot\frac{27}{20}+\frac{1}{20}\cdot\frac{27}{20}+\frac{9}{20}\cdot\frac{10+\frac{27}{10}}{3}+\frac{1}{20}\cdot\frac{10+\frac{27}{10}}{3} = \frac{335}{120}$
$U^1((1,1,0,0),s_{-1}) = U^1((1,1,1,0),s_{-1}) = \frac{27}{10}$

Since no dominated strategy is a best reply to s_{-1} we have that $(1,0,1,0)$ is the only best reply to s_{-1}.

Player 2

$\Pr(\omega_{-2} = (1,1,1,1) \mid s_{-2}) = \frac{9}{20}$
$\Pr(\omega_{-2} = (1,1,2,1) \mid s_{-2}) = \frac{1}{20}$
$\Pr(\omega_{-2} = (2,1,1,1) \mid s_{-2}) = \frac{9}{20}$
$\Pr(\omega_{-2} = (2,1,2,1) \mid s_{-2}) = \frac{1}{20}$

From these probabilities it follows that:

$U^2((0,1,0,1),s_{-2}) = \frac{9}{20}\cdot\frac{10+9}{2}+\frac{1}{20}\cdot\frac{10+9+2}{3}+\frac{9}{20}\cdot\frac{10+9}{3}+\frac{1}{20}\cdot\frac{10+9+2}{4} = \frac{619}{80}$
$U^2((0,1,0,0),s_{-2}) = \frac{9}{20}\cdot 10+\frac{1}{20}\cdot\frac{10+2}{2}+\frac{9}{20}\cdot\frac{10}{2}+\frac{1}{20}\cdot\frac{10+2}{3} = \frac{29}{4}$
$U^2((0,1,1,0),s_{-2}) = \frac{9}{20}\cdot\frac{10+2}{2}+\frac{1}{20}\cdot 2+\frac{9}{20}\cdot\frac{10+2}{3}+\frac{1}{20}\cdot 2 = \frac{47}{10}$
$U^2((0,1,1,1),s_{-2}) = \frac{9}{20}\cdot\frac{10+9+2}{3}+\frac{1}{20}\cdot 2+\frac{9}{20}\cdot\frac{10+9+2}{4}+\frac{1}{20}\cdot 2 = \frac{457}{80}$

It is not difficult to see that any dominated strategy is not a best reply to s_{-2}. Hence $(0,1,0,1)$ it is the only best reply to s_{-2}.

Player 3

$\Pr(\omega_{-3} = (1,1,1,2) \mid s_{-3}) = \frac{9}{10}$
$\Pr(\omega_{-3} = (1,1,2,2) \mid s_{-3}) = \frac{1}{10}$

From these probabilities it follows that:

$U^3((0,1,0,0),s_{-3}) = \frac{9}{10}\cdot\frac{10}{2}+\frac{1}{10}\cdot\frac{10+1}{3} = \frac{73}{15}$
$U^3((1,1,0,0),s_{-3}) = \frac{9}{10}\cdot\frac{10+\frac{191}{39}}{3}+\frac{1}{10}\cdot\frac{10+1+\frac{191}{39}}{4} = \frac{73}{15}$
$U^3((0,1,1,0),s_{-3}) = \frac{9}{10}\cdot\frac{10+1}{3}+\frac{1}{10} = \frac{17}{5}$
$U^3((1,1,1,0),s_{-3}) = \frac{9}{10}\cdot\frac{10+1+\frac{191}{39}}{4}+\frac{1}{10} = \frac{239}{65}$

Furthermore, it is not difficult to see that no dominated strategy is a best reply to s_{-3}. Hence the only two (pure) best replies of player 3 are $(0,1,0,0)$ and $(1,1,0,0)$.

Player 4

$\Pr(\omega_{-4} = (1,2,1,1) \mid s_{-4}) = \frac{1}{2}$

$\Pr(\omega_{-4} = (2,2,1,1) \mid s_{-4}) = \frac{1}{2}$

From these probabilities it follows that:

$U^4((0,0,0,1), s_{-4}) = \frac{1}{2} \cdot \frac{10}{2} + \frac{1}{2} \cdot \frac{10+4}{3} = \frac{29}{6}$

$U^4((0,0,1,1), s_{-4}) = \frac{1}{2} \cdot \frac{10+\frac{34}{7}}{3} + \frac{1}{2} \cdot \frac{10+4+\frac{34}{7}}{4} = \frac{29}{6}$

$U^4((1,0,1,1), s_{-4}) = \frac{1}{2} \cdot \frac{10+4+\frac{34}{7}}{4} + \frac{1}{2} \cdot 4 = \frac{61}{14}$

$U^4((1,0,0,1), s_{-4}) = \frac{1}{2} \cdot \frac{10+4}{3} + \frac{1}{2} \cdot 4 = \frac{13}{3}$

Moreover it is easy to see that no dominated strategy is a best reply to s_{-4}. Hence the only two (pure) best replies of player 4 are $(0,0,0,1)$ and $(0,0,1,1)$.

Hence s is a quasi-strict equilibrium of Γ.