On integrable Hamiltonians for higher spin XXZ chain

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Abstract

Integrable Hamiltonians for higher spin periodic XXZ chains are constructed in terms of the spin generators; explicit examples for spins up to $\frac{3}{2}$ are given. Relations between Hamiltonians for some $U_q(sl_2)$-symmetric and $U(1)$-symmetric universal r-matrices are studied; their properties are investigated. A certain modification of the higher spin periodic chain Hamiltonian is shown to be an integrable $U_q(sl_2)$-symmetric Hamiltonian for an open chain.
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1 Introduction

XXZ spin chains have numerous connections with two-dimensional statistical physics and (1+1)-dimensional quantum field theory. They describe interaction of q-deformed spins sitting at the nodes of a one-dimensional lattice. The spin generators S^+, S^- and S^3 obey the commutation relations of the quantum Lie algebra $U_q(sl_2)$ [1]

$$[S^+, S^-] = \frac{\sin(2\gamma S^3)}{\sin\gamma}, \qquad [S^3, S^\pm] = \pm S^\pm . \tag{1}$$

We will consider only integrable XXZ spin models. The simplest example in this class is a spin-$\frac{1}{2}$ chain with the Hamiltonian given by

$$H = \sum_n \Big(\frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + (\cos\gamma)\, S_n^3 S_{n+1}^3 \Big) .$$

Higher spin XXZ chains have also been studied [2, 3, 4] but only in the spin-1 case an explicit expression for the corresponding Hamiltonian has been found [2].

In the present paper we will construct local integrable Hamiltonians (in terms of the spin generators) for higher spin XXZ chains. In Section 2 we recall some facts about $U_q(sl_2)$-symmetric universal r-matrix $r(\lambda)$. In Section 3 we construct a $U_q(sl_2)$-symmetric local Hamiltonian $H_{n,n+1}$. Its properties and properties of the corresponding closed chain

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Hamiltonian $\mathcal{H}$ are discussed in Section 4. In particular, we observe that $H_{n,n+1}$ decomposes into a $U(1)$-symmetric bulk Hamiltonian $\widehat{H}_{n,n+1}$ plus a universal local boundary term. Section 5 contains explicit expressions for $H_{n,n+1}$ for spins $\frac{1}{2}$, 1 and $\frac{3}{2}$. In Section 6 we find a family of universal r-matrices and Hamiltonians corresponding to an alternative choice of the co-multiplication. In Section 7 we construct another family of ($U(1)$-symmetric) universal r-matrices and local Hamiltonians which contains the Hamiltonian $\widehat{H}_{n,n+1}$ and the corresponding reflection-symmetric universal r-matrix $r_0(\lambda)$. In Section 8 we employ our construction to establish the integrability of certain ($U_q(sl_2)$-symmetric) Hamiltonians for an open chain. Appendix A contains technical details related to $r(\lambda)$ and $r_0(\lambda)$. Appendix B provides some details on computation of the spin-1 and spin-$\frac{3}{2}$ Hamiltonians presented in Section 5. Appendix C explains a q-trace formula for $H_{n,n+1}$ used in Section 8.

For compactness of notations, we will use in the text both the deformation parameter γ introduced in (1) and $q \equiv e^{\mathrm{i}\gamma}$. We assume that q is either real or takes values on the unit circle. In the latter case q is assumed to be generic, i.e., it is not a root of unity.

2 Universal r-matrix

The starting point of the quantum inverse scattering method is the exchange relation

$$R(\lambda)\, L(\lambda+\mu) \otimes L(\mu) = L(\mu) \otimes L(\lambda+\mu)\, R(\lambda)\,, \tag{2}$$

with $\otimes$ understood as the tensor product with respect to an auxiliary space V (below it is the space of $2{\times}2$ matrices) and the usual product of operators in the quantum space $\mathfrak{H}$. The R-matrix belongs to $V \otimes V$. Thus (2) is an equation in $V \otimes V \otimes \mathfrak{H}$.

The following L-operator, i.e., an element of $V \otimes \mathfrak{H}$, is consistent with the algebra (1)

$$L(\lambda) = \frac{1}{\sin\gamma} \begin{pmatrix} \sinh[\gamma(\lambda+\mathrm{i}S^3)] & \mathrm{i}\,\sin\gamma\, e^{\gamma\lambda}\, S^- \\ \mathrm{i}\,\sin\gamma\, e^{-\gamma\lambda}\, S^+ & \sinh[\gamma(\lambda-\mathrm{i}S^3)] \end{pmatrix} \tag{3}$$

in the sense that it satisfies eq. (2) provided that $R(\lambda) = P_V \check{R}(\lambda)$, where

$$\check{R}(\lambda) = \mathrm{i}\, e^{\gamma\lambda}\, \sigma^+ \otimes \sigma^- + \mathrm{i}\, e^{-\gamma\lambda}\, \sigma^- \otimes \sigma^+ + \frac{1}{\sin\gamma} \sinh\Big(\gamma\lambda + \frac{\mathrm{i}\,\gamma}{2}(1 \otimes 1 + \sigma^3 \otimes \sigma^3)\Big)\,. \tag{4}$$

Here σ^a denote the Pauli matrices, $\sigma^{\pm} = \sigma^1 \pm \mathrm{i}\sigma^2$, and $P_V = \frac{1}{2}(1 \otimes 1 + \sum_a \sigma^a \otimes \sigma^a)$ is the permutation matrix in $V \otimes V$.

The L-operator (3) decomposes into two λ-independent Borel components,

$$(2\sin\gamma)\, L(\lambda) = e^{\gamma\lambda} L_+ - e^{-\gamma\lambda} L_-\,, \tag{5}$$

which can be utilized to define [5] the co-multiplication (a linear homomorphism $\Delta : U_q(sl_2) \to [U_q(sl_2)]^{\otimes 2}$) in the matrix form,

$$\Delta(L_\pm) = L_\pm \dot{\otimes} L_\pm\,, \tag{6}$$

where Δ acts on the quantum space and $\dot{\otimes}$ denotes the tensor product with respect to $\mathfrak{H}$ and the usual matrix product in V. In explicit form (6) reads

$$\Delta(S^\pm) = S^\pm \otimes q^{-S^3} + q^{S^3} \otimes S^\pm\,, \qquad \Delta(S^3) = S^3 \otimes 1 + 1 \otimes S^3\,. \tag{7}$$

Observe that (4) can be obtained by evaluating the quantum space of $L(\lambda+\frac{\mathrm{i}}{2})$ in the fundamental representation, where $S^\pm = q^{\pm\frac{1}{2}}\sigma^\pm$ and $S^3 = \frac{1}{2}\sigma^3$. In fact, $\check{R}(\lambda)$ and $L(\lambda)$ are, respectively, $V\otimes V$ and $V\otimes\mathfrak{H}$ representations of a more general object, $\check{\mathbf{R}}(\lambda)\in[U_q(sl_2)]^{\otimes 2}$, that satisfies the Yang-Baxter equation in $\mathfrak{H}\otimes\mathfrak{H}\otimes\mathfrak{H}$,

$$\check{\mathbf{R}}_{12}(\lambda)\,\check{\mathbf{R}}_{13}(\lambda+\mu)\,\check{\mathbf{R}}_{23}(\mu)=\check{\mathbf{R}}_{23}(\mu)\,\check{\mathbf{R}}_{13}(\lambda+\mu)\,\check{\mathbf{R}}_{12}(\lambda)\,. \tag{8}$$

Relation (2) is then a particular case of (8) with the first and second spaces evaluated in the fundamental representation. Evaluating (8) in the $V\otimes\mathfrak{H}\otimes\mathfrak{H}$ representation (and interchanging λ with μ), we obtain

$$r(\lambda)\,L(\lambda+\mu)\dot{\otimes}L(\mu)=L(\mu)\dot{\otimes}L(\lambda+\mu)\,r(\lambda)\,, \tag{9}$$

with $r(\lambda)\equiv\check{\mathbf{R}}(\lambda)\mathbf{P}$, where $\mathbf{P}$ is the permutation in $\mathfrak{H}\otimes\mathfrak{H}$. For a given L-operator, (9) can be regarded as a defining equation on the element $r(\lambda)\in[U_q(sl_2)]^{\otimes 2}$ which we will call the universal r-matrix. Using (5), equation (9) can be equivalently rewritten as follows

$$r(\lambda)\;L_\pm\dot{\otimes}L_\pm=L_\pm\dot{\otimes}L_\pm\;r(\lambda)\,, \tag{10}$$

$$r(\lambda)\,(e^{\gamma\lambda}L_+\dot{\otimes}L_-+e^{-\gamma\lambda}L_-\dot{\otimes}L_+)=(e^{-\gamma\lambda}L_+\dot{\otimes}L_-+e^{\gamma\lambda}L_-\dot{\otimes}L_+)\,r(\lambda)\,. \tag{11}$$

In view of (6), the first line implies that for any element $\xi\in U_q(sl_2)$ we have

$$r(\lambda)\,\Delta(\xi)=\Delta(\xi)\,r(\lambda)\,. \tag{12}$$

Recall that, for a generic q, the tensor product of two irreducible highest weight $U_q(sl_2)$-representations of spin S is completely reducible and decomposes into the sum

$$D_S\otimes D_S=\sum_{j=0}^{2S}D_j\,, \tag{13}$$

where each subspace D_j is a highest weight $U_q(sl_2)$-module with respect to the action of the operators $\Delta(S^\pm)$ and $\Delta(S^3)$.

Eq. (12) implies that $r(\lambda)$ is a function of an operator J such that

$$J\,|j,m\rangle=j\,|j,m\rangle \tag{14}$$

for any vector $|j,m\rangle$ from D_j. In other words,

$$r(\lambda)=\sum_{j=0}^{2S}r_j(\lambda)\,\mathcal{P}_j\,, \tag{15}$$

where $\mathcal{P}_j$ is the projector onto D_j, i.e., $\mathcal{P}_k|j,m\rangle=\delta_{jk}\,|j,m\rangle$. Taking (15) into account, one can solve equation (11) explicitly ([6, 7] and see also Appendix A),

$$r(\lambda)=\mathcal{P}_0+\sum_{j=1}^{2S}\Big(\prod_{k=1}^{j}\frac{\sin[\gamma(k-i\lambda)]}{\sin[\gamma(k+i\lambda)]}\Big)\,\mathcal{P}_j\,. \tag{16}$$

Two obvious consequences of this formula are

$$r(\lambda)\,r(-\lambda)=1\otimes 1\,,\qquad r(\lambda)\,r(\mu)=r(\mu)\,r(\lambda)\,. \tag{17}$$

Introduce a q-analogue of the gamma function satisfying the relation

$$(q^x-q^{-x})\,\Gamma_q(x)=(q-q^{-1})\,\Gamma_q(x+1)$$

and normalized such that $\Gamma_q(1) = 1$. If $|q| \neq 1$, this equation can be solved in terms of a convergent infinite product. For instance, for $|q| < 1$, the solution is given by

$$\Gamma_q(x) = q^{\frac{1}{2}x(1-x)} (q^{-1} - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{2n+2}}{1 - q^{2n+2x}} \,. \tag{18}$$

In terms of the q-gamma function eq. (16) can be rewritten as follows

$$r(\lambda) = \frac{\Gamma_q(J + 1 - i\lambda)}{\Gamma_q(J + 1 + i\lambda)} \frac{\Gamma_q(1 + i\lambda)}{\Gamma_q(1 - i\lambda)} \tag{19}$$

with J defined by (14). Eqs. (16) and (19) are q-deformations of their XXX counterparts [1] found in [8, 4] (see also [9]).

3 Hamiltonian

Having a solution $\check{\mathbf{R}}(\lambda)$ to equation (8) such that $\check{\mathbf{R}}(0) = \mathbf{P}$ (equivalently, $r(0) = 1 \otimes 1$), one can construct an integrable Hamiltonian for a closed chain in the following way [10, 4, 7, 9] (the normalization is chosen for later convenience)

$$\begin{aligned} \mathcal{H} \;\; &= \;\; \mathrm{i}\frac{\sin\gamma}{2\gamma} \sum_{n=1}^{N} \mathbf{P}_{n,n+1} \frac{d}{d\lambda} \check{\mathbf{R}}_{n,n+1}(\lambda)\Big|_{\lambda=0} = \sum_{n=1}^{N} \mathbf{P}_{n,n+1} H_{n,n+1} \mathbf{P}_{n,n+1} = \sum_{n=1}^{N} H_{n+1,n} \,, \\ H_{n,n+1} &= \mathrm{i}\frac{\sin\gamma}{2\gamma} \frac{d}{d\lambda} \ln r_{n,n+1}(\lambda)\Big|_{\lambda=0} \end{aligned} \tag{20}$$

with $H_{N,N+1} \equiv H_{N,1}$ (the periodic boundary conditions). Although $H_{n,n+1} \neq H_{n+1,n}$, we will show below that for the total Hamiltonian we have

$$\mathcal{H} = \sum_{n=1}^{N} H_{n+1,n} = \sum_{n=1}^{N} H_{n,n+1} \,. \tag{21}$$

The Hamiltonian $\mathcal{H}$ commutes with the higher quantum integrals of motion which are constructed as higher derivatives of $\ln r(\lambda)$. Moreover, if the reference state ω for an L-operator in question is such that $\omega \otimes \omega$ is an eigenvector of $r(\lambda)$, then the corresponding Bethe vectors are eigenvectors of the Hamiltonian [11, 10, 4, 9]. In particular, this is the case for the L-operator (3), for which the reference state ω is just the highest weight vector.

Combining (19) with (20), we obtain a compact formula for the Hamiltonian,

$$H_{n,n+1} = \frac{\sin\gamma}{\gamma} \left(\Psi_q(J_{n,n+1} + 1) - \Psi_q(1) \right) , \tag{22}$$

where $\Psi_q(x)$ stands for the logarithmic derivative of $\Gamma_q(x)$.

In order to find the Hamiltonian explicitly in terms of the spin generators we first substitute eq. (16) into eq. (20) and derive

$$H_{n,n+1} = (\sin\gamma) \sum_{j=1}^{2S} \Big(\sum_{k=1}^{j} \frac{\cos\gamma k}{\sin\gamma k} \Big) \mathcal{P}_j \,. \tag{23}$$

[1]Actually, [8] and [4] dealt with the XXX analogue of $\check{\mathbf{R}}(\lambda)$. In the XXZ case, $\mathbf{P}$ and $\check{\mathbf{R}}(\lambda)$ do not commute with Δ and therefore cannot be represented as combinations of the projectors $\mathcal{P}_j$.

Next, we have to construct the projectors $\mathcal{P}_j$ explicitly. Recall that the Casimir operator of the algebra (1) is given by

$$C = S^- S^+ + \frac{\sin\gamma S^3\ \sin\gamma(S^3+1)}{\sin^2\gamma}\,. \tag{24}$$

Its value in the highest weight representation of spin S is

$$C_S = \frac{\sin\gamma S\ \sin\gamma(S+1)}{\sin^2\gamma} = \frac{\cos\gamma - \cos\gamma(2S+1)}{2\sin^2\gamma}\,.$$

Applying the co-multiplication (7) to the Casimir operator (24), we obtain an operator that acts in the tensor product $D_S \otimes D_S$,

$$\begin{aligned} \mathbf{X}_S = \tfrac{1}{2}\Delta C = \tfrac{1}{2}(q^{S^3}S^+)\otimes(S^-q^{-S^3}) + \tfrac{1}{2}(q^{S^3}S^-)\otimes(S^+q^{-S^3}) \\ +\tfrac{1}{4\sin^2\gamma}\Big((1\otimes 1 + q^{2S^3}\otimes q^{-2S^3})\ \cos\gamma - (1\otimes q^{-2S^3} + q^{2S^3}\otimes 1)\ \cos(\gamma(2S+1))\Big)\,. \end{aligned} \tag{25}$$

Here we employed the commutation relations (1) and used (as reflected in the subscript of $\mathbf{X}_S$) that on $D_S \otimes D_S$ we have $C\otimes 1 = 1\otimes C = (1\otimes 1)C_S$. It is now easy to see that

$$\mathbf{X}_S = \frac{\sin\gamma J\ \ \sin\gamma(J+1)}{2\sin^2\gamma} \tag{26}$$

with J defined by (14). Indeed, by construction, $\mathbf{X}_S$ commutes with $\Delta(S^\pm)$. Bearing this in mind, (26) follows from computing the action of $\mathbf{X}_S$ on highest weight vectors.

For a generic q, the eigenvalues of (26) on different subspaces D_j in (13) do not coincide. Therefore, we can utilize (26) to construct the projectors $\mathcal{P}_j$ by means of the Lagrange interpolation,

$$\mathcal{P}_j = \prod_{\substack{l=0\\ l\neq j}}^{2S} \frac{2\mathbf{X}_S - [l]_q[l+1]_q}{[j-l]_q[j+l+1]_q}\,. \tag{27}$$

Here we used the q-numbers defined as $[k]_q \equiv (q^k - q^{-k})/(q-q^{-1}) = \frac{\sin(\gamma k)}{\sin\gamma}$. Note that the denominator in (27) can be written differently with the help of the identity $[j-l]_q[j+l+1]_q = [j]_q[j+1]_q - [l]_q[l+1]_q$. We have chosen the first expression to indicate singularities that can occur if q is a root of unity.

Combining (27) with (23), we obtain a local [2] integrable Hamiltonian for the XXZ spin chain:

$$H_{n,n+1} = (\sin\gamma)\sum_{j=1}^{2S}\Bigg[\Big(\sum_{k=1}^{j}\frac{\cos\gamma k}{\sin\gamma k}\Big)\prod_{\substack{l=0\\ l\neq j}}^{2S}\frac{2(\sin\gamma)^2\,\mathbf{X}_S - \sin\gamma l\ \sin\gamma(l+1)}{\sin\gamma(j-l)\ \sin\gamma(j+l+1)}\Bigg] \tag{28}$$

with $\mathbf{X}_S \in \mathfrak{H}_n \otimes \mathfrak{H}_{n+1}$ given by (25) which can also be rewritten as follows

$$\begin{aligned} \mathbf{X}_S = e^{\mathrm{i}\gamma S_n^3}\Big(\tfrac{1}{2}S_n^+S_{n+1}^- + \tfrac{1}{2}S_n^-S_{n+1}^+ + \sin\gamma S_n^3 \sin\gamma S_{n+1}^3 \tfrac{\cos\gamma S\cos\gamma(S+1)}{\sin^2\gamma} \\ + \cos\gamma S_n^3\cos\gamma S_{n+1}^3\tfrac{\sin\gamma S\sin\gamma(S+1)}{\sin^2\gamma}\Big)\, e^{-\mathrm{i}\gamma S_{n+1}^3}\,. \end{aligned} \tag{29}$$

[2]We call $H_{n,n+1}$ local because it is a lattice analogue of the Hamiltonian density. But $H_{n,n+1}$ is also local in the sense that it involves spines only at two nearest sites of the lattice. We hope that this mixed terminology will not lead to a confusion.

We remark that $2\mathbf{X}_S$ is a q-deformation of square of the sum of two spins in the sense that in the $\gamma \to 0$ limit eq. (29) simplifies to

$$\mathbf{X}_S^0 = S(S+1) + \mathbf{S}_n^0 \cdot \mathbf{S}_{n+1}^0 = \frac{1}{2}(\mathbf{S}_n^0 + \mathbf{S}_{n+1}^0)^2 \,,$$

where S_a^0 are the generators of sl_2. In this limit eq. (28) turns into the integrable XXX Hamiltonian constructed in [12, 4].

Let us consider the large spin S asymptotics of the Hamiltonian (22)-(23) in terms of the spin operator J defined by (14). Recall first that for the logarithmic derivative of the (non-deformed) gamma function we have

$$\Psi(j+1) = \Psi(1) + \sum_{k=1}^{j} \frac{1}{k} = \ln(j + \frac{1}{2}) + O(j^{-2}) \,.$$

Therefore the large spin approximation of the XXX Hamiltonian is

$$H_{n,n+1} = \ln(J + \frac{1}{2}) + \text{const} \,. \tag{30}$$

In the XXZ case, if q is real, eq. (18) (or its counterpart for $|q| > 1$) yields $\Psi_q(x) \approx x\,|\gamma|$ for large x. More precisely, we infer from (22)-(23) that

$$\Psi_q(j+1) = \Psi_q(1) + |\gamma| \sum_{k=1}^{j} \tanh(k|\gamma|) = |\gamma|\, j + \varkappa_\gamma + O(e^{-2|\gamma| j}) \,, \tag{31}$$

where $\varkappa_\gamma$ is a γ-dependent constant. Hence the large spin asymptotics of the XXZ Hamiltonian is

$$H_{n,n+1} = J \,\sinh|\gamma| + \text{const} \,. \tag{32}$$

Here we should remind that the spin operator in (32) is not the same as in (30), Indeed, it is always given by $J = \sum_j j\mathcal{P}_j$ but the projectors are different for different γ. Notice also that the correction to the leading order in (31) decays very fast (because $\tanh(x) = 1 + O(e^{-2x})$). So, if $|\gamma| S$ is not too small, (32) is a good approximation even for not too large S.

Thus, the large spin asymptotics of $H_{n,n+1}$ in the XXZ case differs from that in the XXX case. Moreover, for real γ, i.e., when $|q| = 1$, the function $\Psi_q(x)$ is not monotonous (as seen from (23)) and does not have an asymptotics at all.

4 Properties

Global symmetry: Recall that the co-associativity property, $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, leads to a natural notion of an n-th power of the co-multiplication: $\Delta^{(n+1)} \equiv (\Delta \otimes 1^{\otimes n})\Delta^{(n)}$ with $\Delta^{(1)} \equiv \Delta$. Then the global spin generators for a chain with N nodes are naturally introduced [6] as $\mathcal{S}^{\pm} = \Delta^{(N-1)}(S^{\pm})$ and $\mathcal{S}^3 = \Delta^{(N-1)}(S^3)$; which in explicit form reads

$$\mathcal{S}^3 = \sum_{n=1}^{N} S_n^3 \,, \qquad \mathcal{S}^{\pm} = \sum_{n=1}^{N} q^{S_1^3} \ldots q^{S_{n-1}^3} S_n^{\pm} q^{-S_{n+1}^3} \ldots q^{-S_N^3} \,. \tag{33}$$

By construction (see (12) and (23)), the local Hamiltonian $H_{n,n+1}$ is $U_q(sl_2)$-symmetric, i.e., it commutes with $\Delta(\xi)$ for any $\xi \in U_q(sl_2)$. Furthermore, it is easy to check that

$$[\, H_{n,n+1}, \mathcal{S}^3 \,] = 0 \,, \qquad \text{for } n = 1, \ldots, N \,, \tag{34}$$

$$[\, H_{n,n+1}, \mathcal{S}^{\pm} \,] = 0 \,, \qquad \text{for } n = 1, \ldots, N-1 \,. \tag{35}$$

Since $H_{N,1}$ does not satisfy (35), the total Hamiltonian enjoys only the $U(1)$-symmetry,

$$[\mathcal{H}, \mathcal{S}^3] = 0\,. \tag{36}$$

Actually, the higher quantum integrals of motion also commute with $\mathcal{S}^3$. Therefore in the presence of a constant magnetic field h the corresponding Hamiltonian, $\mathcal{H}_h = \mathcal{H} + h\mathcal{S}^3$, remains integrable and $U(1)$-symmetric.

C and P symmetries: Recall that $\mathbf{P}$ denotes the permutation in $\mathfrak{H} \otimes \mathfrak{H}$. Since the comultiplication Δ does not commute with $\mathbf{P}$, neither does $r(\lambda)$, $\mathbf{X}_S$ or $H_{n,n+1}$. However, we observe that (here and below dependence on γ is shown explicitly only if it is affected by a transformation)

$$\mathbf{P}\,\mathbf{X}_S(\gamma)\,\mathbf{P} = \mathbf{X}_S(-\gamma)\,, \qquad \mathbf{P}\,r(\lambda,\gamma)\,\mathbf{P} = r(\lambda,-\gamma)\,. \tag{37}$$

The first relation is obvious from (29) and yields the second one upon noticing that both the projectors $\mathcal{P}_j(x,\gamma)$ and the coefficients $r_j(\lambda,\gamma)$ in (15) are even functions in γ. Further, the second relation in (37) implies readily that

$$\mathbf{P}_{n,n+1}\,H_{n,n+1}(\gamma)\,\mathbf{P}_{n,n+1} = H_{n,n+1}(-\gamma)\,. \tag{38}$$

Thus $H_{n,n+1}$ does not have the P (reflection) symmetry. But, if q is real, it has the C-symmetry (invariance with respect to the complex conjugation). If $|q| = 1$, then eq. (38) shows that $H_{n,n+1}$ has no C- or P-symmetry separately but it has the CP-symmetry.

Local bulk and boundary terms: As we will see below, $H_{n,n+1}$ decomposes into two parts (which we will refer to as the local bulk term and the local boundary term):

$$H_{n,n+1} = \widehat{H}_{n,n+1} + \mathrm{i}\frac{\sin\gamma}{2}(S^3_n - S^3_{n+1})\,. \tag{39}$$

The local bulk term, $\widehat{H}_{n,n+1}$, has the following properties (see Section 7)

$$\mathbf{P}_{n,n+1}\widehat{H}_{n,n+1}\mathbf{P}_{n,n+1} = \widehat{H}_{n,n+1}\,, \qquad \widehat{H}_{n,n+1}(\gamma) = \widehat{H}_{n,n+1}(-\gamma)\,. \tag{40}$$

Thus, $\widehat{H}_{n,n+1}$ is P- and C-symmetric for real q as well as for $|q| = 1$. In fact (see Section 7), $\widehat{H}_{n,n+1}$ is a local Hamiltonian associated with the L-operator

$$\widehat{L}(\lambda) = \frac{1}{\sin\gamma}\begin{pmatrix} \sinh[\gamma(\lambda + \mathrm{i}S^3)] & \mathrm{i}\,S^-\sin\gamma \\ \mathrm{i}\,S^+\sin\gamma & \sinh[\gamma(\lambda - \mathrm{i}S^3)] \end{pmatrix}. \tag{41}$$

Notice that $\widehat{H}_{n,n+1}$ is not $U_q(sl_2)$-symmetric but only $U(1)$-symmetric.

The local boundary term in (39) has the same form for all positive (half-) integer spins. In the total Hamiltonian of a closed chain all these boundary terms mutually cancel, hence

$$\mathcal{H} = \sum_{n=1}^{N} H_{n,n+1} = \sum_{n=1}^{N} \widehat{H}_{n,n+1}\,. \tag{42}$$

This together with (40) explains why the two sums in (21) coincide.

Properties with respect to $*$–operation: The $*$–structure on $U_q(sl_2)$ corresponding to the compact real form $U_q(su(2))$ is defined as an anti-automorphism such that

$$(S^{\pm})^* = \eta^{\pm 1}\,S^{\mp}\,, \qquad (S^3)^* = S^3 \tag{43}$$

with some real η. Of course, the prefactor $\eta^{\pm 1}$ can be eliminated by rescaling the generators. However it may be convenient to keep it. For instance, we saw in Section 2 that, in the spin-$\frac{1}{2}$ case, it is natural to put $S^{\pm} = q^{\pm\frac{1}{2}}\sigma^{\pm}$. Then $\eta = 1$ if $|q| = 1$ but $\eta = q$ if q is real. In fact, the choice of η is not important for our purposes since $H_{n,n+1}$ and $\widehat{H}_{n,n+1}$ contain $S^{\pm}$ only in homogeneous combinations like $(S^{+} \otimes S^{-})$.

The action of the $*$–operation extends on a tensor product as $(\xi \otimes \zeta)^* = \xi^* \otimes \zeta^*$. It must be remarked that properties of the co-multiplication with respect to the action of the $*$–operation depend on the choice of q, namely we infer from (7) and (43) that [3]

$$(\Delta(\xi))^* = \Delta(\xi^*) \quad \text{if } q \in \mathbb{R} \qquad \text{but} \qquad (\Delta(\xi))^* = \mathbf{P}\Delta(\xi^*)\mathbf{P} \quad \text{if } |q| = 1\,.$$

Therefore the local Hamiltonian (28) has the following properties:

$$(H_{n,n+1})^* = H_{n,n+1} \qquad \text{for } q \in \mathbb{R} \tag{44}$$

$$(H_{n,n+1})^* = \mathbf{P}_{n,n+1} H_{n,n+1} \mathbf{P}_{n,n+1} \qquad \text{for } |q| = 1\,. \tag{45}$$

Nevertheless, we see from eq. (23) that the eigenvalues of $H_{n,n+1}$ are real in the both cases. Furthermore, it follows from (39)-(40) and (44)-(45) that

$$(\widehat{H}_{n,n+1})^* = \widehat{H}_{n,n+1} \tag{46}$$

for the both regimes of q. In combination with (42) this implies that the relation

$$\mathcal{H}^* = \mathcal{H} \tag{47}$$

holds also in the both regimes of q.

It is worth emphasizing that, for $|q| = 1$, objects which have the global $U_q(sl_2)$-symmetry, like $H_{n,n+1}$ for $n \neq N$, are in general not self-conjugate with respect to the $*$–operation (43). Indeed, if $\mathcal{O}^* = \mathcal{O}$ and it commutes with the global spin generators $\mathcal{S}^{\pm}$ given by (33), then it must also commute with $(\mathcal{S}^{\pm})^*$. This imposes a strong extra condition on the structure of $\mathcal{O}$ since for $|q| = 1$ the conjugate of $\mathcal{S}^{\pm}$ belong to $U_{q^{-1}}(sl_2)$ rather than to $U_q(sl_2)$. Vice versa, objects that are self-conjugate for $|q| = 1$ are in general not $U_q(sl_2)$-symmetric, for instance $\widehat{H}_{n,n+1}$ and $\mathcal{H}$.

T symmetry: The definition $(\xi \otimes \zeta)^* = \xi^* \otimes \zeta^*$ is consistent with the property of the matrix transposition, $(\xi \otimes \zeta)^t = \xi^t \otimes \zeta^t$, if ξ and ζ are regarded as matrices. Thus the $*$-operation can be realized as the matrix hermitian conjugation, $\xi^* = \bar{\xi}^{\,t}$ (bar denotes the complex conjugation).

From (44), (45) and (38) we deduce that $(H_{n,n+1})^* = \bar{H}_{n,n+1}$ in the both regimes of q. Therefore finite dimensional matrix representations of the considered above Hamiltonians are symmetric matrices,

$$(H_{n,n+1})^t = H_{n,n+1}\,, \qquad (\widehat{H}_{n,n+1})^t = \widehat{H}_{n,n+1}\,, \qquad (\mathcal{H})^t = \mathcal{H}\,. \tag{48}$$

Here the first equality leads to the second and the third due to (39) and (42), respectively.

5 Examples ($S = \frac{1}{2}, 1, \frac{3}{2}$)

For the spin $S = \frac{1}{2}$, the Hamiltonian (28) is very simple:

$$H_{n,n+1} = (\cos\gamma)\, \mathcal{P}_1 = \mathbf{X}_{\frac{1}{2}}\,, \tag{49}$$

[3] A $*$–structure satisfying $(\Delta(\xi))^* = \Delta(\xi^*)$ for $|q| = 1$ is the *non-compact* real form $U_q(sl(2,\mathbb{R}))$, see [13].

and the spin generators are given by $S^{\pm} = q^{\pm\frac{1}{2}}\sigma^{\pm}$ and $S^3 = \frac{1}{2}\sigma^3$. Using the relations

$$e^{t\sigma^3}\sigma^{\pm} = \sigma^{\pm}e^{-t\sigma^3} = e^{\pm t}\sigma^{\pm}\,, \qquad e^{\mathrm{i}t\sigma^3} = \cos t + \mathrm{i}\sigma^3\,\sin t\,, \tag{50}$$

it is easy to check that (49) acquires the following form

$$H_{n,n+1} = \frac{1}{2}(S_n^+S_{n+1}^- + S_n^-S_{n+1}^+) + (\cos\gamma)\,S_n^3S_{n+1}^3 + \frac{3}{4}\cos\gamma + \mathrm{i}\frac{\sin\gamma}{2}(S_n^3 - S_{n+1}^3)\,. \tag{51}$$

The bulk term here is the well-known XXZ deformation of the Heisenberg spin chain.

For $S=1$ the Hamiltonian (28) looks as follows

$$H_{n,n+1} = (\cos\gamma)\,\mathcal{P}_1 + \frac{\sin 3\gamma}{\sin 2\gamma}\,\mathcal{P}_2 = \frac{1}{4(\cos\gamma)^3}\Big((\cos\gamma + 4\cos^3\gamma)\,\mathbf{X}_1 - (\mathbf{X}_1)^2\Big)\,. \tag{52}$$

In this case the spin generators are given by (98); they are related to the spin-1 sl_2-generators as $S^{\pm} = (\cos\gamma)^{\frac{1}{2}}S_0^{\pm}$, $S^3 = S_0^3$. Rewriting $\mathbf{X}_1$ and its square as polynomials in the spin generators (see Appendix B), we obtain

$$H_{n,n+1} = \frac{1}{4\cos\gamma}\Big(\mathbf{\Omega}_1 - (\mathbf{\Omega}_1)^2 + \mathbf{F}_1\Big) + \mathrm{i}\frac{\sin\gamma}{2}(S_n^3 - S_{n+1}^3)\,, \tag{53}$$

where

$$\mathbf{\Omega}_1 = \frac{1}{2\cos\gamma}(S_n^+S_{n+1}^- + S_n^-S_{n+1}^+) + (2\cos\gamma - 1)\,S_n^3S_{n+1}^3\,, \tag{54}$$

$$\begin{aligned}\mathbf{F}_1 = {} & 2\cos\gamma(\cos\gamma - 1)S_n^3S_{n+1}^3 + 2(\cos\gamma-1)^2(S_n^3S_{n+1}^3)^2 \\ & - 2(\sin\gamma)^2\,((S_n^3)^2 + (S_{n+1}^3)^2) + 4 + 2\cos 2\gamma\,.\end{aligned} \tag{55}$$

The bulk term of the Hamiltonian (53) (i.e., (53) without the last term) coincides (up to a constant) with the Fateev-Zamolodchikov (FZ) Hamiltonian [2]. The FZ Hamiltonian is usually written in a slightly different way in terms of the spin-1 sl_2-generators; we have chosen the above form since it allows for better comparison with the $S=\frac{3}{2}$ case.

In the limit $\gamma\to 0$, (52)-(53) simplifies to the well-know spin-1 XXX Hamiltonian:

$$H^0_{n,n+1} = \tfrac{5}{4}\,\mathbf{X}_1^0 - \tfrac{1}{4}\,(\mathbf{X}_1^0)^2 = \tfrac{1}{4}\,\mathbf{S}_n^0\cdot\mathbf{S}_{n+1}^0 - \tfrac{1}{4}\,(\mathbf{S}_n^0\cdot\mathbf{S}_{n+1}^0)^2 + \tfrac{3}{2}\,.$$

In the $S=\frac{3}{2}$ case the Hamiltonian (23) is given by

$$H_{n,n+1} = (\cos\gamma)\,\mathcal{P}_1 + \frac{\sin 3\gamma}{\sin 2\gamma}\,\mathcal{P}_2 + \Big(\frac{\sin 3\gamma}{\sin 2\gamma} + (\sin\gamma)\frac{\cos 3\gamma}{\sin 3\gamma}\Big)\,\mathcal{P}_3\,. \tag{56}$$

Rewriting the corresponding expression (28) in the polynomial (with respect to the spin generators) form (see Appendix B), we obtain

$$\begin{aligned}H_{n,n+1} = {} & \frac{1}{12(\cos\gamma)^3(1+2\cos 2\gamma)^3}\Big(12(\cos\gamma)\,(\mathbf{\Omega}_{\frac{3}{2}})^3 + (5\cos 4\gamma - \cos 2\gamma + 2)\,(\mathbf{\Omega}_{\frac{3}{2}})^2 \\ & + \frac{1}{2\cos\gamma}(\mathbf{\Omega}_{\frac{3}{2}}\,\mathbf{Q} + \mathbf{Q}\,\mathbf{\Omega}_{\frac{3}{2}}) + \frac{1+2\cos 2\gamma}{64\cos^2\gamma}\mathbf{F}_{\frac{3}{2}}\Big) + \mathrm{i}\frac{\sin\gamma}{2}(S_n^3 - S_{n+1}^3)\,,\end{aligned} \tag{57}$$

where $\mathbf{Q}$ and $\mathbf{F}_{\frac{3}{2}}$ are real symmetric polynomials in S_n^3 and S_{n+1}^3 (see Appendix B) and

$$\mathbf{\Omega}_{\frac{3}{2}} = \frac{1}{2}(S_n^+S_{n+1}^- + S_n^-S_{n+1}^+) + (1+2\cos 2\gamma)\frac{\cos 2\gamma}{3\cos\gamma}\,S_n^3S_{n+1}^3\,. \tag{58}$$

Let us remark that in both Hamiltonians (53) and (57) the two "most non-diagonal" terms are powers of an operator $\mathbf{\Omega}$ which is quadratic in generators. To clarify this fact, we observe that, like for $S=\frac{1}{2}$, the corresponding Casimir operators (24) are actually quadratic:

$$\frac{1}{2}(S^+S^- + S^-S^+) + (\cos\gamma)(S^3)^2 = 2\cos\gamma \qquad \text{for } S=1\,, \tag{59}$$

$$\frac{1}{2}(S^+S^- + S^-S^+) + (\cos\gamma)^2(S^3)^2 = \frac{1}{4}(17\cos^2\gamma - 2) \qquad \text{for } S=\frac{3}{2} \tag{60}$$

(the last expression differs from $C_{\frac{3}{2}}$ by a constant). Eq. (59) is due to the first relation in (99); eq. (60) follows from the relation (107) and the identity $2(\sin\gamma S^3)^2 = 1 - \cos 2\gamma S^3$.

6 Another co-multiplication

As we saw in Section 2, the co-multiplication operation Δ plays a key rôle in the construction of the Hamiltonian $\mathcal{H}$. However, eq. (7) does not exhaust possible definitions of Δ for $U_q(sl_2)$. In fact, a homomorphism $\Delta_\alpha : U_q(sl_2) \to [U_q(sl_2)]^{\otimes 2}$ such that

$$\Delta_\alpha(S^\pm) = S^\pm \otimes q^{(\pm\alpha-1)S^3} + q^{(\pm\alpha+1)S^3} \otimes S^\pm\,, \qquad \Delta_\alpha(S^3) = S^3\otimes 1 + 1 \otimes S^3 \tag{61}$$

satisfies all the properties of the co-multiplication for any real α. We had before $\alpha=0$ which is the most "symmetric" choice. But we may prefer $\alpha=1$ (or $\alpha=-1$) if we want the regime $q<0$ to be on equal footing with $q>0$ in the sense that the terms $q^{(\pm\alpha\pm1)S^3}$ in (61) take only real values.

Notice that Δ_α is obtained from the co-multiplication (7) by twisting:

$$\Delta_\alpha(\xi) = F_\alpha\,\Delta(\xi)\,(F_\alpha)^{-1}\,, \qquad F_\alpha = q^{\alpha S^3\otimes S^3} \in \mathfrak{H}\otimes\mathfrak{H}\,. \tag{62}$$

This twist is rather specific in that it preserves the co-associativity of the co-multiplication (more general twists give rise to the so-called quasi-Hopf algebras [14]; see [15] for their applications in integrable spin models).

In order to construct an integrable Hamiltonian using the new co-multiplication Δ_α in the same way as we used Δ in Sections 2 and 3, we should first find an L-operator, $\tilde{L}_\alpha(\lambda)$, that satisfies (6) with Δ_α. For this purpose we observe that

$$F_\alpha = (\phi_\alpha\otimes\phi_\alpha)^{-1}\,\Delta(\phi_\alpha) \qquad \text{where} \quad \phi_\alpha = q^{\frac{\alpha}{2}(S^3)^2} \in \mathfrak{H}\,. \tag{63}$$

Therefore we can rewrite the l.h.s. of (6) for $\tilde{L}_\alpha(\lambda)$ as

$$\Delta_\alpha(\tilde{L}_\alpha(\lambda)) = (\phi_\alpha\otimes\phi_\alpha)^{-1}\,\Delta(\phi_\alpha\,\tilde{L}_\alpha(\lambda)\,\phi_\alpha^{-1})\,(\phi_\alpha\otimes\phi_\alpha)$$

which makes it obvious that $\tilde{L}_\alpha(\lambda)$ satisfies (6) with Δ_α if it is related to the L-operator (3) as follows

$$\tilde{L}_\alpha(\lambda) = \phi_\alpha^{-1}\,L(\lambda)\,\phi_\alpha \tag{64}$$

$$= \frac{1}{\sin\gamma}\begin{pmatrix} \sinh[\gamma(\lambda+\mathrm{i}S^3)] & \mathrm{i}\,\sin\gamma\, e^{\gamma\lambda} q^{\frac{\alpha}{2}} q^{\alpha S^3}\, S^- \\ \mathrm{i}\,\sin\gamma\, e^{-\gamma\lambda} q^{\frac{\alpha}{2}} q^{-\alpha S^3}\, S^+ & \sinh[\gamma(\lambda-\mathrm{i}S^3)] \end{pmatrix}. \tag{65}$$

Since the map $\xi \to \phi_\alpha^{-1}\xi\phi_\alpha$ is an automorphism of $U_q(sl_2)$, it is clear that the exchange relation (2) holds for $\tilde{L}_\alpha(\lambda)$ with the same R-matrix as for $L(\lambda)$. Consequently, the Bethe ansatz equations for $\tilde{L}_\alpha(\lambda)$ coincide with those for $L(\lambda)$ (the reference state ω has also not changed).

Substituting (64) into (9), we find a universal r-matrix for the new L-operator:

$$\tilde{r}_\alpha(\lambda) = (\phi_\alpha \otimes \phi_\alpha)^{-1}\, r(\lambda)\, (\phi_\alpha \otimes \phi_\alpha) = F_\alpha\, r(\lambda)\, (F_\alpha)^{-1} . \tag{66}$$

The second equality is due to relation (63) and the property (12). The same twists relate the corresponding solutions of the Yang-Baxter equation (8), i.e.,

$$\check{\mathbf{R}}_\alpha(\lambda) = (\phi_\alpha \otimes \phi_\alpha)^{-1} \check{\mathbf{R}}(\lambda)(\phi_\alpha \otimes \phi_\alpha) = F_\alpha\, \check{\mathbf{R}}(\lambda)(F_\alpha)^{-1} . \tag{67}$$

Notice that the first equality is consistent with (64) since ϕ_α in the fundamental representation is just a constant. Further, evaluating the r.h.s. of (67) in $V \otimes \mathfrak{H}$ representation, we see that $\tilde{L}_\alpha(\lambda)$ can be constructed also as a twist by 2×2 matrix,

$$\tilde{L}_\alpha(\lambda) = f_\alpha\, L(\lambda)\, f_\alpha^{-1} , \qquad f_\alpha = q^{\frac{\alpha}{2}\sigma^3 \otimes S^3} .$$

Let us underline that existence of the two ways of constructing $\tilde{L}_\alpha(\lambda)$ and, as a consequence, of the relation

$$[L(\lambda), \phi_\alpha\, f_\alpha] = 0$$

is due to the property of the universal r-matrix (12) applied to $\xi = \phi_\alpha$ (indeed, $q^{\frac{\alpha}{2}}\phi_\alpha f_\alpha$ is $\Delta(\phi_\alpha)$ evaluated in $V \otimes \mathfrak{H}$).

According to (20), the local Hamiltonian corresponding to the universal r-matrix (66) is

$$\tilde{H}^{(\alpha)}_{n,n+1} = (\phi^\alpha_n\, \phi^\alpha_{n+1})^{-1}\, H_{n,n+1}\, (\phi^\alpha_n\, \phi^\alpha_{n+1}) = F^\alpha_{n,n+1}\, H_{n,n+1}\, (F^\alpha_{n,n+1})^{-1} . \tag{68}$$

This transformation does not modify the $S = \frac{1}{2}$ Hamiltonian (49) since in this case ϕ_α is trivial. But already for $S = 1$ we find (see Appendix B)

$$\tilde{H}^{(\alpha)}_{n,n+1} - H_{n,n+1} = \frac{1 - \cos(\alpha\gamma)}{2\cos\gamma} \{\Upsilon_{n,n+1}, S^3_n S^3_{n+1}\} - \mathrm{i}\frac{\sin(\alpha\gamma)}{2\cos\gamma} [\Upsilon_{n,n+1}, S^3_n S^3_{n+1}] , \tag{69}$$

where $[,]$ and $\{,\}$ stand for commutator and anticommutator, respectively, and $\Upsilon_{n,n+1} \equiv \frac{1}{2}(S^+_n S^-_{n+1} + S^-_n S^+_{n+1})$. Notice that the terms on the r.h.s. of (69) are non-diagonal.

Finally, the total Hamiltonian is given by

$$\tilde{\mathcal{H}}^{(\alpha)} = \sum_{n=1}^{N} \tilde{H}^{(\alpha)}_{n,n+1} = (\Phi_\alpha)^{-1}\, \mathcal{H}\, \Phi_\alpha = \mathcal{F}_\alpha\, \mathcal{H}\, (\mathcal{F}_\alpha)^{-1} , \qquad \Phi_\alpha \equiv \prod_{n=1}^{N} \phi_\alpha , \quad \mathcal{F}_\alpha \equiv q^{\alpha \sum\limits_{n<m} S^3_n S^3_m} .$$

The Φ–twist here follows easily from (68). The $\mathcal{F}$–twist yields the same result because $\mathcal{H}$ commutes with $\mathcal{S}^3$ (36) and hence with $\Phi_\alpha \mathcal{F}_\alpha = q^{\frac{\alpha}{2}(\mathcal{S}^3)^2}$. Since $\tilde{\mathcal{H}}^{(\alpha)}$ and $\mathcal{H}$ are related by a twist they have the same set of eigenvalues; this agrees with the fact that the Bethe ansatz equations have not changed.

Let us conclude this section with a remark: eqs. (66), (67) and (68) may appear to suggest that $\tilde{L}_\alpha(\lambda)$ and $\tilde{r}_\alpha(\lambda)$ are related to another twisted co-multiplication, $\tilde{\Delta}_\theta(\xi) \equiv \theta^{-1}\,\Delta(\xi)\,\theta$, where $\theta = \phi_\alpha \otimes \phi_\alpha$. But $\tilde{\Delta}_\theta$ fails to satisfy the necessary property of a co-multiplication [14], $(\epsilon \otimes 1)\Delta(\xi) = \xi$, where ϵ is the co-unit. So $\tilde{\Delta}_\theta$ is not a co-multiplication of a (quasi) Hopf algebra. (Moreover, $\tilde{\Delta}_\theta$ is not co-associative.)

7 From $H_{n,n+1}$ to $\widehat{H}_{n,n+1}$

Consider now another family of L-operators,

$$\widehat{L}_\beta(\lambda) = \frac{1}{\sin\gamma} \begin{pmatrix} \sinh[\gamma(\lambda + \mathrm{i}S^3)] & \mathrm{i}\sin\gamma\, e^{(\gamma-\beta)\lambda}\, S^- \\ \mathrm{i}\sin\gamma\, e^{(\beta-\gamma)\lambda}\, S^+ & \sinh[\gamma(\lambda - \mathrm{i}S^3)] \end{pmatrix} \tag{70}$$

which are obtained from the L-operator (3) as twists by certain $2{\times}2$ matrices,

$$\widehat{L}_\beta(\lambda) = K_\lambda^{-1}\, L(\lambda)\, K_\lambda\,, \qquad K_\lambda = e^{\frac{1}{2}\beta\lambda\,\sigma^3} \in V\,. \tag{71}$$

For $\beta = \gamma$ this gives the L-operator (41) (which is most often used in applications of the Bethe ansatz). The map $S^\pm \to e^{\pm\beta\lambda} S^\pm$, $S^3 \to S^3$ is an automorphism of $U_q(sl_2)$ but, unlike the case treated in Section 6, it is λ-dependent. Therefore, the $\check{R}$-matrix corresponding to $\widehat{L}_\beta(\lambda)$ differs from (4). Namely, as seen from (71), exchange relation (2) holds for $\widehat{L}_\beta(\lambda)$ with $R_\beta(\lambda) = (1 \otimes K_\lambda^{-1}) R(\lambda) (K_\lambda \otimes 1)$, i.e., $R_\beta(\lambda) = P\check{R}_\beta(\lambda)$, where

$$\check{R}_\beta(\lambda) = \mathrm{i}\, e^{(\gamma-\beta)\lambda}\, \sigma^+ \otimes \sigma^- + \mathrm{i}\, e^{(\beta-\gamma)\lambda}\, \sigma^- \otimes \sigma^+ + \frac{1}{\sin\gamma} \sinh\Bigl(\gamma\lambda + \frac{\mathrm{i}\,\gamma}{2}(1 \otimes 1 + \sigma^3 \otimes \sigma^3)\Bigr)\,. \tag{72}$$

In order to find a universal r-matrix for $\widehat{L}_\beta(\lambda)$, we will apply the approach which we used in Section 6. Namely, we observe that $\varphi_\lambda \equiv e^{\beta\lambda S^3} \in \mathfrak{H}$ is a complementary twist to (71) in the sense that

$$[L(\lambda), K_\lambda\, \varphi_\lambda] = 0$$

as follows from the property of the universal r-matrix (12) for $\xi = \varphi_\lambda$ (in fact, K_λ is just φ_λ evaluated in the fundamental representation; hence $K_\lambda \varphi_\lambda$ is $\Delta(\varphi_\lambda)$ evaluated in $V \otimes \mathfrak{H}$).

Thus, instead of the twist (71) in the auxiliary space V, the L-operator (70) can be obtained as a twist in the quantum space $\mathfrak{H}$,

$$\widehat{L}_\beta(\lambda) = \varphi_\lambda\, L(\lambda)\, \varphi_\lambda^{-1}\,, \qquad \varphi_\lambda = e^{\beta\lambda S^3} \in \mathfrak{H}\,. \tag{73}$$

Substituting (73) into (9) and taking again into account that $[r(\lambda), \Delta(\varphi_\mu)] = 0$, we find a universal r-matrix for $\widehat{L}_\beta(\lambda)$,

$$\widehat{r}_\beta(\lambda) = (1 \otimes \varphi_\lambda)\, r(\lambda)\, (\varphi_\lambda^{-1} \otimes 1)\,. \tag{74}$$

Unlike eq. (66) this relation is not a twist. Notice however that the corresponding solutions of the Yang-Baxter equation (8) are related by a twist,

$$\check{\mathbf{R}}_\beta(\lambda) = (1 \otimes \varphi_\lambda) \check{\mathbf{R}}(\lambda) (1 \otimes \varphi_\lambda^{-1})\,. \tag{75}$$

It must be stressed now that $\widehat{L}_\beta(\lambda)$ does not possess a decomposition of the type (5)-(6), and $\widehat{r}_\beta(\lambda)$ does not commute with Δ (or Δ_α), i.e., (12) does not hold for $\widehat{r}_\beta(\lambda)$ for a generic ξ. As a consequence, $\widehat{r}_\beta(\lambda)$ does not have a representation of the type (15). Moreover, for $\beta \neq 0$ we have in general $[\widehat{r}_\beta(\lambda), \widehat{r}_\beta(\mu)] \neq 0$ (except for the fundamental representation in the case $\beta = \gamma$). Nevertheless, the general recipe for constructing a local integrable Hamiltonian applies (because the Yang-Baxter equation for $\check{\mathbf{R}}_\beta(\lambda)$ is valid). So we substitute (74) into the formula (20) and derive

$$\widehat{H}^{(\beta)}_{n,n+1} = H_{n,n+1} - \mathrm{i}\, \frac{\beta\, \sin\gamma}{2\gamma}\, (S^3_n - S^3_{n+1})\,. \tag{76}$$

Thus the new local Hamiltonian differs from $H_{n,n+1}$ only in the local boundary term. Hence the total Hamiltonian $\widehat{\mathcal{H}} = \sum_{n=1}^{N} \widehat{H}^{\beta}_{n,n+1}$ coincides with $\mathcal{H}$. Note that the Bethe ansatz equations, describing the spectrum of $\widehat{\mathcal{H}}$, have also not changed, which is not entirely trivial since the corresponding R-matrix has changed. The reason is that in the derivation of the Bethe ansatz equations the non-diagonal entries of (72) appear only in the so-called "unwanted terms" that cancel each other [10, 11, 9].

In Section 4 we asserted that the local Hamiltonian $H_{n,n+1}$ decomposes into a local boundary term and a local bulk term $\widehat{H}_{n,n+1}$ which is associated with the L-operator (41). Now this is obvious from (70) and (76) if we put $\beta = \gamma$. Since we have found the corresponding universal r-matrix, we are in a position to prove the properties of $\widehat{H}_{n,n+1}$ stated in Section 4. For brevity we denote $r_0(\lambda) \equiv \widehat{r}_\gamma(\lambda)$.

As seen from (72) for $\beta = \gamma$, the auxiliary R-matrix associated with the L-operator (41) is P-symmetric, i.e., it commutes with the $4{\times}4$ permutation matrix P_V. The corresponding universal r-matrix, $r_0(\lambda)$, has analogous properties. Namely, (as we prove in Appendix A) $r_0(\lambda)$ satisfies the following relations

$$\mathbf{P}\, r_0(\lambda)\, \mathbf{P} = r_0(\lambda)\,, \qquad r_0(\lambda,\gamma) = r_0(\lambda,-\gamma)\,, \tag{77}$$

$$(r_0(\lambda))^t = r_0(\lambda)\,, \qquad r_0(\lambda)\, r_0(-\lambda) = 1\otimes 1\,, \tag{78}$$

where $\mathbf{P}$ is the permutation in $\mathfrak{H}\otimes\mathfrak{H}$ and t denotes transposition. The last equality follows from the formula (74), the first relation in (17) for $r(\lambda)$, and the relation $[r(\lambda), \Delta(\varphi_\lambda)] = 0$. Taking logarithmic derivative of (77) at $\lambda = 0$, we establish the symmetries (40) of the local bulk term $\widehat{H}_{n,n+1}$. The first relation in (78) yields in the same way the second relation in (48).

8 Open chain

As we saw in Section 4, $H_{N,1}$ is the only term in $\mathcal{H}$ which does not commute with the global spin generators $\mathcal{S}^{\pm}$. Omitting it, we obtain a Hamiltonian for an *open* spin chain,

$$\mathcal{H}' = \sum_{n=1}^{N-1} H_{n,n+1} = \sum_{n=1}^{N-1} \widehat{H}_{n,n+1} + \mathrm{i}\frac{\sin\gamma}{2}(S_1^3 - S_N^3) \tag{79}$$

which is apparently $U_q(sl_2)$-symmetric, i.e., $[\mathcal{H}', \mathcal{S}^{\pm}] = [\mathcal{H}', \mathcal{S}^3] = 0$. It is however not immediately evident whether the Hamiltonian (79) remains *integrable*.

Let us refer to the sum on the r.h.s. of (79) as the bulk Hamiltonian, $\widehat{\mathcal{H}}'$. The remaining part can be called the surface term; it is a sum of $(N-1)$ local boundary terms we dealt with before. As we discussed above, the bulk Hamiltonian $\widehat{\mathcal{H}}'$ corresponds to the L-operator (41) and it is only $U(1)$-symmetric. The fact that adding the surface term to $\widehat{\mathcal{H}}'$ restores the $U_q(sl_2)$-symmetry was observed for spin $\frac{1}{2}$ in [16] and for spin 1 in [17]. Integrability of the corresponding total Hamiltonians was established in [18, 19, 20] in the framework of boundary integrable lattice models. We will prove below that $\mathcal{H}'$ is integrable for the higher spins as well.

Let us briefly recall the construction of an integrable Hamiltonian for a chain with boundaries [18]. Let $\check{\mathbf{R}}(\lambda)$ be a solution of the Yang-Baxter equation (8), and $T(\lambda)$ be a monodromy matrix obeying the exchange relation (2) with R-matrix $\mathbf{P}\check{\mathbf{R}}(\lambda)$. Introduce a boundary monodromy matrix, $Z(\lambda) \equiv T(\lambda)\mathcal{K}^-(\lambda)(T(-\lambda))^{-1}$. The boundary matrix,[4] $\mathcal{K}^-(\lambda) \in \mathfrak{H}$, repre-

[4]Strictly speaking, $\mathcal{K}^-(\lambda)$ is an element of $\mathfrak{H}\otimes\mathfrak{H}$ but with trivial second component. We had a similar situation in Section 7 where the twist φ_λ had trivial first component, see eq. (73).

sents non-periodic boundary conditions ($\check{\mathbf{R}}$ and $\mathcal{K}^-$ are close analogues of the bulk and boundary scattering matrices [21]). Now, if $\mathcal{K}^-(\lambda)$ satisfies the so-called reflection equation, namely

$$\begin{aligned}\check{\mathbf{R}}(\lambda-\mu)\,(\mathcal{K}^-(\lambda)\otimes 1)\,(\check{\mathbf{R}}(-\lambda-\mu))^{-1}\,(1\otimes\mathcal{K}^-(\mu))\\ =(1\otimes\mathcal{K}^-(\mu))\,\check{\mathbf{R}}(\lambda+\mu)\,(\mathcal{K}^-(\lambda)\otimes 1)\,(\check{\mathbf{R}}(-\lambda+\mu))^{-1}\,,\end{aligned} \tag{80}$$

then $Z(\lambda)$ also satisfies this equation. Using this fact one can show that a special trace of $Z(\lambda)$, $\tau(\lambda)=\mathrm{tr}_0\mathcal{K}^+(\lambda)Z(\lambda)$, is a generating function for quantum integrals of motion if the boundary matrix $\mathcal{K}^+(\lambda)\in\mathfrak{H}$ satisfies a "dual" reflection equation [18, 22],

$$\begin{aligned}\check{\mathbf{R}}(-\lambda+\mu)\,((\mathcal{K}^+)^t(\lambda)\otimes 1)\,\check{\mathbf{R}}(-\lambda-\mu-2\delta)\,(1\otimes(\mathcal{K}^+)^t(\mu))\\ =(1\otimes(\mathcal{K}^+)^t(\mu))\,\check{\mathbf{R}}(-\lambda-\mu-2\delta)\,((\mathcal{K}^+)^t(\lambda)\otimes 1)\,\check{\mathbf{R}}(-\lambda+\mu)\,,\end{aligned} \tag{81}$$

and $\check{\mathbf{R}}$ has the following properties

$$\mathbf{P}\,\check{\mathbf{R}}(\lambda)\,\mathbf{P}=\check{\mathbf{R}}(\lambda)\,,\quad \check{\mathbf{R}}(\lambda)\,\check{\mathbf{R}}(-\lambda)=\rho_1(\lambda)\,(1\otimes 1)\,,\quad (\check{\mathbf{R}}(\lambda))^t=\check{\mathbf{R}}(\lambda)\,, \tag{82}$$

$$(\check{\mathbf{R}}(\lambda))^{t_1}\,(\check{\mathbf{R}}(-\lambda-2\delta))^{t_1}=\rho_2(\lambda)\,(1\otimes 1)\,, \tag{83}$$

where δ is a constant, and $\rho_1(\lambda)$ and $\rho_2(\lambda)$ are scalar function.

With all these conditions, an integrable Hamiltonian is given [18] by the following analogue of the formula (20) (we keep the same normalization as in (20))

$$\begin{aligned}\mathcal{H}'' &= \sum_{n=1}^{N-1} h_{n,n+1}+\mathrm{i}\frac{\sin\gamma}{4\gamma}\,\frac{d}{d\lambda}\mathcal{K}_1^-(\lambda)\Big|_{\lambda=0}+\frac{\mathrm{tr}_0(\mathcal{K}_0^+(0)\,h_{0,N})}{\mathrm{tr}\,\mathcal{K}^+(0)}\,,\\ & h_{n,n+1}=\mathrm{i}\frac{\sin\gamma}{2\gamma}\,\mathbf{P}_{n,n+1}\,\frac{d}{d\lambda}\check{\mathbf{R}}_{n,n+1}(\lambda)\Big|_{\lambda=0}\end{aligned} \tag{84}$$

Deriving (84) one assumes that $\check{\mathbf{R}}_{n,n+1}(0)=\mathbf{P}$ and $\mathcal{K}^-(0)=1$, which is consistent with (80).

Let us try to identify the Hamiltonian (79) as a particular case of (84). First, we can put $h_{n,n+1}=\widehat{H}_{n,n+1}$ if we choose $\check{\mathbf{R}}(\lambda)=\check{\mathbf{R}}_\gamma(\lambda)$, where $\check{\mathbf{R}}_\gamma(\lambda)=r_0(\lambda)\mathbf{P}$ is given by (75) with $\beta=\gamma$. For this R-matrix the properties (82) follow from (77); in particular, the second relation (unitarity) holds with $\rho_1(\lambda)=1$. The crossing unitarity (83) holds for $\check{\mathbf{R}}_\gamma(\lambda)$ with $\delta=\mathrm{i}$.

The derivation of the reflection equation (80) for $\mathcal{K}^-$ does not use the conditions (82)-(83). So, let us look first for an R-matrix for which the reflection equation has a trivial solution, $\mathcal{K}^-(\lambda)=1$. In this case (80) turns into

$$\check{\mathbf{R}}(\lambda-\mu)\,(\check{\mathbf{R}}(-\lambda-\mu))^{-1}=\check{\mathbf{R}}(\lambda+\mu)\,(\check{\mathbf{R}}(-\lambda+\mu))^{-1}\,. \tag{85}$$

A solution to this equation is given by $\check{\mathbf{R}}(\lambda)=r(\lambda)\mathbf{P}$, where $r(\lambda)$ is the universal r-matrix we discussed in Section 2. Indeed, (85) follows from the second relation in (17). Using this observation, we can find a solution to the reflection equation for any R-matrix $\check{\mathbf{R}}_\beta(\lambda)$ given by (75). Indeed, substituting (75) in (85), we derive

$$\begin{aligned}\check{\mathbf{R}}_\beta(\lambda-\mu)\,(1\otimes\varphi_\lambda^2)\,(\check{\mathbf{R}}_\beta(-\lambda-\mu))^{-1}\,(1\otimes\varphi_{-\mu}^2)\\ =(1\otimes\varphi_{-\mu}^2)\,\check{\mathbf{R}}_\beta(\lambda+\mu)\,(1\otimes\varphi_\lambda^2)\,(\check{\mathbf{R}}_\beta(-\lambda+\mu))^{-1}\,.\end{aligned} \tag{86}$$

Multiplying this relation by $\Delta(\varphi_\lambda^{-2})=(\varphi_\lambda\otimes\varphi_\lambda)^{-2}$ (which commutes with $\check{\mathbf{R}}_\beta(\lambda)$), we bring it to the form of (80). Thus, a solution of the reflection equation for $\check{\mathbf{R}}_\beta$ is

$$\mathcal{K}_\beta^-(\lambda)=(\varphi_\lambda)^{-2}=e^{-2\beta\lambda S^3}\,.$$

Notice that for $\beta = \gamma$ the R-matrix $\check{\mathbf{R}}$ is symmetric with respect to γ (the second relation in (77)). Therefore in this case we have another solution (which can be derived directly from (86) by applying the permutation and then multiplying by $(\varphi_\mu \otimes \varphi_\mu)^2$),

$$\mathcal{K}^-(\lambda) = e^{2\gamma\lambda S^3} . \tag{87}$$

This is the boundary matrix we need since its derivative in (84) gives exactly the S_1^3 term in (79).

In general, solutions of the reflection equation and of its "dual" are independent of each other. However, as was noted already in [18], there exist several isomorphisms that allow us to construct $\mathcal{K}^+$ if we know $\mathcal{K}^-$. In particular, it is easy to see that a possible solution for (81) is $\mathcal{K}^+(\lambda) = (\mathcal{K}^-(-\lambda - \delta))^t$. For $\mathcal{K}^-$ given by (87) this yields

$$\mathcal{K}^+(\lambda) = e^{-2\gamma(\lambda+\mathrm{i})S^3} . \tag{88}$$

It turns out that substitution of $\mathcal{K}^+(0) = q^{-2S^3}$ into (84) gives exactly the S_N^3 term in (79). To prove this assertion, we first observe (see Appendix C) that for the q-trace of the local Hamiltonian $H_{n,n+1}$ we have

$$\mathrm{tr}_n(q^{2S_n^3}\, H_{n,n+1}) = \tilde{\rho}_S\, 1_{n+1} , \tag{89}$$

where $\tilde{\rho}_S$ is a scalar constant.

Taking into account the relation (39) between $H_{n,n+1}$ and $\widehat{H}_{n,n+1}$, we infer from (89) that $\mathrm{tr}_0(q^{2S_0^3}\, \widehat{H}_{0,N}) = (\mathrm{tr}\, q^{2S^3})\, \mathrm{i}\frac{\sin\gamma}{2}\, S_N^3 + \text{const}$. Replacing now q with q^{-1} and using that $\widehat{H}_{n,n+1}$ is an even function of γ (40), we find that

$$\frac{\mathrm{tr}_0(\mathcal{K}^+(0)\, \widehat{H}_{0,N})}{\mathrm{tr}\, \mathcal{K}^+(0)} = \frac{\mathrm{tr}_0(q^{-2S_0^3}\, \widehat{H}_{0,N})}{\mathrm{tr}\, q^{-2S^3}} = -\mathrm{i}\frac{\sin\gamma}{2} S_N^3 \tag{90}$$

holds (up to an additive constant). Thus, the boundary matrix (88) gives the S_N^3 term in (79). This completes the proof that the $U_q(sl_2)$-symmetric open chain Hamiltonian (79) is integrable for all (half-) integer spins.

Conclusion

In summary, we have constructed explicitly (in terms of the spin generators) higher spin closed chain Hamiltonians for two families of XXZ-type L-operators including the two L-operators most often used in the literature. We have investigated properties of these Hamiltonians, described their interrelations, and discussed the connection with $U_q(sl_2)$-symmetric open chain Hamiltonians.

We have emphasized a key rôle of the underlying quantum algebraic structure, especially the universal r-matrix and the co-multiplication, for constructing integrable Hamiltonians and investigating their symmetries. The technique presented in this paper can be applied also for constructing the higher quantum integrals of motion.

Our construction does not in general extend to the case when q is a root of unity. However it appears that we could allow $q^p = 1$ if p is sufficiently large in comparison with the spin S. For instance, examining the denominator of (27) we see that the projectors $\mathcal{P}_j$ do not become singular at roots of unity if $p > 8S$. If the ratio p/S is not large, then an appropriate modification of the presented construction is needed; it will be certainly of interest and use.

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Appendix A

Here we give some technical details on the universal r-matrices used in the text. First we recall the derivation of the universal r-matrix (16) along the lines of [6, 7] (see also [9, 23]). The explicit form of the off-diagonal entries of equation (11) is

$$r(\lambda)\,(e^{\mp\gamma\lambda} S^{\pm} \otimes q^{S^3} + e^{\pm\gamma\lambda} q^{-S^3} \otimes S^{\pm}) = (e^{\pm\gamma\lambda} S^{\pm} \otimes q^{S^3} + e^{\mp\gamma\lambda} q^{-S^3} \otimes S^{\pm})\, r(\lambda)\,. \qquad (91)$$

A solution to this equation is unique up to a scalar factor [7]. Notice that the co-multiplication (7) satisfies the following relations

$$[S^{\varepsilon} \otimes q^{S^3}, \Delta(S^{\varepsilon})] = [q^{-S^3} \otimes S^{\varepsilon}, \Delta(S^{\varepsilon})] = 0\,, \qquad \varepsilon = \pm\,.$$

Therefore, for the highest/lowest weight vectors we deduce that

$$S^{\pm} \otimes q^{S^3}\, |j, \pm j\rangle = g_{\pm}(j) |j+1, \pm(j+1)\rangle\,, \qquad q^{-S^3} \otimes S^{\pm}\, |j, \pm j\rangle = h_{\pm}(j) |j+1, \pm(j+1)\rangle\,, \quad (92)$$

where $g_{\pm}(j)$ and $h_{\pm}(j)$ are scalar functions. Furthermore, we have

$$\begin{aligned} &(S^{\pm} \otimes q^{S^3} + q^{\pm(2+2j)} q^{-S^3} \otimes S^{\pm})\, |j, \pm j\rangle = \Big((1 \otimes q^{2S^3})\, S^{\pm} \otimes q^{-S^3} \\ &+ q^{\pm(2+2j)} (q^{S^3} \otimes S^{\pm} q^{2S^3}) q^{-2S^3} \otimes q^{-2S^3} \Big)\, |j, \pm j\rangle = (1 \otimes q^{2S^3})\, \Delta(S^{\pm})\, |j, \pm j\rangle = 0\,. \end{aligned} \qquad (93)$$

Applying now (91) to $|j, j\rangle$ (for the upper signs) or to $|j, -j\rangle$ (for the lower signs) and using (15), (92) and (93), we deduce that

$$r_{j+1}(\lambda)\,(e^{\pm\gamma\lambda} - q^{\pm(2+2j)} e^{\mp\gamma\lambda}) = (e^{\mp\gamma\lambda} - q^{\pm(2+2j)} e^{\pm\gamma\lambda})\; r_j(\lambda)\,. \qquad (94)$$

Both relations in (94) yield the same functional equation on $r_j(\lambda)$ (which arises also for universal r-matrices in the lattice sine-Gordon model [24] and in the lattice Virasoro algebra [25])

$$r_{j+1}(\lambda) = \frac{\sin[\gamma(j+1-i\lambda)]}{\sin[\gamma(j+1+i\lambda)]}\; r_j(\lambda)\,. \qquad (95)$$

Upon imposing the normalization condition $r(0) = 1$, we obtain the expression (16). A proof that this $r(\lambda)$ does satisfy all the relations in (11) is given in [6, 7]. Of course, one is still free to multiply $r(\lambda)$ by a scalar function, $\rho(\lambda)$, such that $\rho(0) = 1$. For instance, the spin-$\frac{1}{2}$ representation of $\check{R}(\lambda)$ given by (4) corresponds to $\rho(\lambda) = \sin[\gamma(1+i\lambda)]/\sin\gamma$.

Consider now the universal r-matrix $r_0(\lambda)$ introduced in Section 7. It is a solution to equation (9) for the L-operator (41). Unlike the previous case, this L-operator does not have a Borel decomposition of the type (5). Therefore the off-diagonal entries of (9) give us four λ-dependent relations:

$$r_0(\lambda)\,(S^{\pm} \otimes q^{S^3} + e^{\pm\gamma\lambda} q^{-S^3} \otimes S^{\pm}) = (e^{\pm\gamma\lambda} S^{\pm} \otimes q^{S^3} + q^{-S^3} \otimes S^{\pm})\, r_0(\lambda)\,, \qquad (96)$$

$$r_0(\lambda)\,(S^{\pm} \otimes q^{-S^3} e^{\pm\gamma\lambda} + q^{S^3} \otimes S^{\pm}) = (e^{\pm\gamma\lambda} q^{S^3} \otimes S^{\pm} + S^{\pm} \otimes q^{-S^3})\, r_0(\lambda)\,. \qquad (97)$$

It can be proven along the lines of [7] that solution to this equation is unique up to a scalar factor. As we have already shown in Section 7, this solution is given by $r_0(\lambda) = (1 \otimes e^{\gamma\lambda S^3}) r(\lambda)(e^{-\gamma\lambda S^3} \otimes 1)$, where $r(\lambda)$ solves (91).

Now we observe that $r_1(\lambda) \equiv \mathbf{P} r_0(\lambda)\mathbf{P}$ and $r_2(\lambda) \equiv (r_0(\lambda))^t$ solve the same set of equations (96)-(97). By the above mentioned uniqueness, this implies that $r_i(\lambda) = c_i(\lambda) r_0(\lambda)$, where $c_i(\lambda)$, $i = 1, 2$ are scalar functions. Since $\mathbf{P}^2 = 1 \otimes 1$, and $((r_0)^t)^t = r_0$, we conclude that $c_i(\lambda) = \pm 1$. Imposing the condition $r_0(0) = 1 \otimes 1$, we have to put $c_i(\lambda) = 1$. Thus, we have proven the relations on the l.h.s of (77) and (78). Employing the first of them and the property (37) of $r(\lambda)$, we prove the second relation in (77) as follows:

$$\begin{aligned} r_0(\lambda, -\gamma) &= (1 \otimes e^{-\gamma\lambda S^3}) r(\lambda, -\gamma)(e^{\gamma\lambda S^3} \otimes 1) = (1 \otimes e^{-\gamma\lambda S^3})\mathbf{P} r(\lambda, \gamma)\mathbf{P}(e^{\gamma\lambda S^3} \otimes 1) \\ &= \mathbf{P}(e^{-\gamma\lambda S^3} \otimes 1) r(\lambda, \gamma)(1 \otimes e^{\gamma\lambda S^3})\mathbf{P} = \mathbf{P}(1 \otimes e^{\gamma\lambda S^3}) r(\lambda, \gamma)(e^{-\gamma\lambda S^3} \otimes 1)\mathbf{P} \\ &= \mathbf{P} r_0(\lambda, \gamma)\mathbf{P} = r_0(\lambda, \gamma)\,. \end{aligned}$$

In the second line we used that $[r(\lambda), \Delta(e^{\gamma\lambda S^3})] = 0$.

Appendix B

Here we provide some details on computation of the Hamiltonians in the cases of spin 1 and spin $\frac{3}{2}$. For $S = 1$, a matrix representation of the spin generators is (as was discussed in Section 4, one has a freedom of rescaling $S^\pm \to \eta^{\mp\frac{1}{2}} S^\pm$ with any real $\eta \neq 0$)

$$S^+ = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad S^- = (S^+)^t, \quad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{98}$$

where t denotes transposition and $a = \sqrt{2\cos\gamma}$. Since $(S^3)^3 = S^3$, any function of S^3 is a polynomial in S^3 of a degree not exceeding two. In particular, we have

$$\sin(tS^3) = S^3 \sin t\,, \qquad \cos(tS^3) = 1 - 2(S^3)^2 \sin^2 \frac{t}{2}\,. \tag{99}$$

Denote $\Upsilon = \frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+)$. With the help of formulae (99) we rewrite (29) as follows

$$\begin{aligned} \mathbf{X}_1 &= q^{S_n^3}\, \Upsilon\, q^{-S_{n+1}^3} + (\cos\gamma)\Big(\Lambda + \frac{1}{2} + 2(\cos\gamma)^2\Big)\,, \qquad (100) \\ \Lambda &= \frac{3}{2} - 2(\cos\gamma)^2 + (\cos\gamma)^2 (S_n^3 S_{n+1}^3) + (\sin\gamma)^2\Big((S_n^3 S_{n+1}^3)^2 - 2(S_n^3)^2 - 2(S_{n+1}^3)^2\Big) \\ &\quad + \mathrm{i}(\sin 2\gamma)\Big(S_n^3 - S_{n+1}^3 + \frac{1}{2}(S_n^3)^2 S_{n+1}^3 - \frac{1}{2} S_n^3 (S_{n+1}^3)^2\Big)\,. \end{aligned}$$

Substituting (100) into (52), applying several times formulae (99) and (59), we obtain the Hamiltonian in the following form

$$\begin{aligned} H_{n,n+1} &= \frac{1}{4\cos\gamma}\Big(-\frac{1}{\cos^2\gamma}(\Upsilon)^2 - \frac{1}{\cos\gamma} q^{S_n^3}\,(\Upsilon\,\Lambda + \Lambda\,\Upsilon)\, q^{-S_{n+1}^3} \qquad (101) \\ &\quad + 4 + 2\cos 2\gamma + (\cos 2\gamma)(S_n^3 S_{n+1}^3 - (S_n^3 S_{n+1}^3)^2) \\ &\quad - 2(\sin\gamma)^2((S_n^3)^2 + (S_{n+1}^3)^2)\Big) + \mathrm{i}\frac{\sin\gamma}{2}\,(S_n^3 - S_{n+1}^3)\,. \end{aligned}$$

Deriving the first term here we used the following analogue of relation (50)

$$e^{tS^3}\,(S^\pm)^2 = (S^\pm)^2\,e^{-tS^3} = e^{\pm t}\,(S^\pm)^2\,. \tag{102}$$

Next, we observe that the following identity holds (as can be checked directly in terms of matrices)

$$-F_\alpha\,(q^{S^3_n}\,(\Upsilon\,\Lambda + \Lambda\,\Upsilon)\,q^{-S^3_{n+1}})\,(F_\alpha)^{-1} = \Theta^-_\alpha\,\Upsilon + \Upsilon\,\Theta^+_\alpha\,, \tag{103}$$

with F_α as in (62) and

$$\Theta^\pm_\alpha = \frac{1}{2}(1\otimes 1) + (1 - 2q^{\pm\alpha}\cos\gamma)\,S^3_n S^3_{n+1}\,. \tag{104}$$

Substituting (103)-(104) with $\alpha = 0$ into (101), we obtain the Hamiltonian (53).

The Hamiltonian $\tilde{H}^{(\alpha)}_{n,n+1}$ discussed in Section 6 is obtained from (101) by the twist (68). Notice that the term $(\Upsilon)^2$ in (101) is not affected due to (102). The diagonal part of Hamiltonian apparently commutes with the twist. So the only part of (101) which changes is the second term. It transforms according to (103), which yields the Hamiltonian (69).

Consider now the case $S = \frac{3}{2}$. The spin generators are given by

$$S^+ = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^- = (S^+)^t\,, \quad S^3 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}, \tag{105}$$

with $a_1 = (2\cos 2\gamma + 1)^{\frac{1}{2}}$ and $a_2 = 2\cos\gamma$. Any function of S^3 is a polynomial in S^3 of a degree not exceeding three. In particular,

$$\begin{aligned} \sin(tS^3) &= (2\sin\frac{t}{2} + \frac{1}{3}\sin^3\frac{t}{2})\,S^3 - \frac{4}{3}(\sin^3\frac{t}{2})\,(S^3)^3\,, && (106)\\ \cos(tS^3) &= \cos\frac{t}{2} + \frac{1}{4}\sin\frac{t}{2}\sin t - (\sin\frac{t}{2}\sin t)\,(S^3)^2\,. && (107) \end{aligned}$$

The Hamiltonian (56) for $S = \frac{3}{2}$ takes the following form in terms of $\mathbf{X}_{\frac{3}{2}}$

$$\begin{aligned} H_{n,n+1} = \; & \frac{1}{4(\cos\gamma)^3(1+2\cos 2\gamma)^3}\Big(4(\cos\gamma)\,(\mathbf{X}_{\frac{3}{2}})^2 \\ & - (\mathbf{X}_{\frac{3}{2}})^2\,(13 + 20\cos 2\gamma + 8\cos 4\gamma + 2\cos 6\gamma) \\ & + \mathbf{X}_{\frac{3}{2}}\,(68\cos\gamma + 48\cos 3\gamma + 23\cos 5\gamma + 7\cos 7\gamma + \cos 9\gamma)\Big)\,. \end{aligned} \tag{108}$$

This Hamiltonian can be rewritten in the same way as we did for $S = 1$ using, in particular, relations (60), (106) and (107), and appropriate analogues of (102) and (103). The final form is given by (57), with

$$\begin{aligned} \mathbf{Q} = \; & \frac{1}{192}\,(1\otimes 1)\,(815 - 5832\cos\gamma + 2352\cos 2\gamma - 3888\cos 3\gamma \\ & +2542\cos 4\gamma - 1620\cos 5\gamma + 1600\cos 6\gamma - 324\cos 7\gamma + 467\cos 8\gamma) \\ & +\frac{2}{3}(\sin\frac{\gamma}{2})^2\,S^3_n S^3_{n+1}\,(106 + 37\cos\gamma + 186\cos 2\gamma + 29\cos 3\gamma \\ & +88\cos 4\gamma + 13\cos 5\gamma + 28\cos 6\gamma + 5\cos 7\gamma) \end{aligned}$$

$$
\begin{aligned}
&+(\sin\frac{\gamma}{2})^2\,(S_n^3 S_{n+1}^3)^2\,(28-53\cos\gamma+54\cos 2\gamma-31\cos 3\gamma \\
&\quad +28\cos 4\gamma-11\cos 5\gamma+10\cos 6\gamma-\cos 7\gamma) \\
&+\frac{1}{12}(\sin\frac{\gamma}{2})^2\left((S_n^3)^2+(S_{n+1}^3)^2\right)(412+1687\cos\gamma+606\cos 2\gamma \\
&\quad +1085\cos 3\gamma+268\cos 4\gamma+433\cos 5\gamma+58\cos 6\gamma+83\cos 7\gamma)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{\frac{3}{2}} \;=\; &\frac{\cos^2\gamma}{4}\,(1\otimes 1)\,(10328+17080\cos 2\gamma+9071\cos 4\gamma+3220\cos 6\gamma+621\cos 8\gamma) \\
&-\frac{1}{9}(\sin\frac{\gamma}{2})^2\,S_n^3 S_{n+1}^3\,(8364+5020\cos\gamma+12752\cos 2\gamma+2499\cos 3\gamma+7150\cos 4\gamma \\
&\quad +1023\cos 5\gamma+3320\cos 6\gamma+447\cos 7\gamma+814\cos 8\gamma+83\cos 9\gamma) \\
&-\frac{4}{3}(\sin\frac{\gamma}{2})^2\,(S_n^3 S_{n+1}^3)^2\,(536-472\cos\gamma+864\cos 2\gamma-347\cos 3\gamma+650\cos 4\gamma \\
&\quad -163\cos 5\gamma+272\cos 6\gamma-65\cos 7\gamma+78\cos 8\gamma-9\cos 9\gamma) \\
&-\frac{64}{9}(\sin\frac{\gamma}{2})^4\,(S_n^3 S_{n+1}^3)^3\,(50-200\cos\gamma+48\cos 2\gamma-168\cos 3\gamma+45\cos 4\gamma \\
&\quad -76\cos 5\gamma+28\cos 6\gamma-20\cos 7\gamma+13\cos 8\gamma) \\
&-(\sin 2\gamma)^2\left((S_n^3)^2+(S_{n+1}^3)^2\right)(169+274\cos 2\gamma+106\cos 4\gamma+27\cos 6\gamma) \\
&+\frac{4}{9}(\sin\frac{\gamma}{2})^2\left((S_n^3)^3 S_{n+1}^3+S_n^3(S_{n+1}^3)^2\right)(444-2516\cos\gamma+560\cos 2\gamma-2037\cos 3\gamma \\
&\quad +334\cos 4\gamma-1137\cos 5\gamma+200\cos 6\gamma-417\cos 7\gamma+46\cos 8\gamma-85\cos 9\gamma)\,.
\end{aligned}
$$

The equivalence of (108) and (57) as 16×16 matrices has been verified with the help of the program Mathematica.

Appendix C

Here we explain the origin of the equation (89) that was important for our discussion on the open chain Hamiltonian in Section 8.

Recall that, by definition (see e.g., [14, 26]), a Hopf algebra $\mathcal{A}$ possesses the antipode map $s:\mathcal{A}\to\mathcal{A}$ which is an anti-homomorphism consistent with the co-multiplication and the co-unit in the sense that $m((s\otimes id)\Delta(\xi)) = m((id\otimes s)\Delta(\xi)) = \epsilon(\xi)\cdot 1$ (here $m:\mathcal{A}^{\otimes 2}\to\mathcal{A}$ is the multiplication). Assume that there exists an element $\chi\in\mathcal{A}$ which realizes the square of the antipode (which is a homomorphism) as an inner automorphism, i.e., $s(s(\xi)) = \chi\,\xi\,\chi^{-1}$ for any $\xi\in\mathcal{A}$. Then, as was proven in [26],

$$
\mathrm{tr}_1((\chi^{-1}\otimes 1)\,b) \tag{109}
$$

belongs to the center of $\mathcal{A}$ if an element $b\in\mathcal{A}^{\otimes 2}$ satisfies $[b,\Delta(\xi)]=0$ for any $\xi\in\mathcal{A}$.

For $\mathcal{A}=U_q(sl_2)$ the antipode consistent with the co-multiplication (7) is given by

$$
s(S^{\pm}) = -q^{\mp 1}\,S^{\pm}\,, \qquad s(S^3) = -S^3\,.
$$

It is easy to see that in this case $\chi = q^{-2S^3}$. Since the universal r-matrix $r(\lambda)$ has the property (12), we can apply (109) and infer that its q-trace, $\mathrm{tr}_1((q^{2S^3}\otimes 1)\,r(\lambda))$, belongs to the center of $U_q(sl_2)$. Consequently, the same holds for the local Hamiltonian $H_{n,n+1}$ constructed as in (20). Being evaluated in an irreducible representation, the q-trace of $H_{n,n+1}$ becomes just a constant, as was stated in eq. (89).

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